Today’s agenda:

Magnetic Fields Due To A Moving Charged Particle.

You must be able to calculate the magnetic field due to a moving charged particle.

Biot-Savart Law: Magnetic Field due to a Current Element.

You must be able to use the Biot-Savart Law to calculate the magnetic field of a current-carrying conductor (for example: a long straight wire).

Force Between Current-Carrying Conductors.

You must be able to calculate forces between current-carrying conductors.

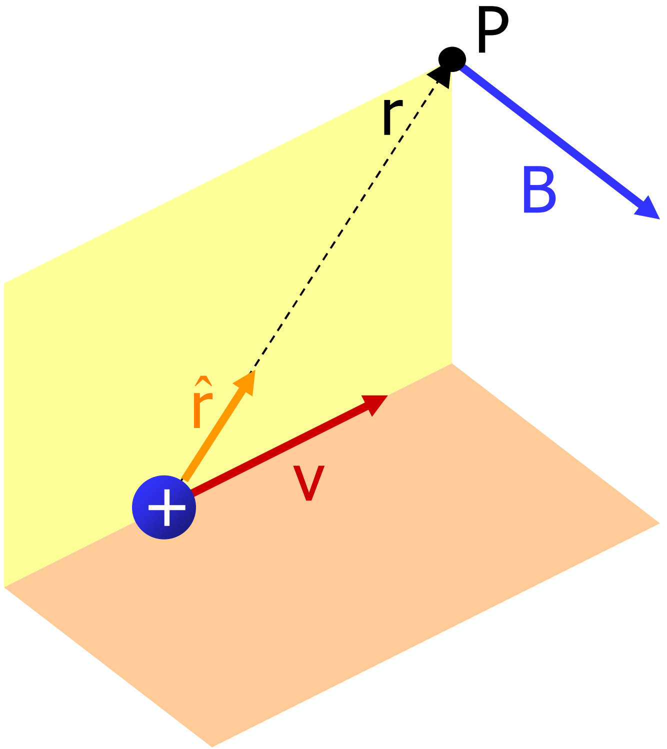
*****last week we studied the effects of magnetic fields on charges, today we learn how to produce magnetic fields

Magnetic Field of a Moving Charged Particle

- moving charge creates magnetic field

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}.}$$

μ_0 is a constant, $\mu_0=4\pi x10^{-7}$ T·m/A



As in lecture 14: Motion with respect to what? You, the earth, the sun?
Highly nontrivial, leads to Einstein's theory of relativity.

Remember:
$\hat{r}$ is unit vector from source point (the thing that causes the field) to the field point P
(location where the field is being measured).

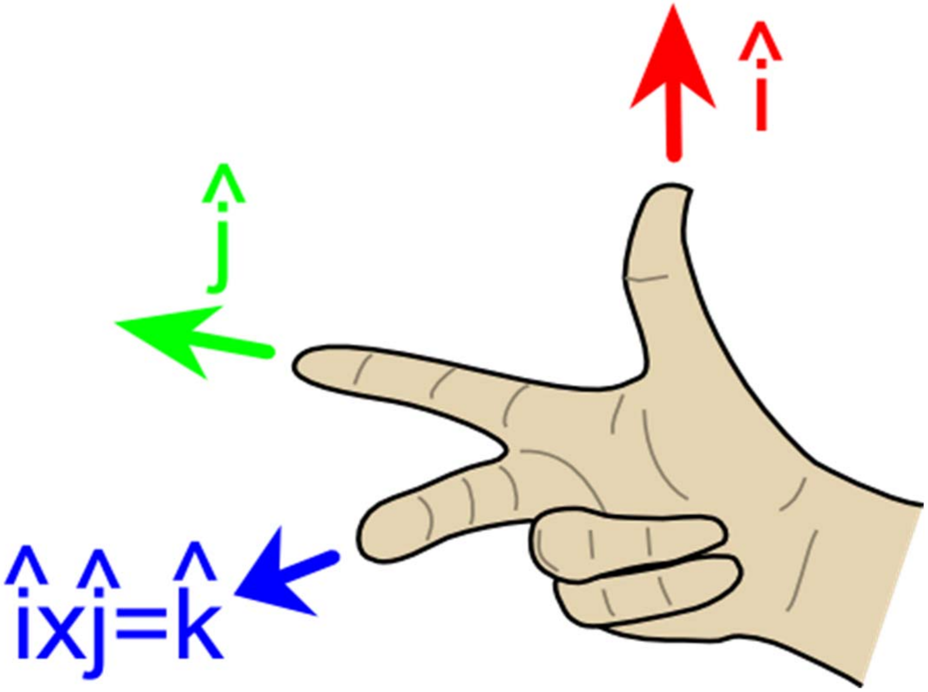
Detour: cross products of unit vectors

- need lots of cross products of unit vectors $\hat{i}, \hat{j}, \hat{k}$

Work out determinant:

Example: $$\hat{k}\times\left(-\hat{j}\right)=\det\begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}=\hat{i}\left(0-(-1)\right)=\hat{i}$$

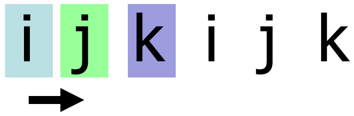
Use right-hand rule:



Detour: cross products of unit vectors

Cyclic property:

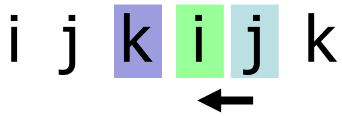
“forward”



i j k i j k

$$\hat{i}\times\hat{j}=\hat{k}$$

“backward”

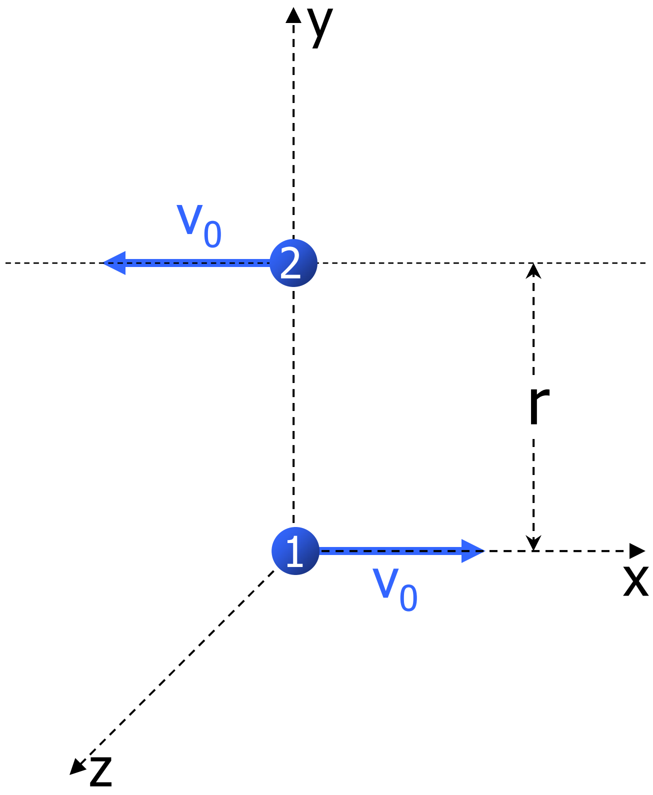


i j k i j k

$$\hat{j}\times\hat{i}=-\hat{k}$$

Example: proton 1 has a speed v_0 (v_0<<c) and is moving along the x-axis in the +x direction. Proton 2 has the same speed and is moving parallel to the x-axis in the –x direction, at a distance r directly above the x-axis. Determine the electric and magnetic forces **on** proton 2 at the instant the protons pass closest to each other.

This is example 28.1 in your text.



Homework Hint: this and the next 3 slides!

