International Journal of Difference Equations
ISSN 0973-6069, Volume 9, Number 2, pp. 135–145 (2014)
http://campus.mst.edu/ijde

Positive Solutions of a Nonlinear q-fractional Difference Equation with Integral Boundary Conditions

Rui A. C. Ferreira
Lusophone University of Humanities and Technologies
Department of Mathematics
Lisbon, 1749-024, Portugal
and Center for Research and Development in
Mathematics and Applications, University of Aveiro
ruiacferreira@ulusofona.pt

Abstract

In this paper, we investigate the existence of positive solutions to the nonlinear q-fractional boundary value problem

$$({}^C D_q^\alpha y)(x) = -f(x, y(x)), \quad 0 < x < 1,$$
$$y(0) = (D_q^2 y)(0) = \ldots = (D_q^{p-1} y)(0) = 0, \quad y(1) = \lambda \int_0^1 y(s) d_q s,$$

by applying a fixed point theorem in cones.

AMS Subject Classifications: 39A13, 34B18, 34A08.
Keywords: Fractional q-difference equations, integral boundary condition, positive solution.

1 Introduction

The q-difference calculus or *quantum* calculus is an old discipline that started to be developed by Jackson in the manuscripts [10, 11]. It is rich in history and in applications as the reader can confirm in the work by T. Ernst [5].

The origin of the fractional q-difference calculus can be traced back to the works by Al-Salam [2] and Agarwal [1]. More recently, perhaps due to the explosion in research within the fractional calculus setting (see the books [15, 16]), new developments in the

Received February 20, 2014; Accepted July 10, 2014
Communicated by Martin Bohner

theory of fractional q-difference calculus were made (see for example [4, 17]). Particularly, the study about the existence of solutions to equations depending on q-fractional derivatives had its start in the works by the author [6,7] and, since then, some more novel results appeared in the literature [3, 9, 14, 18, 19, 21]. This work consists of studying a higher-order boundary value problem with integral boundary value conditions.

This paper is organized as follows: in Section 2 we introduce some notation and provide to the reader the definitions of the q-fractional integral and difference operators together with some basic properties. In Section 3 we present our main results.

2 Preliminaries on Fractional q-calculus

Let $0 < q < 1$. The intervals considered in this text should be understood as the set $\{q^n : n \in \mathbb{N}_0\} \cup \{0\}$.

Define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}_0$ is

$$(a-b)^0 = 1, \quad (a-b)^{(n)} = \prod_{k=0}^{n-1}(a-bq^k), \quad n \in \mathbb{N}, \quad a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R}\backslash\{0, -1, -2, \ldots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \text{ and } \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Similarly as done for derivatives, it can be defined an operator I_q^n, namely,

$$(I_q^0 f)(x) = f(x) \text{ and } (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [12] (see also [4]). We point out now four identities that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$\begin{aligned}
(t-s)^{(\alpha+\beta)} &= (t-s)^{(\alpha)}(t-q^\alpha s)^{(\beta)} \\
[a(t-s)]^{(\alpha)} &= a^\alpha (t-s)^{(\alpha)}, \\
{}_tD_q(t-s)^{(\alpha)} &= [\alpha]_q (t-s)^{(\alpha-1)}, \\
\left({}_xD_q \int_0^x f(x,t)d_qt\right)(x) &= \int_0^x {}_xD_q f(x,t)d_qt + f(qx,x).
\end{aligned}$$

Remark 2.1. We note that if $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$ [6].

The following definition was firstly considered in [1].

Definition 2.2. Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional q-integral of the Riemann–Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)d_qt, \quad \alpha > 0, \quad x \in [0, 1].$$

Definition 2.3. The Caputo fractional q-derivative of order $\alpha \geq 0$ is defined by

$$(D_q^0 f)(x) = f(x),$$

and $({}^C D_q^\alpha f)(x) = (I_q^{m-\alpha} D_q^m f)(x)$ for $\alpha > 0$, where m is the smallest integer greater or equal than α.

