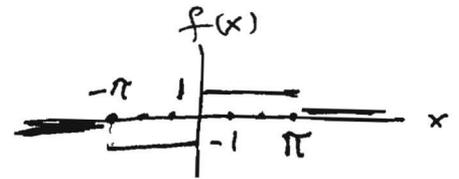


Phys 208 - HW 19

7.12.3

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ 0 & |x| > \pi \end{cases}$$



$$\begin{aligned} g(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx = \frac{1}{2\pi} \left\{ \int_{-\pi}^0 (-1) e^{-i\alpha x} dx + \int_0^{\pi} (1) e^{-i\alpha x} dx \right\} \\ &= \frac{1}{2\pi} \left\{ -\frac{e^{-i\alpha x}}{(-i\alpha)} \Big|_{-\pi}^0 + \frac{e^{-i\alpha x}}{(-i\alpha)} \Big|_0^{\pi} \right\} \\ &= \frac{1}{2\pi i\alpha} \left\{ \underbrace{1 - e^{-i\alpha\pi} - e^{i\alpha\pi} + 1}_{2 - 2\cos\alpha\pi} \right\} \end{aligned}$$

Thus

$$g(\alpha) = \frac{1}{i\alpha\pi} (1 - \cos\alpha\pi)$$

Therefore

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x} d\alpha = \int_{-\infty}^{\infty} \left[\frac{1 - \cos(\alpha\pi)}{i\alpha\pi} \right] e^{i\alpha x} d\alpha$$

7.12.17 The function in problem 7.12.3 is an odd function so it can be represented by a Fourier Sine Transform.

$$\begin{aligned} g_s(\alpha) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(x) \sin(\alpha x) dx = \sqrt{\frac{2}{\pi}} \int_0^{\pi} (1) \sin(\alpha x) dx \\ &= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos(\alpha x)}{\alpha} \right] \Big|_0^{\pi} = \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos(\alpha\pi)}{\alpha} \right] \end{aligned}$$

Thus

$$\begin{aligned} f_s(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_s(\alpha) \sin(\alpha x) d\alpha \\ &= \frac{2}{\pi} \int_0^{\infty} \left[\frac{1 - \cos(\alpha\pi)}{\alpha} \right] \sin(\alpha x) d\alpha \end{aligned}$$

7.12.17 (cont)

In problem 7.12.3 $f(x) = \int_{-\infty}^{\infty} \left[\frac{1 - \cos(\alpha\pi)}{i\alpha\pi} \right] e^{i\alpha x} d\alpha$

Use the Euler identity
to write

$$f(x) = \int_{-\infty}^{\infty} \underbrace{\left[\frac{1 - \cos(\alpha\pi)}{i\alpha\pi} \right]}_{\text{odd in } \alpha} \left(\underbrace{\cos(\alpha x)}_{\text{even in } \alpha} + i \underbrace{\sin(\alpha x)}_{\text{odd in } \alpha} \right) d\alpha$$

$\underbrace{\hspace{10em}}_{\text{overall odd}} \qquad \underbrace{\hspace{10em}}_{\text{overall even}}$

so $\int_{-\infty}^{\infty} (\text{odd}) d\alpha = 0$ so $\int_{-\infty}^{\infty} (\text{even}) d\alpha$

$$= 2 \int_0^{\infty} (\text{even}) d\alpha$$

Therefore

$$f(x) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \left[\frac{1 - \cos(\alpha\pi)}{\alpha} \right] i \sin(\alpha x) d\alpha$$

$$= \frac{2}{\pi} \int_0^{\infty} \left[\frac{1 - \cos(\alpha\pi)}{\alpha} \right] \sin(\alpha x) d\alpha = f_S(x)$$

which is the same result as that obtained
in the Fourier Sine Transform

$$2.12.21 \quad f(x) = e^{-x^2/(2\sigma^2)} \quad -\infty < x < \infty$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-i\alpha x} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-[x^2 + 2i\sigma^2\alpha x]/(2\sigma^2)} dx$$

Now

$$x^2 + 2(i\alpha\sigma^2)x = x^2 + 2(i\alpha\sigma^2)x + (i\alpha\sigma^2)^2 - (i\alpha\sigma^2)^2$$

$$\text{Thus} \quad = [x + i\alpha\sigma^2]^2 + \alpha^2\sigma^4$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-[x + i\alpha\sigma^2]^2/(2\sigma^2)} e^{-\frac{\alpha^2\sigma^4}{2}} dx$$

$$\text{Let } y = x + i\alpha\sigma^2$$

Then

$$g(\alpha) = \frac{1}{2\pi} e^{-\frac{\alpha^2\sigma^4}{2}} \int_{-\infty}^{\infty} e^{-y^2/(2\sigma^2)} dy$$