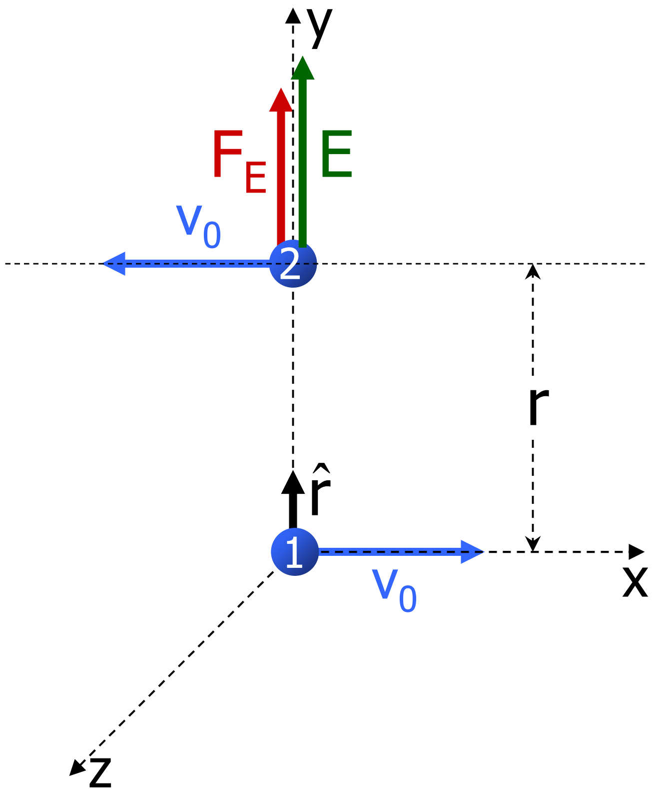
Electric force on proton 2:

- Electric field due to proton 1 at the position of proton 2:

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{e}{r^2}\hat{j}$$

- this electric field exerts a force on proton 2

$$\vec{F}_E = q\vec{E}_1 = e\frac{1}{4\pi\varepsilon_0}\frac{e}{r^2}\hat{j} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\hat{j}$$



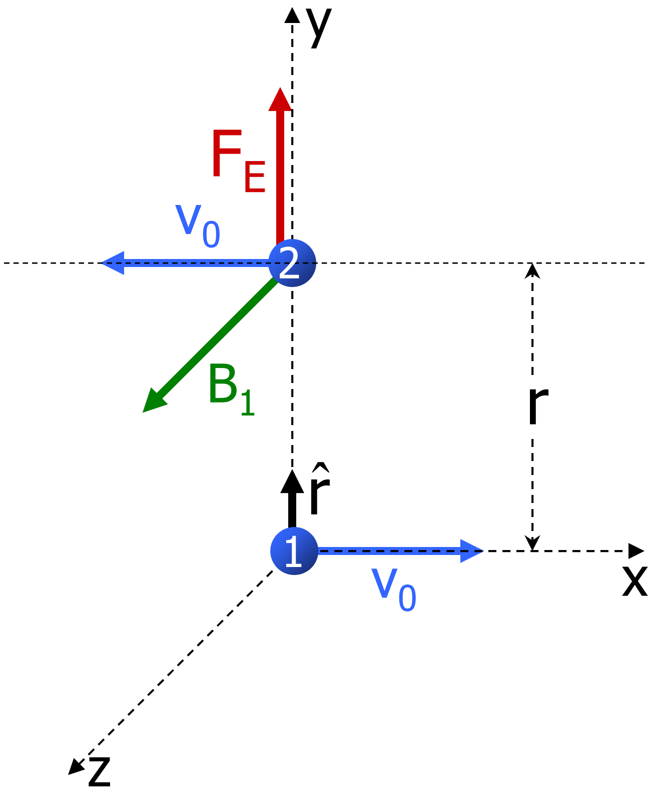
Magnetic force on proton 2:

- magnetic field due to proton 1 at the position of proton 2

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{q_1\vec{v}_1 \times \hat{r}}{r^2}$$

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{ev_0\hat{i} \times \hat{j}}{r^2}$$

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\frac{ev_0}{r^2}\hat{k}$$

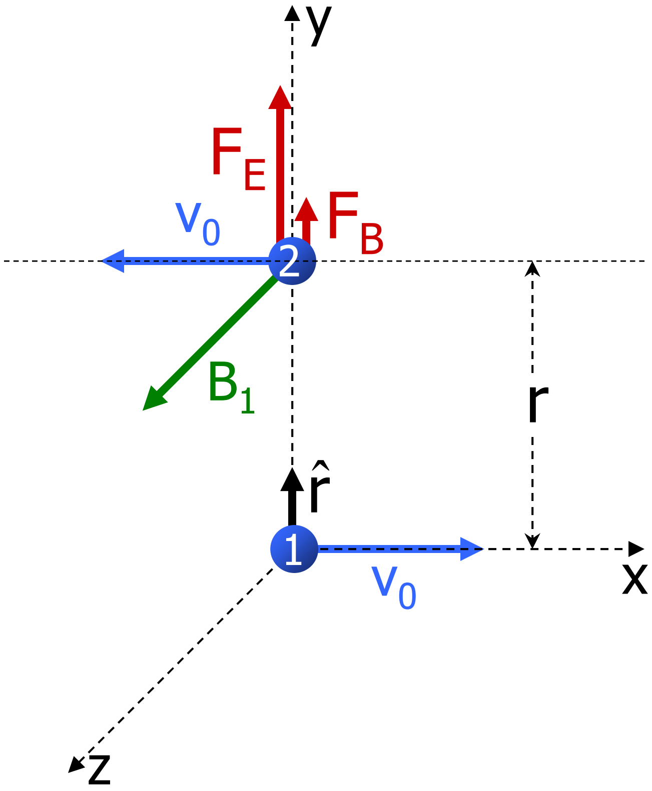


Proton 2 “feels” a magnetic force due to the magnetic field of proton 1.

$$\vec{F}_B = q_2 \vec{v}_2 \times \vec{B}_1$$

$$\vec{F}_B = ev_0\left(-\hat{i}\right)\times\left(\frac{\mu_0}{4\pi}\frac{ev_0}{r^2}\hat{k}\right)$$

$$\vec{F}_B = \frac{\mu_0}{4\pi}\frac{e^2v_0^2}{r^2}\hat{j}$$



What would proton 1 “feel?”

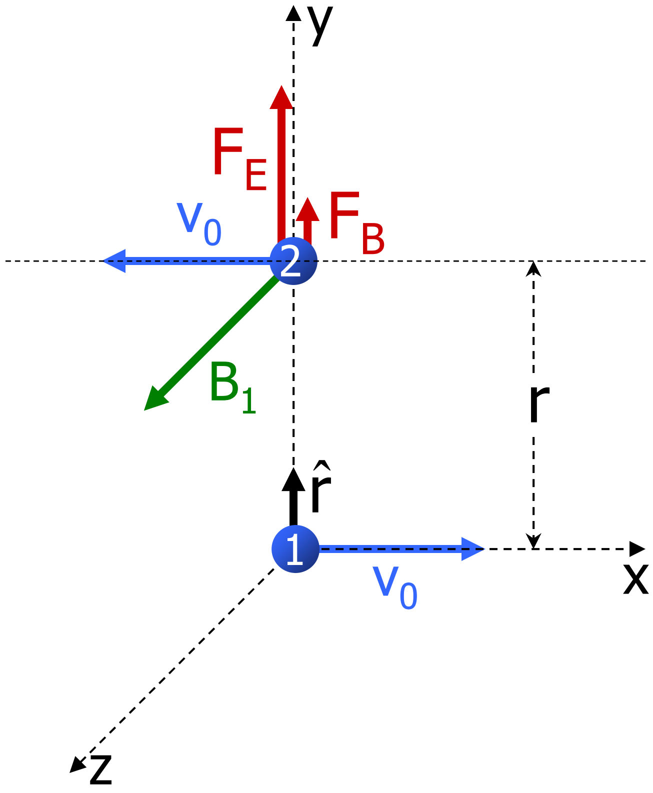
- both forces are in the +y direction
- ratio of their magnitudes:

$$\frac{F_B}{F_E} = \frac{\left(\dfrac{\mu_0}{4\pi}\dfrac{e^2 v_0^2}{r^2}\right)}{\left(\dfrac{1}{4\pi\varepsilon_0}\dfrac{e^2}{r^2}\right)}$$

$$\frac{F_B}{F_E} = \mu_0\varepsilon_0 v_0^2$$

Later we will find that

$$\mu_0\varepsilon_0 = \frac{1}{c^2}$$



Thus $$\frac{F_B}{F_E} = \frac{v_0^2}{c^2}$$

If $v_0=10^6$ m/s, then

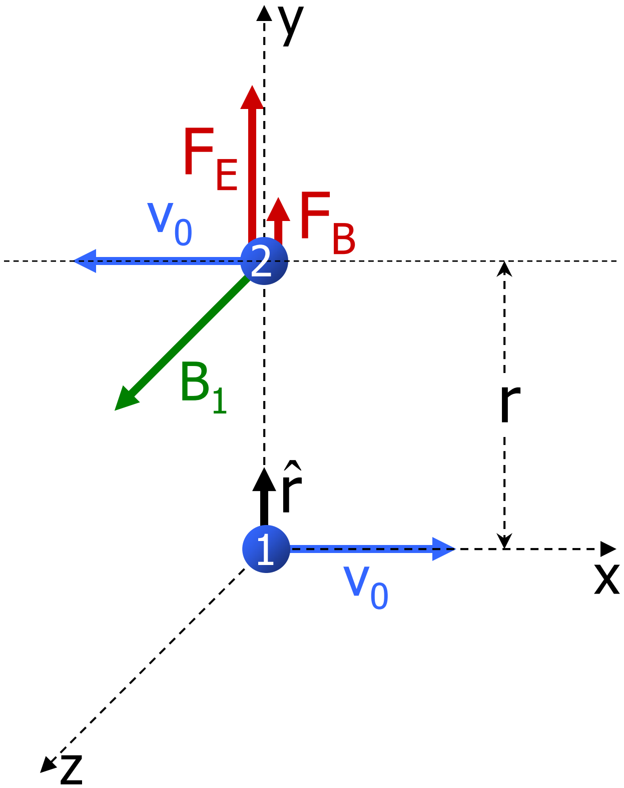
$$\frac{F_B}{F_E} = \frac{\left(10^6\right)^2}{\left(3\times10^8\right)^2} = 1.11\times10^{-5}$$

What if you are a nanohuman, lounging on proton 1. You rightfully claim you are at rest. There is no magnetic field from your proton, and no magnetic force on 2.

Another nanohuman riding on proton 2 would say "I am at rest, so there is no magnetic force on **my** proton, even though there **is** a magnetic field from proton 1."

This calculation says there is a magnetic field **and** force. Who is right? Take Physics 2305/107 to learn the answer.

Or see here, here, and here for a hint about how to resolve the paradox.



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Biot-Savart Law: Magnetic Field due to a Current Element.

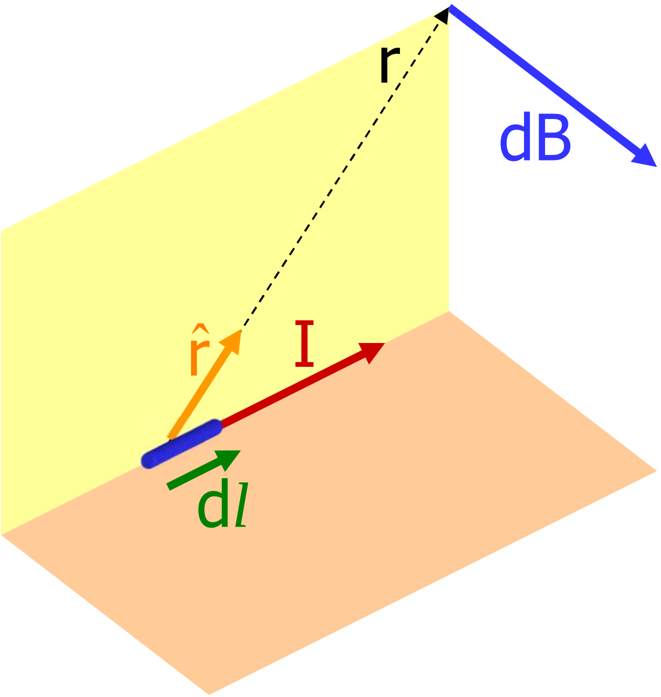
You must be able to use the Biot-Savart Law to calculate the magnetic field of a current-carrying conductor (for example: a long straight wire).

Force Between Current-Carrying Conductors.

You must be able to begin with starting equations and calculate forces between current-carrying conductors.

Biot-Savart Law: magnetic field of a current element

current I in infinitesimal length $d\vec{\ell}$ of wire gives rise to magnetic field $d\vec{B}$



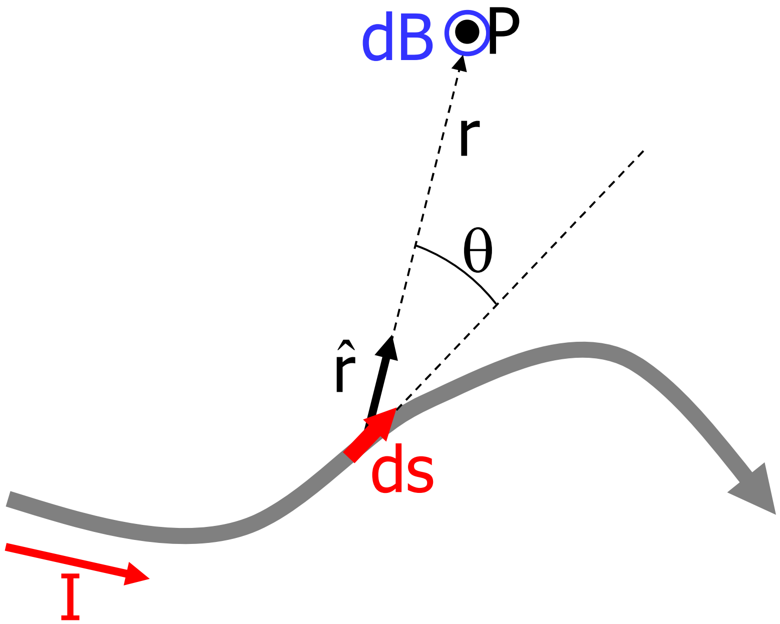
$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\ell}\times\hat{r}}{r^2}}$$

Biot-Savart Law

Derived, as in lecture 15, by summing contributions of all charges in wire element

You may see the equation written using $\vec{r} = r\,\hat{r}$.