Let us now list some properties that are already known in the literature. Their proofs can be found in [1, 20].

Lemma 2.4. *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then, the next formulas hold:*

1. $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,
2. $({}^C D_q^\alpha I_q^\alpha f)(x) = f(x)$.

The next result will be of much importance in the sequel. Its proof can be seen in [20].

Lemma 2.5. *Let $\alpha > 0$ and $p = \lceil \alpha \rceil$. Then, the following equality holds:*

$$(I_q^{\alpha C} D_q^\alpha f)(x) = f(x) - \sum_{k=0}^{p-1} \frac{x^k}{\Gamma_q(k+1)} (D_q^k f)(0). \tag{2.1}$$

3 Fractional Boundary Value Problem

We shall consider now the question of existence of nontrivial solutions to the following problem:

$$({}^C D_q^\alpha y)(x) = -f(x, y(x)), \quad 0 < x < 1, \tag{3.1}$$

subject to the boundary conditions

$$y(0) = (D_q^2 y)(0) = \ldots = (D_q^{p-1} y)(0) = 0, \quad y(1) = \lambda \int_0^1 y(s) d_q s, \tag{3.2}$$

where p is an integer greater or equal than 3, $p - 1 < \alpha \leq p$, $\lambda \neq q + 1$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. To that end we need the following result [13].

Theorem 3.1. *Let $\mathcal{B}$ be a Banach space, and let $C \subset \mathcal{B}$ be a cone. Assume Ω_1, Ω_2 are open disks contained in $\mathcal{B}$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $T : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$ be a completely continuous operator such that*

$$\|Ty\| \geq \|y\|, \ y \in C \cap \partial\Omega_1 \text{ and } \|Ty\| \leq \|y\|, \ y \in C \cap \partial\Omega_2.$$

Then T has at least one fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Next we obtain the exact expression of Green's function associated to BVP (3.1)–(3.2).

Lemma 3.2. *y is the solution of the boundary value problem* (3.1)–(3.2) *if, and only if, y satisfies the integral equation*

$$y(x) = \int_0^1 G(x, qt) f(t, y(t)) d_q t,$$

where

$G(x,s)$

$$= \begin{cases} \dfrac{(q+1)x(1-s)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda q^{\alpha-1}s) - [\alpha]_q(q+1-\lambda)(x-s)^{(\alpha-1)}}{(q+1-\lambda)\Gamma_q(\alpha+1)} \\ \dfrac{(q+1)x(1-s)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda q^{\alpha-1}s)}{(q+1-\lambda)\Gamma_q(\alpha+1)}, \end{cases}$$

where the first line is defined for $s \le x$ and the second line for $s \ge x$.

Proof. We start by applying Lemma 2.4 and Lemma 2.5 to transform (3.1) into the equivalent integral equation:

$$y(x) = -\int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs + \sum_{k=0}^{p-1} \frac{x^k}{\Gamma_q(k+1)}(D_q^k y)(0). \tag{3.3}$$

Using the boundary conditions $y(0) = (D_q^2 y)(0) = \ldots = (D_q^{p-1}y)(0) = 0$ we get from (3.3),

$$y(x) = -\int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs + x(D_q y)(0).$$

Now the boundary condition $y(1) = \lambda \int_0^1 y(s)d_qs$ yields

$$(D_q y)(0) = \lambda \int_0^1 y(s)d_qs + \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs,$$

from which follows that

$$y(x) = -\int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs + \int_0^1 \frac{x(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs + \lambda x \int_0^1 y(s)d_qs. \tag{3.4}$$

