Phys 208 – Theoretical Physics – Final Exam (May 4, 2012)

1.(20 pts) Consider the function $f(x)=\begin{cases}1, & \pi/2<|x|<\pi \\ 0, & \text{otherwise}\end{cases}$

a) Give a sketch of the function.

b) Find the Fourier transform $g(\alpha)$ of the function.

c) Substitute your result into the equation for $f(x)$ and use parity arguments (even/odd) to obtain the integral, $I(x)=\int_0^\infty \left[\frac{\sin(\alpha\pi)-\sin(\alpha\pi/2)}{\alpha}\right]\cos(\alpha x)\,d\alpha$.

d) List all the possible values for $I(x)$ as well as for what possible values of $x$.

2.(20 pts) a) Solve the following differential equation by using a power series: $(1-x^2)\,y''-x\,y'+k^2\,y=0$. What is the recursion relation for the coefficients? Since the equation is second order, there will be two independent coefficients or two different series corresponding to the two independent coefficients. Just find the first three terms of each series.

b) As is often the case, both series will diverge if $k$ is not an integer. However for $k=0,1,2,3,\cdots$ one series will truncate while the other series will still diverge, so in order to get solutions the independent coefficient of the divergent series is set to zero. Using this approach determine the first four possible solutions to the differential equation; namely, $y_0(x), y_1(x), y_2(x), y_3(x)$.

c) The solutions of this equation with the independent coefficients chosen to make $y_k(1)=1$ are called Chebyshev polynomials $T_k(x)$. Determine the first four; namely, $T_0(x), T_1(x), T_2(x), T_3(x)$.

3.(20pts) A ball of mass $M$ and radius $R$ rolls without slipping down an inclined plane under the action of gravity. The incline plane is at an angle α. The moment of inertia of the ball about an axis through its center is given by $I=\frac{2}{5}MR^2$.

a) Determine a Lagrangian which describes the motion of the ball.

b) Determine Lagrange's equation of motion for the ball.

c) Determine the acceleration of the center-of-mass of the ball down the incline plane.

4.(20 pts) Consider the "particle in a box" problem. The potential energy is zero for $0 \le x \le L$ and infinite otherwise. Suppose the wave function in the box is such that

$$\Phi(x) = \begin{cases} N & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases} .$$

a) Normalize $\Phi(x)$.

b) Expand $\Phi(x)$ in terms of the basis functions (eigenfunctions) of the box $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$, *i.e*, expand $\Phi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$.

c) What is the probability, $\wp_n$, that the particle is in a given eigenstate?

5.(20 pts) A hydrogen-like atom has a nucleus of charge *Ze* and an electron of charge $-e$. *Z* is a constant and the attractive potential is the Coulomb potential between the nuclear charge and the electron charge. The distance between the nucleus and the electron is the radial coordinate *r*. The radial Hamiltonian for this hydrogen-like atom in an exited state is given by

$$H_{op} R(r) = -\frac{\hbar^2}{2mr^2} \frac{d}{dr}\left(r^2 \frac{dR(r)}{dr}\right) - \frac{Ze^2}{4\pi\varepsilon_0 r} R(r) + \frac{\hbar^2}{mr^2} = E R(r),$$

where $\hbar$ is Planck's constant divided by $2\pi$, *m* is the effective electron mass, *E* is the bound state energy, and $R(r)$ is the bound state wavefunction.

a) Show that $R(r) = N r e^{-ar}$ (*N* is a constant) is a solution. Hint: You need to determine *a* and *E* by setting coefficients of powers of *r* equal to zero. Using your result for *a*, what value do you obtain for *E* ? You may find it useful to define a constant, $a_0 = \frac{\hbar^2 4\pi\varepsilon_0}{me^2}$.

b) Recall the radial probability density function is defined such that $f(r)dr = r^2[R(r)]^2 dr$ is the probability that the electron is between *r* and $r + dr$ and *r* varies from 0 to $\infty$. Use this information to determine *N*. Recall: $\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$.

c) Calculate $\Delta r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2}$ for this wavefunction.

6.(10 pts) Bonus problem. Starting with the integral $\int_0^\infty e^{-ax}dx = \frac{1}{a}$ use Leibniz' Rule for differentiation of an integral to prove that $\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$.