Applying the Biot-Savart Law



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{s}\times\hat{r}}{r^2}$$

$$\left|d\vec{s}\times\hat{r}\right| = \left|d\vec{s}\right|\left|\hat{r}\right|\sin\theta$$

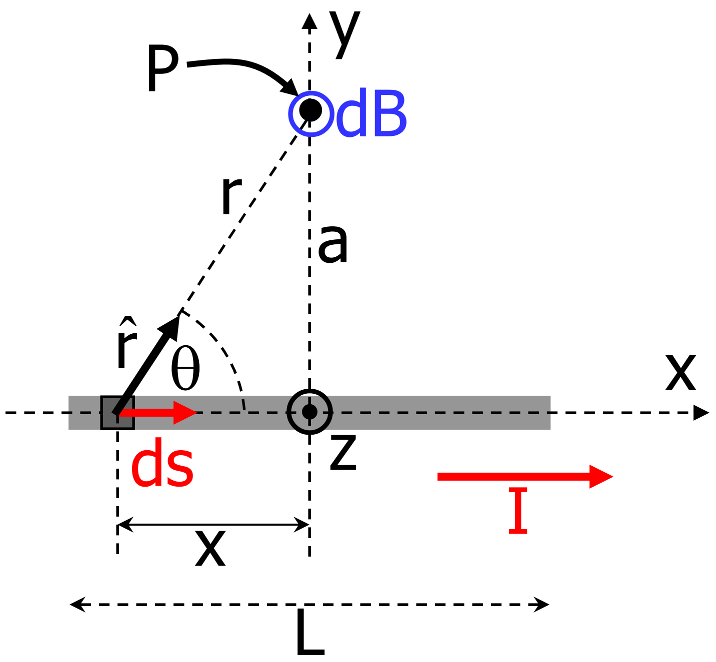
$$= ds\sin\theta \quad \text{because } \left|\hat{r}\right| = 1$$

$$dB = \frac{\mu_0}{4\pi}\frac{I\,ds\sin\theta}{r^2}$$

$$\vec{B} = \int d\vec{B}$$

Homework Hint: if you have a tiny piece of a wire, just calculate dB; no need to integrate.

Example: calculate the magnetic field at point P due to a thin straight wire of length L carrying a current I. (P is on the perpendicular bisector of the wire at distance a.)



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{s}\times\hat{r}}{r^2}$$

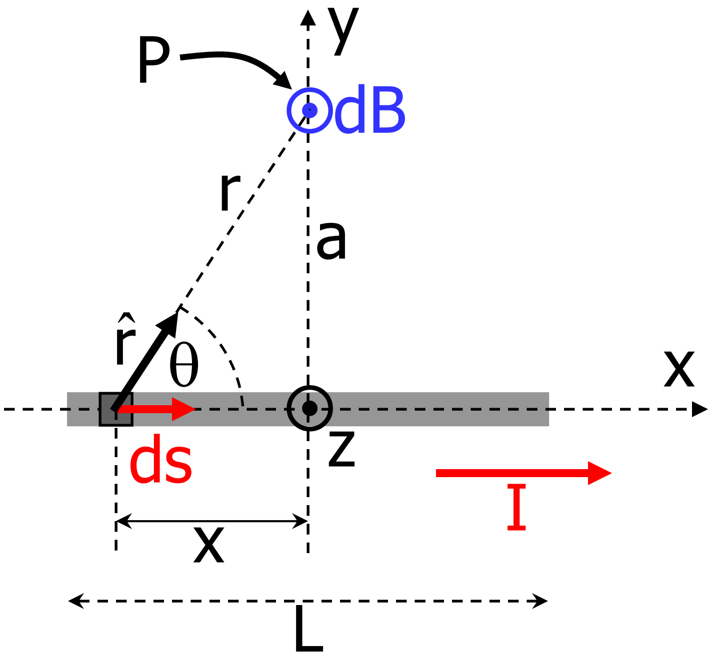
$$d\vec{s}\times\hat{r} = ds\,\sin\theta\,\hat{k}$$

$$dB = \frac{\mu_0}{4\pi}\frac{I\,ds\,\sin\theta}{r^2}$$

ds is an infinitesimal quantity in the direction of dx, so

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dx\,\sin\theta}{r^2}$$

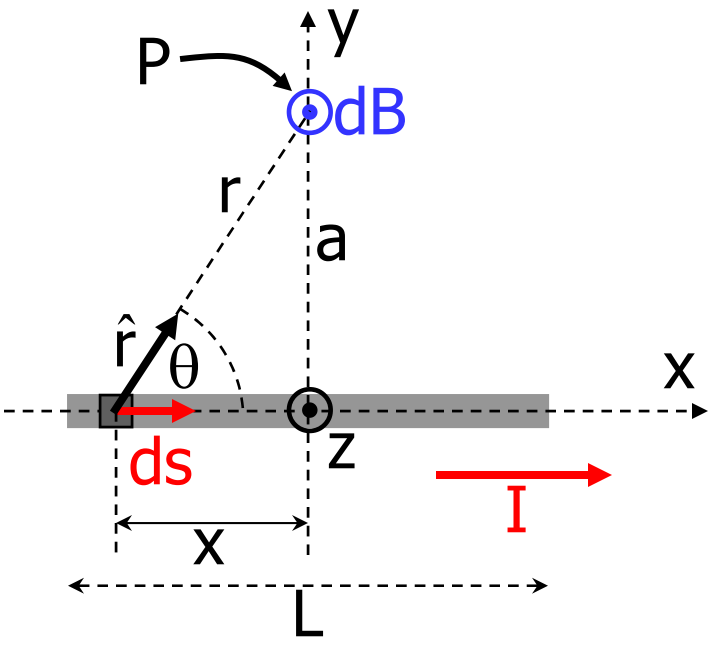
$$\sin\theta = \frac{a}{r} \qquad r = \sqrt{x^2 + a^2} \qquad dB = \frac{\mu_0}{4\pi}\frac{I\,dx\,\sin\theta}{r^2}$$



$$dB = \frac{\mu_0}{4\pi}\frac{I\,dx\,a}{r^3} = \frac{\mu_0}{4\pi}\frac{I\,dx\,a}{\left(x^2 + a^2\right)^{3/2}}$$

$$B = \int_{-L/2}^{L/2} \frac{\mu_0}{4\pi}\frac{I\,dx\,a}{\left(x^2 + a^2\right)^{3/2}}$$

$$B = \frac{\mu_0 I\,a}{4\pi}\int_{-L/2}^{L/2} \frac{dx}{\left(x^2 + a^2\right)^{3/2}}$$



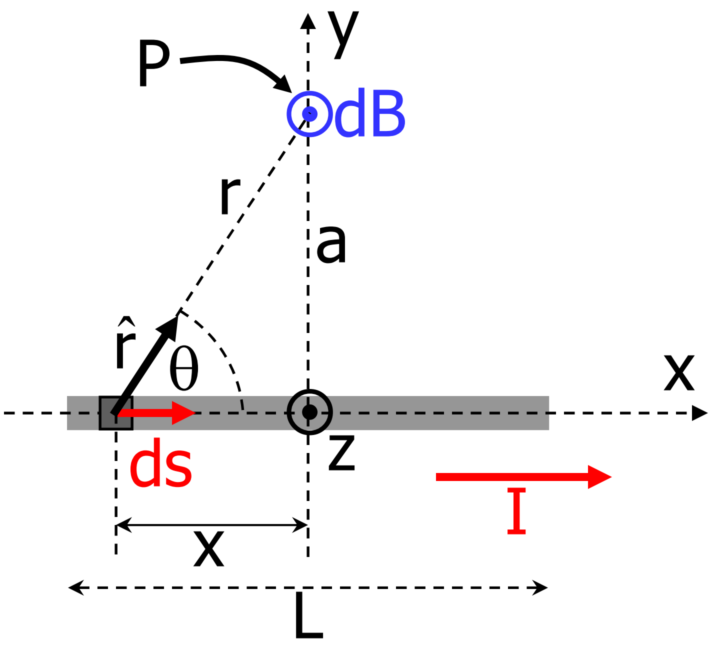
$$B = \frac{\mu_0 I\, a}{4\pi}\int_{-L/2}^{L/2}\frac{dx}{\left(x^2+a^2\right)^{3/2}}$$

look integral up in tables, use the web,or use trig substitutions

$$\int\frac{dx}{\left(x^2+a^2\right)^{3/2}} = \frac{x}{a^2\left(x^2+a^2\right)^{1/2}}$$

$$B = \frac{\mu_0 I\, a}{4\pi}\left.\frac{x}{a^2\left(x^2+a^2\right)^{1/2}}\right|_{-L/2}^{L/2}$$

$$= \frac{\mu_0 I\, a}{4\pi}\left[\frac{L/2}{a^2\left(\left(L/2\right)^2+a^2\right)^{1/2}} - \frac{-L/2}{a^2\left(\left(-L/2\right)^2+a^2\right)^{1/2}}\right]$$