Integrating both sides of (3.4) from 0 to 1 we get

$$\begin{aligned} \int_0^1 y(s)d_qs &= -\int_0^1 \int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs d_qx \\ &\quad + \int_0^1 \int_0^1 \frac{x(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s,y(s))d_qs d_qx + \int_0^1 \lambda x \int_0^1 y(s)d_qs d_qx \\ &= -\int_0^1 \frac{(1-qs)^{(\alpha)}}{\Gamma_q(\alpha+1)} f(s,y(s))d_qs + \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{(q+1)\Gamma_q(\alpha)} f(s,y(s))d_qs \\ &\quad + \frac{\lambda}{q+1}\int_0^1 y(s)d_qs. \end{aligned}$$

Therefore,

$$\int_0^1 y(s)d_qs = -\int_0^1 \frac{(q+1)(1-qs)^{(\alpha)}}{(q+1-\lambda)\Gamma_q(\alpha+1)}f(s,y(s))d_qs + \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{(q+1-\lambda)\Gamma_q(\alpha)}f(s,y(s))d_qs.$$

Substituting the last expression for $\int_0^1 y(s)d_qs$ in (3.4) we obtain (sometimes we write only f to mean $f(s,y(s))$),

$$\begin{aligned}
y(x) &= -\int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,y(s))d_qs + \int_0^1 \frac{x(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,y(s))d_qs \\
&\quad +\lambda x\Bigg(-\int_0^1 \frac{(q+1)(1-qs)^{(\alpha)}}{(q+1-\lambda)\Gamma_q(\alpha+1)}f(s,y(s))d_qs \\
&\quad +\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{(q+1-\lambda)\Gamma_q(\alpha)}f(s,y(s))d_qs\Bigg) \\
&= -\int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,y(s))d_qs \\
&\quad +\int_0^1 \frac{[\alpha]_q(q+1-\lambda)x(1-qs)^{(\alpha-1)}}{(q+1-\lambda)\Gamma_q(\alpha+1)}f(s,y(s))d_qs \\
&\quad -\int_0^1 \frac{\lambda x(q+1)(1-qs)^{(\alpha-1)}(1-q^\alpha s)}{(q+1-\lambda)\Gamma_q(\alpha+1)}f(s,y(s))d_qs \\
&\quad +\int_0^1 \frac{\lambda x[\alpha]_q(1-qs)^{(\alpha-1)}}{(q+1-\lambda)\Gamma_q(\alpha+1)}f(s,y(s))d_qs \\
&= -\int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,y(s))d_qs \\
&\quad +\int_0^1 \frac{(q+1)x(1-qs)^{(\alpha-1)}([\alpha]_q-\lambda+\lambda q^\alpha s)}{(q+1-\lambda)\Gamma_q(\alpha+1)}f(s,y(s))d_qs \\
&= \int_0^x \frac{(q+1)x(1-qs)^{(\alpha-1)}([\alpha]_q-\lambda+\lambda q^\alpha s)-[\alpha]_q(q+1-\lambda)(x-qs)^{(\alpha-1)}}{(q+1-\lambda)\Gamma_q(\alpha+1)}fd_qs \\
&\quad +\int_x^1 \frac{(q+1)x(1-qs)^{(\alpha-1)}([\alpha]_q-\lambda+\lambda q^\alpha s)}{(q+1-\lambda)\Gamma_q(\alpha+1)}f(s,y(s))d_qs,
\end{aligned}$$

which finishes the proof. □

Some properties of the function G needed in the sequel are now stated and proved.

Lemma 3.3. *Suppose that $p \in \{3,4,\ldots\}$ and $p-1<\alpha\le p$. Assume that $q\in(0,1)$ and $0\le\lambda<q+1$. Then, the Green function G defined above satisfies the following*

conditions:

$$G(x, qs) \geq 0, \tag{3.5}$$

and

$$x^{\alpha-1}G(1, qs) \leq G(x, qs) \leq \frac{(q+1)[\alpha]_q}{\lambda([\alpha]_q - (q+1))}G(1, qs), \tag{3.6}$$

for all $0 \leq x, s \leq 1$.

