

$$\begin{aligned}
 7.4.5 \text{ Avg of } \cos^2\left(\frac{x}{2}\right) \text{ on } (0, \pi/2) &= \frac{2}{\pi} \int_0^{\pi/2} \cos^2\left(\frac{x}{2}\right) dx \\
 &= \frac{2}{\pi} \cdot \frac{1}{2} \int_0^{\pi/2} (1 + \cos x) dx \\
 &= \frac{1}{\pi} (x + \sin x) \Big|_0^{\pi/2} = \frac{1}{\pi} \left(1 + \frac{\pi}{2}\right)
 \end{aligned}$$

$$\begin{aligned}
 7.4.13 \quad \int_a^b \sin^2(kx) dx &= \frac{1}{2} \int_a^b [1 - \cos(2kx)] dx \\
 &= \frac{1}{2} \left[ x - \frac{\sin(2kx)}{2k} \right] \Big|_a^b \\
 &= \frac{1}{2} (b-a) - \frac{1}{2k} [\sin(2kb) - \sin(2ka)]
 \end{aligned}$$

i) If  $k(b-a) = n\pi$  or  $kb = n\pi + ka$   
 where  $n$  is an integer, then

$$\begin{aligned}
 \sin(2kb) - \sin(2ka) &= \sin(2ka + 2n\pi) - \sin(2ka) \\
 &= \sin(2ka) \cancel{\cos(2n\pi)}^1 + \cos(2ka) \cancel{\sin(2n\pi)}^0 \\
 &\quad - \sin(2ka) \\
 &= \sin 2ka - \sin 2ka = 0
 \end{aligned}$$

ii) If  $kb = n\pi/2$  and  $ka = m\pi/2$ ;  $n, m$  integers;  
 then  $\sin(2kb) - \sin(2ka) = \sin(n\pi) - \sin(m\pi)$

7.4.13 (cont) Thus  $\int_a^b \sin^2(kx) dx = \frac{1}{2}(b-a)$

in either case.

Likewise  $\int_a^b \cos^2(kx) dx = \int_a^b [1 - \sin^2(kx)] dx$   
 $= (b-a) - \frac{1}{2}(b-a) = \frac{1}{2}(b-a)$

7.4.15

a)  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 \pi x dx = \frac{1}{2} \left( \frac{11}{4} + \frac{1}{4} \right) = \frac{1}{2} \cdot \frac{12}{4} = \frac{3}{2}$

D/C  $\pi \left( \frac{11}{4} + \frac{1}{4} \right) = 3\pi = k(b-a)$

b)  $\int_{-1}^2 \sin^2 \left( \frac{\pi x}{3} \right) dx = \frac{1}{2} [2 - (-1)] = \frac{3}{2}$

D/C  $k(b-a) = \frac{\pi}{3} (2 - \underbrace{(-1)}_{3}) = \pi$

$$7.5.7 \quad f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases} \quad HW17 \quad 3/5$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left\{ \int_{-\pi}^0 0 dx + \int_0^{\pi} x dx \right\} = \frac{1}{\pi} \frac{\pi^2}{2} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x \cos nx dx = \frac{1}{\pi n^2} \int_0^{n\pi} u \cos u du$$

Let  $u = nx$

$$= \frac{1}{\pi n^2} \left\{ u \cancel{\sin u} \Big|_0^{n\pi} - \int_0^{n\pi} \sin u du \right\} = \frac{1}{\pi n^2} \cancel{\cos u} \Big|_0^{n\pi}$$

$$= \frac{1}{\pi n^2} (\cos n\pi - 1) = \frac{1}{\pi n^2} [(-1)^n - 1] = \begin{cases} 0 & \text{even} \\ -\frac{2}{\pi n^2} & n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x \sin nx dx = \frac{1}{\pi n^2} \int_0^{n\pi} u \cancel{\sin u} du$$

Let  $u = nx$

$$= \frac{1}{\pi n^2} \left\{ -u \cos u \Big|_0^{n\pi} + \int_0^{n\pi} \cos u du \right\} = \frac{1}{\pi n^2} \left\{ -n\pi \cos n\pi + \cancel{\sin u} \Big|_0^{n\pi} \right\}$$

$$= \frac{1}{\pi n^2} (-n\pi) \cos n\pi = \frac{(-1)^{n+1}}{n}$$

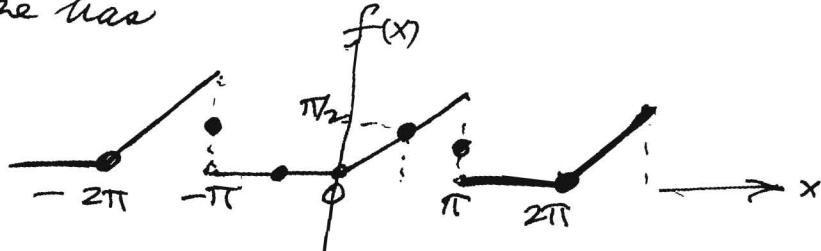
Therefore  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$   
 becomes

$$\begin{aligned} f(x) &= \frac{\pi}{4} - \frac{2}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\cos nx}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \\ &= \frac{\pi}{4} - \frac{2}{\pi} \left( \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right) \\ &\quad + \left( \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \right) \end{aligned}$$

$$7.6.7. \quad f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

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Thus one has



According to Dirichlet's theorem

at  $x = \pm 2\pi$  the series converges to 0

$$x = \pm \pi \quad " \quad " \quad " \quad " \quad \frac{\pi}{2}$$

$$x = -\frac{\pi}{2} \quad " \quad " \quad " \quad " \quad 0$$

$$x = +\frac{\pi}{2} \quad " \quad " \quad " \quad " \quad \frac{\pi}{2}$$

$$x = 0 \quad " \quad " \quad " \quad " \quad 0$$

7.6.14 The series for the function in 7.5.7 & 7.6.7  
to

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$$

At  $x=0$ , the series converges to 0

and  $\sin nx|_{x=0} = 0$  and  $\cos nx|_{x=0} = 1$

Thus

$$0 = \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

7.6.14 (cont.)

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At  $x = \pi$ , the series converges to  $\frac{\pi}{2}$

$$\sin nx \Big|_{x=\pi} = \sin n\pi = 0$$

and

$$\cos nx \Big|_{x=\pi} = \cos n\pi = (-1)^n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

Thus

$$\frac{\pi}{2} = \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} = \frac{\pi}{4} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\text{or } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{2} \left[ \frac{\pi}{2} - \frac{\pi}{4} \right] = \frac{\pi}{2} \left( \frac{\pi}{4} \right) = \frac{\pi^2}{8}$$

At  $x = \frac{\pi}{2}$ , the series converges to  $\frac{\pi}{2}$

$$\cos nx \Big|_{x=\frac{\pi}{2}} = \cos n \frac{\pi}{2} = 0 \text{ odd}$$

and

$$\sin nx \Big|_{x=\frac{\pi}{2}} = \sin n \frac{\pi}{2} = 0 \text{ even}$$

Thus

$$\text{or } \frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} \text{ since } (-1)^{n+1} = 1$$

$$\frac{\pi}{4} = \cancel{\sin \frac{\pi}{2}}^1 + \cancel{\frac{\sin(3\pi/2)}{3}}^{-1} + \cancel{\frac{\sin(5\pi/2)}{5}}^{+1} \text{ if } n \text{ is odd} + \dots$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

or

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$$