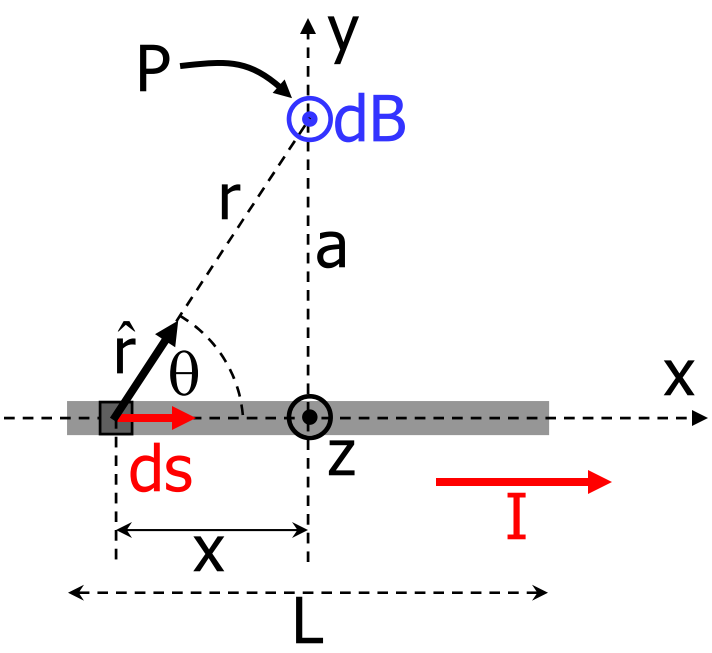


$$B = \frac{\mu_0 I\, a}{4\pi}\left[\frac{2L/2}{a^2\left(L^2/4+a^2\right)^{1/2}}\right]$$

$$B = \frac{\mu_0 I\, L}{4\pi a}\frac{1}{\left(L^2/4+a^2\right)^{1/2}}$$

$$B = \frac{\mu_0 I\, L}{2\pi a}\frac{1}{\sqrt{L^2+4a^2}}$$

$$B = \frac{\mu_0 I}{2\pi a}\frac{1}{\sqrt{1+\frac{4a^2}{L^2}}}$$



$$B = \frac{\mu_0 I}{2\pi a}\frac{1}{\sqrt{1+\frac{4a^2}{L^2}}}$$

When L→∞, $B = \frac{\mu_0 I}{2\pi a}$.

Solution beginning with vector notation

$$\vec{r} = -x\,\hat{i} + a\,\hat{j}$$

$$r = \sqrt{x^2 + a^2}$$

$$d\,\vec{s} = dx\,\vec{i}$$

$$\hat{r} = \frac{-x}{\sqrt{x^2 + a^2}}\hat{i} + \frac{a}{\sqrt{x^2 + a^2}}\hat{j}$$

$$d\vec{B}=\frac{\mu_o I}{4\pi}\frac{d\vec{s}\times\hat{r}}{r^2}$$

$$d\vec{B}=\frac{\mu_o I}{4\pi}\frac{dx\,\hat{i}\times\left(\frac{-x}{\sqrt{x^2+a^2}}\hat{i}+\frac{a}{\sqrt{x^2+a^2}}\hat{j}\right)}{x^2+a^2}$$

$$d\vec{B}=\frac{\mu_o I}{4\pi}\frac{a\,dx}{\left(x^2+a^2\right)^{3/2}}\hat{k}$$

$$\vec{B}=\frac{\mu_o I a}{4\pi}\int_{-L/2}^{L/2}\frac{dx}{\left(x^2+a^2\right)^{3/2}}\hat{k}$$

$$\vec{B}=\frac{\mu_o I a}{4\pi}\left[\frac{x}{a^2\sqrt{x^2+a^2}}\hat{k}\right]_{-L/2}^{L/2}$$

$$\vec{B}=\frac{\mu_o I\,a}{4\pi}\left[\frac{x}{a^2\sqrt{x^2+a^2}}\hat{k}\right]_{-L/2}^{L/2}$$

$$\vec{B}=\frac{\mu_o I\,a}{4\pi}\left[\frac{L/2}{a^2\sqrt{\frac{L^2}{4}+a^2}}-\frac{-L/2}{a^2\sqrt{\frac{L^2}{4}+a^2}}\right]\hat{k}$$

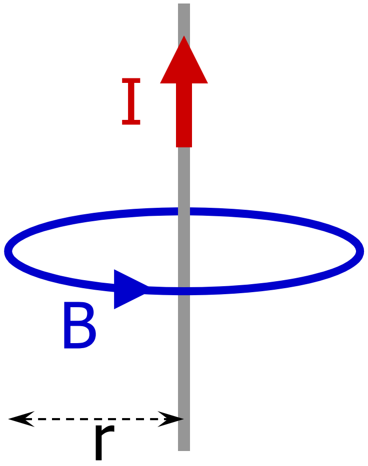
$$\boxed{\vec{B}=\frac{\mu_o I\,a}{4\pi}\left[\frac{L}{a^2\sqrt{\frac{L^2}{4}+a^2}}\right]\hat{k}}$$

Magnetic Field of a Long Straight Wire

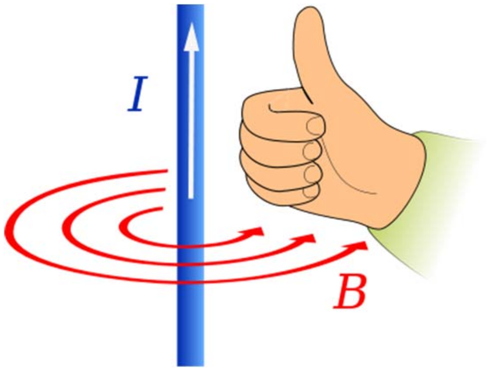
We've just derived the equation for the magnetic field around a long, straight* wire…

$$\boxed{B = \frac{\mu_0 I}{2\pi r}}$$

r is shortest (perpendicular) distance between field point and wire



…with a direction given by a "new" right-hand rule.



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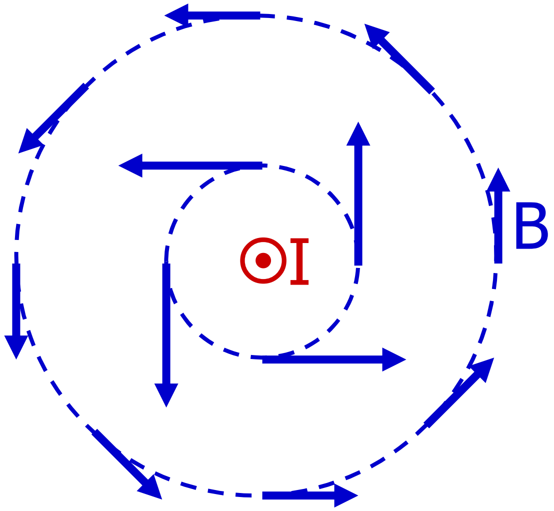
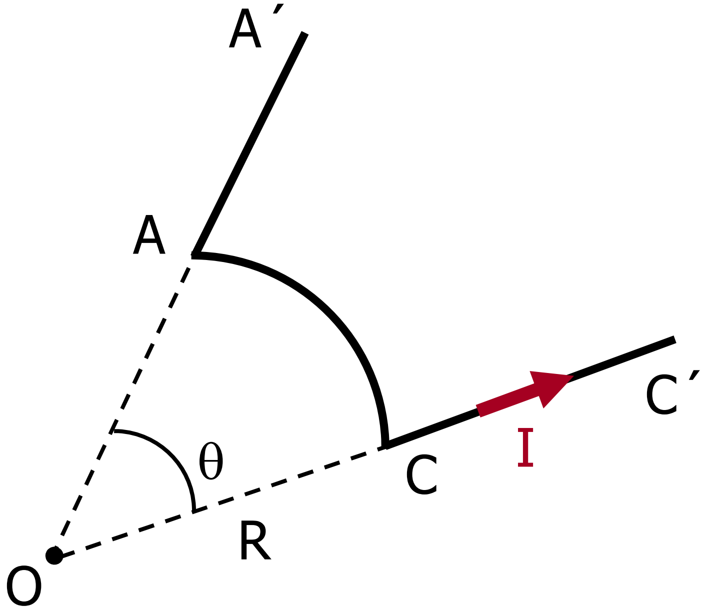
*Don't use this equation unless you have a long, straight wire!