Proof. We start noticing that under the hypothesis on α, λ, q the following inequalities hold:

$$[\alpha]_q - \lambda + \lambda q^\alpha s \geq [\alpha]_q - \lambda > [\alpha]_q - (q+1) > 0, \quad 0 \leq s \leq 1.$$

Therefore, in order to prove (3.5) we need to show that

$$(q+1)x(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda q^\alpha s) - [\alpha]_q(q+1-\lambda)(x-qs)^{(\alpha-1)} \geq 0, \ 0 \leq s \leq x \leq 1. \tag{3.7}$$

Now, if $x = 0$ then the result is immediate. So, let $x \neq 0$. Then,

$$\begin{aligned}
&(q+1)x(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda q^\alpha s) - [\alpha]_q(q+1-\lambda)(x-qs)^{(\alpha-1)} \\
&= (q+1)x(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda q^\alpha s) - [\alpha]_q(q+1-\lambda)x^{\alpha-1}\left(1 - q\frac{s}{x}\right)^{(\alpha-1)} \\
&\geq x^{\alpha-1}(1-qs)^{(\alpha-1)}\{(q+1)([\alpha]_q - \lambda + \lambda q^\alpha s) - [\alpha]_q(q+1-\lambda)\} \\
&\geq x^{\alpha-1}(1-qs)^{(\alpha-1)}\lambda([\alpha]_q - (q+1)) > 0,
\end{aligned}$$

and (3.7) is proved.

Let us now turn our attention to the inequalities in (3.6). Suppose firstly that $0 < x \leq s \leq 1$. Then, $qs < 1$ and for $x \leq qs$ we obtain

$$\frac{G(x, qs)}{G(1, qs)} = \frac{(q+1)x(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda q^\alpha s)}{(1-qs)^{(\alpha-1)}\lambda((q+1)(q^\alpha s - 1) + [\alpha]_q)} = \frac{(q+1)x([\alpha]_q - \lambda + \lambda q^\alpha s)}{\lambda((q+1)(q^\alpha s - 1) + [\alpha]_q)},$$

from which follows that

$$x^{\alpha-1} < x < \frac{q+1}{\lambda}x \leq \frac{G(x, qs)}{G(1, qs)} \leq \frac{(q+1)[\alpha]_q}{\lambda([\alpha]_q - (q+1))}.$$

Suppose now that $0 < s \leq x \leq 1$. Since

$$\begin{aligned}
&x^{\alpha-1}(1-qs)^{(\alpha-1)}\lambda([\alpha]_q - (q+1)) \\
&\quad \leq \ (q+1)x(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda q^\alpha s) - [\alpha]_q(q+1-\lambda)(x-qs)^{(\alpha-1)} \\
&\quad \leq \ (q+1)x(1-qs)^{(\alpha-1)}([\alpha]_q - \lambda + \lambda q^\alpha s),
\end{aligned}$$

we conclude immediately by what was done before that

$$x^{\alpha-1} \leq \frac{G(x, qs)}{G(1, qs)} \leq \frac{(q+1)[\alpha]_q}{\lambda([\alpha]_q - (q+1))}.$$

The proof is done. □

Remark 3.4. If we let $0 < \tau < 1$, then $0 < \tau^{\alpha-1} < 1$ and

$$\min_{x\in[\tau,1]} G(x,qs) \geq \tau^{\alpha-1}G(1,qs) \text{ for } s \in [0,1]. \tag{3.8}$$

Let $\mathcal{B} = C[0,1]$ be the Banach space endowed with norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Let $\tau = q^n$ for a given $n \in \mathbb{N}$ and, for α, γ, q as in Lemma 3.3, define the cone $C \subset \mathcal{B}$ by

$$C = \left\{ u \in \mathcal{B} : u(x) \geq 0 \text{ and } \min_{x\in[\tau,1]} u(x) \geq \tau^{\alpha-1}\frac{\lambda([\alpha]_q - (q+1))}{(q+1)[\alpha]_q}\|u\| \right\}.$$

