Formulae for Solutions to Laplace's equation, $\nabla^2 V = 0$

Cartesian Coordinates (no $z$ dependence):

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

If $f(y) = \sum_{n=1}^{\infty} \{A_n \cos(\frac{n\pi y}{a}) + B_n \sin(\frac{n\pi y}{a})\}$, then

$$A_n = \frac{2}{a}\int_0^a f(y)\cos(\tfrac{n\pi y}{a})dy \quad \text{and} \quad B_n = \frac{2}{a}\int_0^a f(y)\sin(\tfrac{n\pi y}{a})dy \ .$$

Cylindrical Coordinates (no $z$ dependence):

$$V(s,\phi) = A_0 + B_0 \ln s + \sum_{m=1}^{\infty} s^m (A_m \cos m\phi + B_m \sin m\phi) + \sum_{m=1}^{\infty} \frac{1}{s^m}(C_m \cos m\phi + D_m \sin m\phi)$$

If $f(\phi) = \sum_{m=1}^{\infty} \{A_m \cos(m\phi) + B_m \sin(m\phi)\}$, then

$$A_m = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos(m\phi)d\phi \quad \text{and} \quad B_m = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin(m\phi)d\phi \ .$$

Spherical coordinates (no $\phi$ dependence): $V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

If $f(\theta) = \sum_{\ell=0}^{\infty} C_\ell P_\ell(\cos(\theta))$, then $C_\ell = \frac{2\ell+1}{2}\int_0^{\pi} f(\theta)P_\ell(\cos(\theta))\sin(\theta)d\theta$ .

Legendre polynomials:

$$P_0 = 1$$
$$P_1 = \cos\theta$$
$$P_2 = \tfrac{1}{2}(3\cos^2\theta - 1)$$
$$P_3 = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

$$P_\ell(1) = 1 \qquad P_\ell(-1) = (-1)^\ell$$

$$\int_0^{\pi} P_\ell(\cos\theta)\,P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$