Looking “down” along the wire:



- magnetic field is not constant
- at fixed distance r from wire, **magnitude** of field is constant (but vector magnetic field is not uniform).
- magnetic field **direction** is a tangent to imaginary circles around wire

Example: calculate the magnetic field at point O due to the wire segment shown. The wire carries uniform current I, and consists of two radial straight segments and a circular arc of radius R that subtends angle θ.



I see three "parts" to the wire.

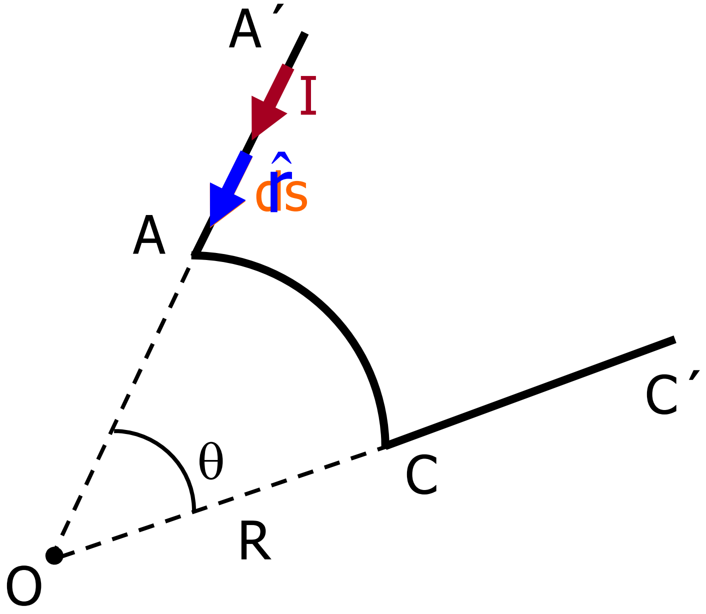
A′ to A

A to C

C to C′

As usual, break the problem up into simpler parts.

Example: calculate the magnetic field at point O due to the wire segment shown. The wire carries uniform current I, and consists of two radial straight segments and a circular arc of radius R that subtends angle θ.



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\, d\vec{s}\times\hat{r}}{r^2}$$

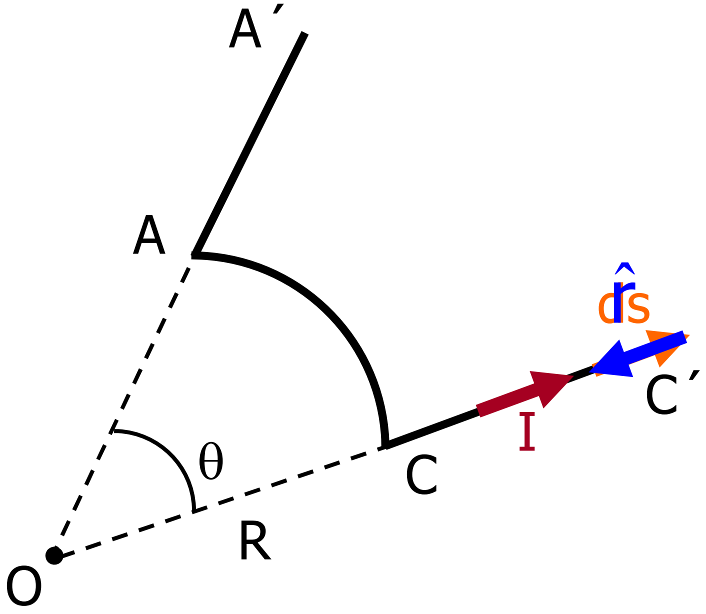
For segment A′ to A:

$$\left|d\vec{s}\times\hat{r}\right| = ds\,\left|\hat{r}\right|\sin 0 = 0$$

$$\left|d\vec{B}_{A'A}\right| = 0$$

$$\vec{B}_{A'A} = 0$$

Example: calculate the magnetic field at point O due to the wire segment shown. The wire carries uniform current I, and consists of two radial straight segments and a circular arc of radius R that subtends angle θ.



$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{s}\times\hat{r}}{r^2}$$

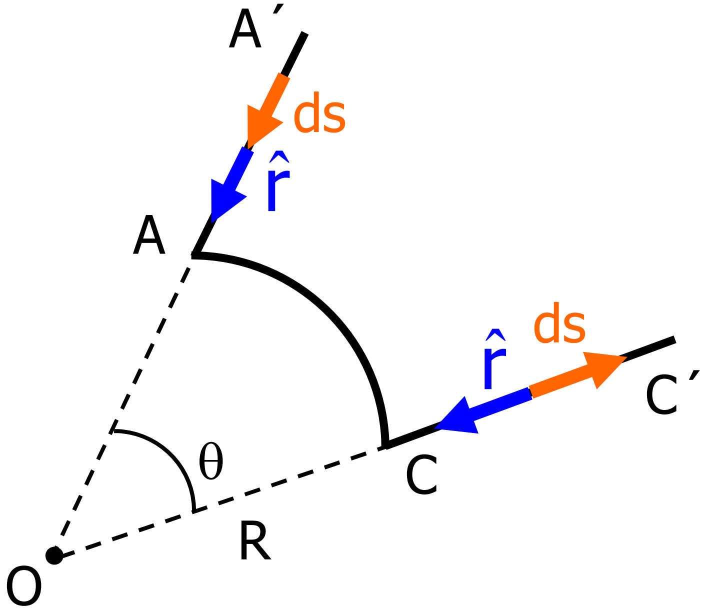
For segment C to C′:

$$\left|d\vec{s}\times\hat{r}\right| = ds\,\left|\hat{r}\right|\sin 180^\circ = 0$$

$$\left|d\vec{B}_{CC'}\right| = 0$$

$$\vec{B}_{CC'} = 0$$

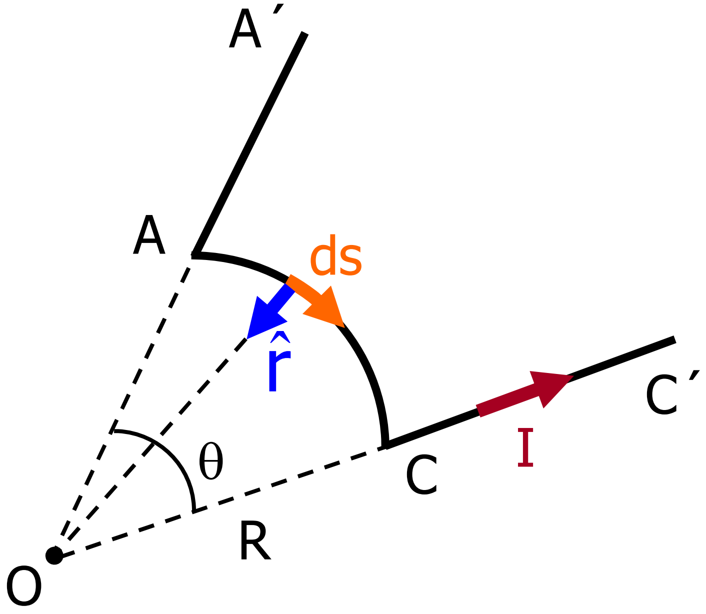
Example: calculate the magnetic field at point O due to the wire segment shown. The wire carries uniform current I, and consists of two straight segments and a circular arc of radius R that subtends angle θ.