Consider the operator $T : C \to \mathcal{B}$ defined by

$$(Ty)(x) = \int_0^1 G(x,qs)f(s,y(s))d_qs.$$

For $y \in C$, $(Ty)(x) \geq 0$ on $[0,1]$ and

$$\begin{aligned}
\min_{x\in[\tau,1]}(Ty)(x) &= \min_{x\in[\tau,1]} \int_0^1 G(x,qt)f(t,y(t))d_qt \\
&\geq \tau^{\alpha-1}\int_0^1 G(1,qt)f(t,y(t))d_qt \\
&= \tau^{\alpha-1}\frac{\lambda([\alpha]_q - (q+1))}{(q+1)[\alpha]_q}\frac{(q+1)[\alpha]_q}{\lambda([\alpha]_q - (q+1))}\int_0^1 G(1,qt)f(t,y(t))d_qt \\
&= \tau^{\alpha-1}\frac{\lambda([\alpha]_q - (q+1))}{(q+1)[\alpha]_q}\|Ty\|,
\end{aligned}$$

that is, $T(C) \subset C$.

For our purposes, let us define two constants

$$M = \frac{\lambda([\alpha]_q - (q+1))}{(q+1)[\alpha]_q}\int_0^1 G(1,qt)d_qt, \quad N = \max_{x\in[0,1]}\int_\tau^1 G\left(x,qt\right)d_qt.$$

Our existence result is now presented.

Theorem 3.5. *let α, γ, q be as in Lemma 3.3 and put $\tau = q^n$ with $n \in \mathbb{N}$. Suppose that $f(t,u)$ is a nonnegative continuous function on $[0,1] \times [0,\infty)$. If there exists two positive constants $r_2 > r_1 > 0$ such that*

$$M \max_{(t,u)\in[0,1]\times[0,r_1]} f(t,u) \leq r_1, \tag{3.9}$$

$$N \min_{(t,u)\in[\tau,1]\times[Rr_2,r_2]} f(t,u) \geq r_2, \tag{3.10}$$

where $R = \tau^{\alpha-1}\dfrac{\lambda([\alpha]_q - (q+1))}{(q+1)[\alpha]_q} < 1$, then problem (3.1)–(3.2) *has a solution y satisfying $y(x) > 0$ for $x \in (0,1]$.*

Proof. A standard procedure shows that the operator $T : C \to C$ is completely continuous. Therefore we only have to show that the operator equation $y = Ty$ has a solution satisfying $y(x) > 0$ for all $x \in (0, 1]$.

Let $\Omega_1 = \{y \in C : \|y\| < r_1\}$. For $y \in C \cap \partial\Omega_1$, we have $0 \le y(x) \le r_1$ on $[0, 1]$. Using (3.5) and (3.9) we obtain,

$$\begin{aligned}\|Ty\| &= \max_{x\in[0,1]} \int_0^1 G(x, qt)f(t, y(t))d_qt \\ &= \frac{\lambda([\alpha]_q - (q + 1))}{(q + 1)[\alpha]_q} \int_0^1 G(1, qt)f(t, y(t))d_qt \\ &\le M \max_{(t,u)\in[0,1]\times[0,r_1]} f(t, u) \le r_1 = \|y\|.\end{aligned}$$

Let $\Omega_2 = \{y \in C : \|y\| < r_2\}$. For $y \in C \cap \partial\Omega_2$, we have $Rr_2 \le y(x) \le r_2$ on $[\tau, 1]$. Using (3.8) and (3.10), and the fact that $\tau = q^n$, we obtain (see page 282 in [8]),

$$\begin{aligned}\|Ty\| &= \max_{x\in[0,1]} \int_0^1 G(x, qt) f(t, y(t))d_qt \\ &\ge \max_{x\in[0,1]} \int_\tau^1 G(x, qt) d_qt \min_{(t,u)\in[\tau,1]\times[Rr_2,r_2]} f(t, u) \\ &\ge r_2 = \|y\|.\end{aligned}$$