Use the integral derived in class $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$
or look it up in table.

$$\text{Thus} \quad \int_{-\infty}^{\infty} e^{-y^2/(2\sigma^2)} dy = \sigma\sqrt{2\pi} \quad \text{since } a = \frac{1}{2\sigma^2}$$

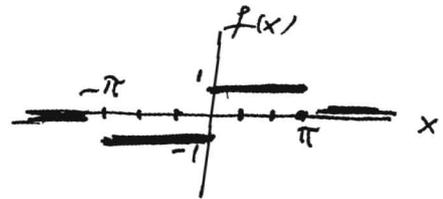
This gives

$$g(\alpha) = \frac{\sigma}{\sqrt{2\pi}} e^{-\alpha^2\sigma^2/2}$$

7.12.23 In problem 7.12.17 one finds that HW19 4/9

$$\int_0^{\infty} \left[\frac{1 - \cos(\alpha\pi)}{\alpha} \right] \sin(\alpha x) d\alpha = \frac{\pi}{2} f(x)$$

where $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ 0 & |x| > \pi \end{cases}$



By Dirichlet's Thm at $x=1$ the integral converges to $\frac{\pi}{2} f(1)$ since $f(x)$ is continuous at $x=1$, i.e.

$$\int_0^{\infty} \left(\frac{1 - \cos \alpha\pi}{\alpha} \right) \sin \alpha d\alpha = \frac{\pi}{2}$$

Likewise at $x=\pi$ the function $f(x)$ is discontinuous so the integral will converge to $\frac{\pi}{2}$ (midpoint) $= \frac{\pi}{2} \left(\frac{1}{2} \right) = \frac{\pi}{4}$ by Dirichlet's Thm.

Thus
$$\int_0^{\infty} \left(\frac{1 - \cos \alpha\pi}{\alpha} \right) \sin(\alpha\pi) d\alpha = \frac{\pi}{4}$$

$$7.12.24 a) f(x) = e^{-|x|} \text{ for } -\infty < x < \infty$$

$$g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|x|} e^{-i\alpha x} dx$$

Now $|x| = x$ for $0 < x < \infty$

and $|x| = -x$ for $-\infty < x < 0$

Therefore

$$g(\alpha) = \frac{1}{2\pi} \left\{ \int_0^{\infty} e^{-(1+i\alpha)x} dx + \int_{-\infty}^0 e^{(1-i\alpha)x} dx \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{e^{-(1+i\alpha)x}}{-(1+i\alpha)} \Big|_0^{\infty} + \frac{e^{(1-i\alpha)x}}{(1-i\alpha)} \Big|_{-\infty}^0 \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{1}{1+i\alpha} + \frac{1}{1-i\alpha} \right\} = \frac{1}{\pi} \left(\frac{1}{1+\alpha^2} \right)$$

Thus

$$f(x) = \int_{-\infty}^{\infty} g(\alpha) e^{i\alpha x} d\alpha = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{1+\alpha^2} \right) (\underbrace{\cos \alpha x}_{\substack{\text{even} \\ \text{wrt } \alpha}} + i \underbrace{\sin \alpha x}_{\text{odd}}) d\alpha$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2 + 1} d\alpha$$

$$\text{or } \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2 + 1} d\alpha = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-|x|}$$

7.12.24 (cont)

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b) Since $f(x) = e^{-|x|}$ is an even function, we can use the Fourier cosine transform eqns. Thus

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\alpha x) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos \alpha x dx \quad \text{since } |x| = x$$

Now $\cos \alpha x = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}$ for $0 < x < \infty$

This yields

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{2} \left\{ \int_0^{\infty} e^{-(1-i\alpha)x} dx + \int_0^{\infty} e^{-(1+i\alpha)x} dx \right\}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \left\{ \frac{1}{1-i\alpha} + \frac{1}{1+i\alpha} \right\} = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$$

Therefore

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(\alpha) \cos \alpha x d\alpha = \frac{2}{\pi} \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2 + 1} d\alpha$$

or $\int_0^{\infty} \frac{\cos \alpha x}{\alpha^2 + 1} d\alpha = \frac{\pi}{2} f(x)$ as in part a.

c) If $f(x) = \frac{1}{1+x^2}$, the Fourier cosine transform is

$$g_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\cos \alpha x}{1+x^2} dx = \sqrt{\frac{\pi}{2}} e^{-|\alpha|}$$

Same integral as in part b with $\alpha \leftrightarrow x$ interchanged

7.12.33 We have $f(x) = e^{-|x|}$ and

$g(\alpha) = \frac{1}{\pi} \left(\frac{1}{1+\alpha^2} \right)$ from problem 7.12.24(a).