Important technique, handy for homework and exams:

The magnetic field due to wire segments A'A and CC' is zero because $d\vec{s}$ is either parallel or antiparallel to $\hat{r}$ along those paths.

Example: calculate the magnetic field at point O due to the wire segment shown. The wire carries uniform current I, and consists of two radial straight segments and a circular arc of radius R that subtends angle θ.



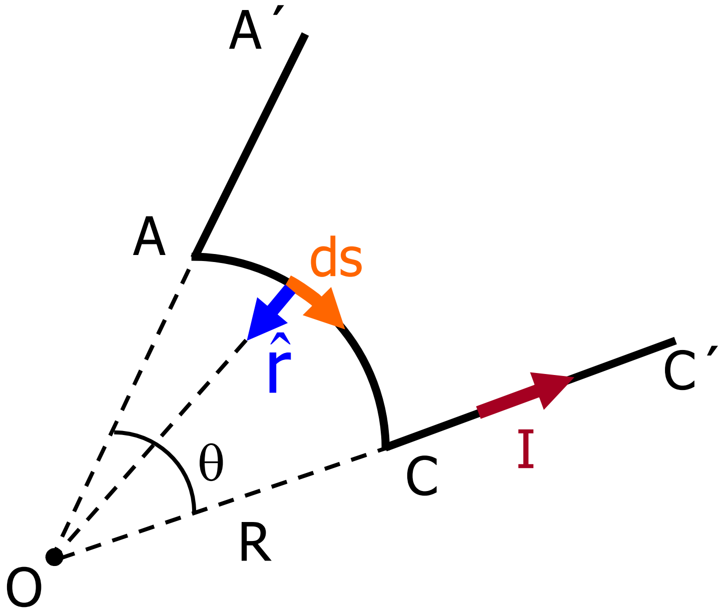
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{s}\times\hat{r}}{r^2}$$

For segment A to C:

$$\left|d\vec{s}\times\hat{r}\right| = ds\,\left|\hat{r}\right|\sin 90^\circ$$
$$= ds\,(1)\,(1)$$
$$= ds$$

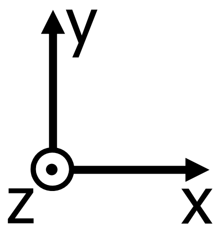
$$\left|d\vec{B}_{AC}\right| = dB_{AC} = \frac{\mu_0}{4\pi}\frac{I\,ds}{R^2}$$

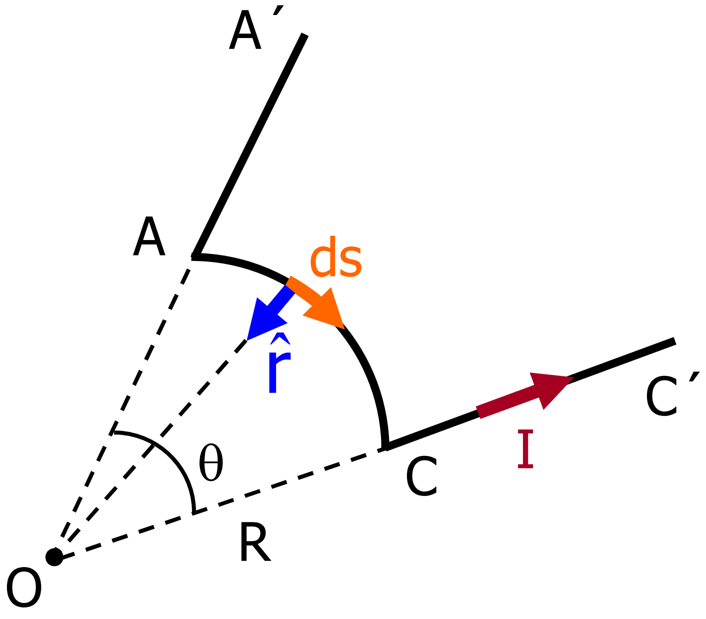
Direction of $d\vec{B}$



Cross $d\vec{s}$ into $\hat{r}$.
Direction is “into” the page, or ⊗,
for all wire elements.

If we use the standard xyz axes,
the direction is -$\hat{k}$.





$$dB_{AC} = \frac{\mu_0}{4\pi}\frac{I\,ds}{R^2}$$

$$B_{AC} = \int_{arc} dB_{AC} = \int_{arc} \frac{\mu_0 I}{4\pi}\frac{ds}{R^2}$$

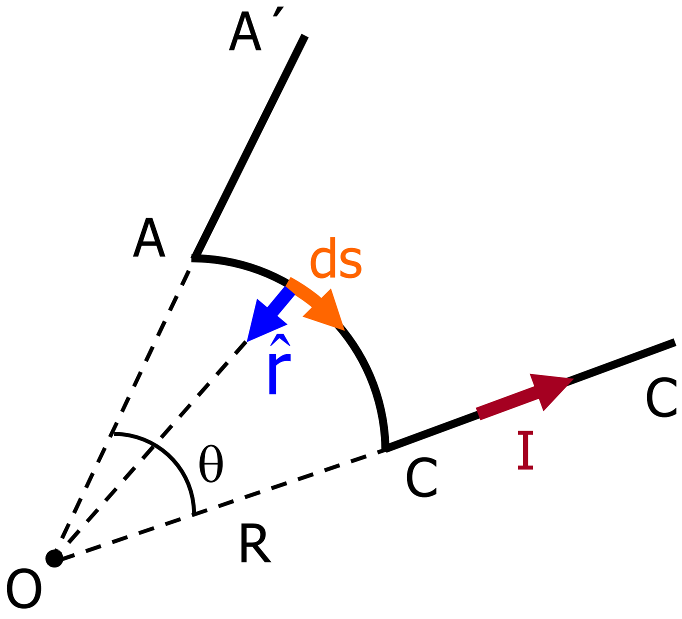
$$B_{AC} = \frac{\mu_0 I}{4\pi R^2}\int_{arc} ds$$

The integral of ds is just the arc length; just use that if you already know it.

$$B_{AC} = \frac{\mu_0 I}{4\pi R^2}\int_{arc} R\,d\theta$$

$$B_{AC} = \frac{\mu_0 I}{4\pi R}\theta$$

Final answer: $\vec{B} = \vec{B}_{A'A} + \vec{B}_{AC} + \vec{B}_{CC'} = -\frac{\mu_0 I\theta}{4\pi R}\hat{k}$



Important technique, handy for exams:

Along path AC, $d\vec{s}$ is perpendicular to $\hat{r}$.

$$\left|d\vec{s}\times\hat{r}\right| = \left|d\vec{s}\right|\ \left|\hat{r}\right|\sin 90^\circ$$

$$\left|d\vec{s}\times\hat{r}\right| = ds$$

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Force Between Current-Carrying Conductors.

You must be able to begin with starting equations and calculate forces between current-carrying conductors.

Magnetic Field of a Current-Carrying Wire

It is experimentally observed that parallel wires exert forces on each other when current flows.

Example: use the expression for B due to a current-carrying wire to calculate the force between two current-carrying wires.

