Complex Impedance

We will consider three circuit elements. They are a resistor *R*, an inductor *L*, and a capacitor *C*. The resistor *R* has units of ohms (Ω); the inductor *L* has units of henries (H); and the capacitor has units of farads (F).

The physical source voltage is taken to be $V_p(t) = V_0 \sin(\omega t)$, where $\omega = 2\pi f$ and f is the source frequency in hertz (Hz) and ω is the angular frequency in radians/second. We define the complex voltage as:

$\tilde{V} = V_0 e^{i\omega t}$ and the actual physical voltage as $V_p = \mathrm{Im}(\tilde{V}) = V_0\,\mathrm{Im}(e^{i\omega t}) = V_0 \sin(\omega t)$.

The complex impedances for the three elements are: $\tilde{Z}_R = R$; $\tilde{Z}_L = i\omega L$; $\tilde{Z}_C = \dfrac{1}{i\omega C} = -\dfrac{i}{\omega C}$. The units for the complex impedances are all in ohms (Ω). Algebraically the complex impedances work like simple resistors. Thus the equivalent complex impedance for impedances in series is the sum of the impedances and the equivalent complex impedance for impedances in parallel is the sum of the inverses. Thus

$\tilde{Z}_{eq} = \sum_n \tilde{Z}_n$ for a series combination and $\dfrac{1}{\tilde{Z}_{eq}} = \sum_n \dfrac{1}{\tilde{Z}_n}$ for a parallel combination.

If the impedance for a circuit is known such that $\tilde{Z} = \mathrm{Re}\,(\tilde{Z}) + i\,\mathrm{Im}\,(\tilde{Z})$, then one defines the impedance in polar form as

$$\tilde{Z} = |\tilde{Z}|e^{i\phi}, \text{ where } |\tilde{Z}| = \sqrt{[\mathrm{Re}\,(\tilde{Z})]^2 + [\mathrm{Im}\,(\tilde{Z})]^2} \text{ and } \tan\phi = \frac{\mathrm{Im}\,(\tilde{Z})}{\mathrm{Re}\,(\tilde{Z})}.$$

Then the complex current in the circuit is given by

$$\tilde{I} = \frac{V_0}{\tilde{Z}} = \frac{V_0}{|\tilde{Z}|}e^{-i\phi} = I_0 e^{-i\phi} \text{ , where } I_0 = \frac{V_0}{|\tilde{Z}|}.$$

The physical current is obtained by multiplying by $e^{i\omega t}$ and taking the imaginary part; that is

$$I_p = \mathrm{Im}\,(\tilde{I}\,e^{i\omega t}) = \mathrm{Im}\,(I_0 e^{i(\omega t - \phi)}) = I_0 \sin(\omega t - \phi) = \frac{V_0}{|\tilde{Z}|}\sin(\omega t - \phi).$$

The power is given by the time average of the instantaneous power.

Thus $P = \langle I_p(t) V_p(t)\rangle = \langle I_0 V_0 \sin(\omega t)\sin(\omega t - \phi)\rangle$

or $P = \dfrac{I_0 V_0}{T}\displaystyle\int_0^T \sin(\omega t)[\sin(\omega t)\cos\phi - \cos(\omega t)\sin\phi]dt$, where $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$ is a period of the motion.

This yields $P = \dfrac{I_0 V_0}{2}\cos\phi$ and $\cos\phi$ is called the power factor (p.f.).

The circuit is said to be in resonance if it is possible to have $\phi = 0$ or $\cos\phi = 1$.