Parseval's theorem states that

$$\int_{-\infty}^{\infty} |g(\alpha)|^2 d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Now $g(\alpha)$ is even w.r.t α
and $f(x)$ is even w.r.t x

Thus we need to verify that

$$\begin{aligned} \frac{1}{\pi^2} \int_0^{\infty} \left(\frac{1}{1+\alpha^2} \right)^2 d\alpha &= \frac{1}{2\pi} \int_0^{\infty} e^{-2x} dx = \frac{1}{2\pi} \left(\frac{e^{-2x}}{-2} \right) \Big|_0^{\infty} \\ &= \frac{1}{4\pi} \end{aligned}$$

Consider

$$\int_0^{\infty} \frac{d\alpha}{(1+\alpha^2)^2} = \lim_{a \rightarrow 1} \int_0^{\infty} \frac{d\alpha}{(a+\alpha^2)^2}$$

$$\text{Then } \frac{\partial}{\partial a} \left(\frac{1}{a+\alpha^2} \right) = -\frac{1}{(a+\alpha^2)^2}$$

$$\begin{aligned} \text{Thus } \int_0^{\infty} \frac{d\alpha}{(1+\alpha^2)^2} &= \lim_{a \rightarrow 1} \left[(-1) \frac{\partial}{\partial a} \left(\int_0^{\infty} \frac{d\alpha}{a+\alpha^2} \right) \right] \\ &= \lim_{a \rightarrow 1} \left[(-1) \frac{\partial}{\partial a} \left(\frac{1}{\sqrt{a}} \tan^{-1} \left(\frac{\alpha}{\sqrt{a}} \right) \right) \right] \Big|_0^{\infty} \\ &= \lim_{a \rightarrow 1} \left[(-1) \frac{\partial}{\partial a} \left(\frac{1}{\sqrt{a}} \frac{\pi}{2} \right) \right] = \lim_{a \rightarrow 1} \left[\frac{\pi}{2} \frac{1}{2a^{3/2}} \right] \end{aligned}$$

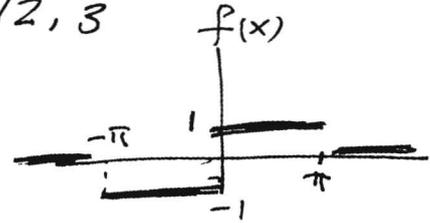
Therefore

$$\frac{1}{\pi^2} \int_0^{\infty} \left(\frac{1}{1+\alpha^2} \right)^2 d\alpha = \frac{1}{\pi^2} \frac{\pi}{4} = \frac{1}{4\pi} \quad \text{Q.E.D.}$$

HW 19.1 From problem 7.12, 3

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$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \\ 0 & |x| > \pi \end{cases}$$



and

$$g(\alpha) = \frac{1}{i\alpha\pi} (1 - \cos \alpha\pi)$$

From Parseval's theorem we have

$$\int_{-\infty}^{\infty} |g(\alpha)|^2 d\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Thus

$$\frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{[1 - \cos \alpha\pi]^2}{\alpha^2} d\alpha = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1) dx = \frac{1}{2\pi} (2\pi) = 1$$

so

$$\int_{-\infty}^{\infty} \frac{[1 - \cos \alpha\pi]^2}{\alpha^2} d\alpha = \pi^2$$

or

$$\int_0^{\infty} \frac{[1 - \cos \alpha\pi]^2}{\alpha^2} d\alpha = \frac{\pi^2}{2}$$



HW 19.2

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$$a) \int_2^6 (3x^2 - 2x - 1) \delta(x-3) dx = 27 - 6 - 1 = 20$$

$x=3$ between 2 & 6

$$b) \int_0^5 \cos x \delta(x-\pi) dx = \cos \pi = -1$$

$x=\pi$ between 0 & 5

$$c) \int_0^3 x^3 \delta(x+1) dx = 0$$

$x=-1$ is not between 0 & 3

$$d) \int_{-\infty}^{\infty} \ln(x+3) \delta(x+2) dx = \ln 1 = 0$$

$x=-2$ between $-\infty$ & ∞

$$e) \int_{-2}^2 (2x+3) \delta(3x) dx = \frac{1}{3}(3) = 1$$

$\frac{1}{3} \delta(x)$
 $x=0$ between -2 & 2

$$f) \int_{-1}^1 9x^2 \delta(3x+1) dx = \frac{1}{3} 9 \left(-\frac{1}{3}\right)^2 = \frac{1}{3}$$

$\frac{1}{3} \delta\left(x + \frac{1}{3}\right)$
 $x = -\frac{1}{3}$ between -1 & 1