Force on wire 1 produced by wire 2

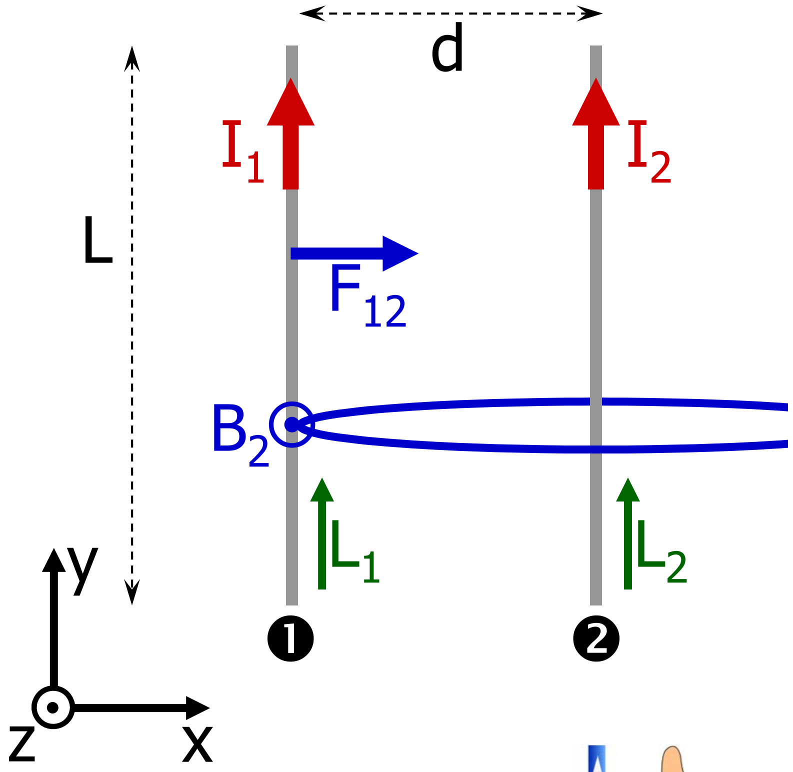
$$\vec{F}_{12} = I_1 \vec{L}_1 \times \vec{B}_2$$

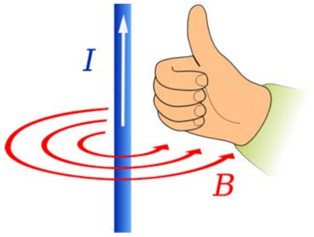
$$\hat{B}_2 = \frac{\mu_0 I_2}{2\pi d}\hat{k}$$

$$\vec{F}_{12} = I_1 L\hat{j} \times \frac{\mu_0 I_2}{2\pi d}\hat{k}$$

$$\vec{F}_{12} = \frac{\mu_0 I_1 I_2 L}{2\pi d}\hat{i}$$

force per unit length of wire i: $\dfrac{\vec{F}_{12}}{L} = \dfrac{\mu_0 I_1 I_2}{2\pi d}\hat{i}.$





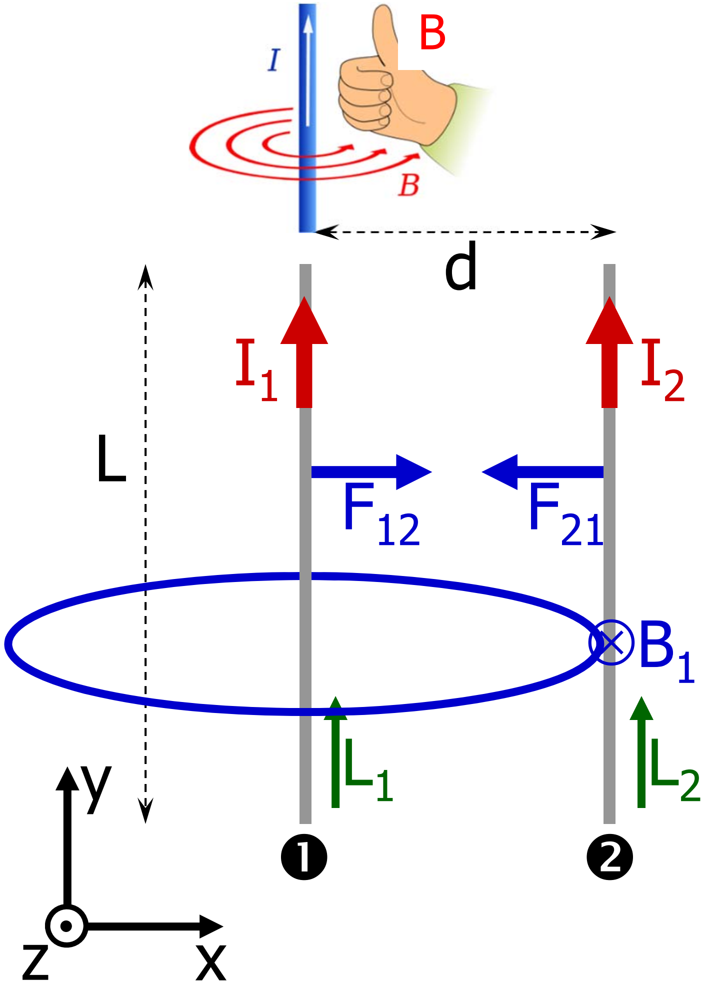
Force on wire 2 produced by wire 1

$$\vec{F}_{21} = I_2\vec{L}_2 \times \vec{B}_1$$

$$\vec{B}_1 = -\frac{\mu_0 I_1}{2\pi d}\hat{k}$$

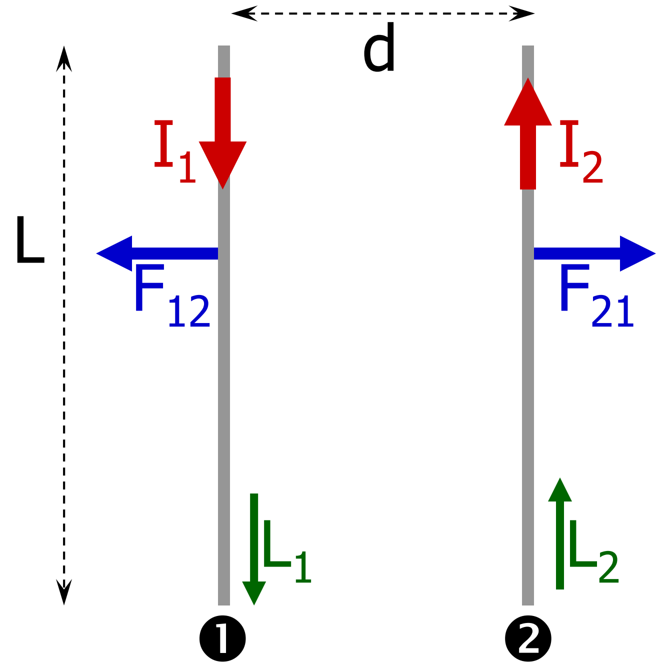
$$\vec{F}_{21} = I_2L\hat{j}\times\left(-\frac{\mu_0 I_1}{2\pi d}\hat{k}\right)$$

$$\vec{F}_{21} = -\frac{\mu_0 I_1 I_2 L}{2\pi d}\hat{i}$$



The force per unit length of wire is $\frac{\vec{F}_{21}}{L} = -\frac{\mu_0 I_1 I_2}{2\pi d}\hat{i}.$

Analogously:
If currents are in opposite directions, force is repulsive.



$$F_{12} = F_{21} = \frac{\mu_0 I_1 I_2 L}{2\pi d}$$

$$F_{12} = F_{21} = \frac{4\pi \times 10^{-7} I_1 I_2 L}{2\pi d} = 2 \times 10^{-7} I_1 I_2 \frac{L}{d}$$

Official definition of the Ampere:

1 A is the current that produces a force of $2x10^{-7}$ N per meter of length between two long parallel wires placed 1 meter apart in empty space.