Now, Theorem 3.1 assures the existence of a fixed point y of T such that $r_1 \le \|y\| \le r_2$. To finish the proof, note that by (3.6)

$$\begin{aligned}y(x) &= \int_0^1 G(x, qt) f(t, y(t))d_qt \\ &\ge x^{\alpha-1} \frac{\lambda([\alpha]_q - (q + 1))}{(q + 1)[\alpha]_q} \frac{(q + 1)[\alpha]_q}{\lambda([\alpha]_q - (q + 1))} \int_0^1 G(1, qt) f(t, y(t))d_qt \\ &= x^{\alpha-1} \frac{\lambda([\alpha]_q - (q + 1))}{(q + 1)[\alpha]_q} \|y\|,\end{aligned}$$

which implies that $y(x) \ge x^{\alpha-1} \frac{\lambda([\alpha]_q - (q + 1))}{(q + 1)[\alpha]_q} r_1$. Therefore, $y(x) > 0$ for $x \in (0, 1]$ and the proof is done. □

Acknowledgments

This work was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (FCT-Fundação para a Ciência e a Tecnologia), within project PEst-OE/MAT/UI4106/2014.

References

[1] R. P. Agarwal, Certain fractional q-integrals and q-derivatives, Proc. Cambridge Philos. Soc. **66** (1969), 365–370.

[2] W. A. Al-Salam, Some fractional q-integrals and q-derivatives, Proc. Edinburgh Math. Soc. (2) **15** (1966/1967), 135–140.

[3] R. Almeida and N. Martins, Existence results for fractional q-difference equations of order $\alpha \in]2, 3[$ with three-point boundary conditions, Commun. Nonlinear Sci. Numer. Simul. **19** (2014), no. 6, 1675–1685.

[4] F. M. Atici and P. W. Eloe, Fractional q-calculus on a time scale, J. Nonlinear Math. Phys. **14** (2007), no. 3, 333–344.

[5] T. Ernst, The History of q-Calculus and a New Method, U. U. D. M. Report 2000:16, ISSN 1101-3591, Department of Mathematics, Uppsala University, 2000.

[6] R. A. C. Ferreira, Nontrivial solutions for fractional q-difference boundary value problems, Electron. J. Qual. Theory Differ. Equ. **2010**, No. 70, 10 pp.

[7] R. A. C. Ferreira, Positive solutions for a class of boundary value problems with fractional q-differences, Comput. Math. Appl. **61** (2011), no. 2, 367–373.

[8] H. Gauchman, Integral inequalities in q-calculus, Comput. Math. Appl. **47** (2004), no. 2-3, 281–300.

[9] J. R. Graef and L. Kong, Positive solutions for a class of higher order boundary value problems with fractional q-derivatives, Appl. Math. Comput. **218** (2012), no. 19, 9682–9689.

[10] F. H. Jackson, On q-functions and a certain difference operator. Trans. Royal Soc. Edin. 46, (1908), 253–281.

[11] F. H. Jackson, On q-definite integrals. Quart. J. Pure and Appl. Math. 41 (1910), 193–203.

[12] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2002.

[13] M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron, P. Noordhoff Ltd. Groningen, 1964.

[14] J. Ma and J. Yang, Existence of solutions for multi-point boundary value problem of fractional q-difference equation, Electron. J. Qual. Theory Differ. Equ. **2011**, No. 92, 10 pp.

[15] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[16] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.

[17] P. M. Rajković, S. D. Marinković and M. S. Stanković, Fractional integrals and derivatives in q-calculus, Appl. Anal. Discrete Math. **1** (2007), no. 1, 311–323.

[18] M. El-Shahed and F. M. Al-Askar, Positive solutions for boundary value problem of nonlinear fractional q-difference equation, ISRN Math. Anal. **2011**, Art. ID 385459, 12 pp.

[19] M. El-Shahed and M. Al-Yami, On the existence and uniqueness of solutions for Q-fractional boundary value problem, Int. J. Math. Anal. (Ruse) **5** (2011), no. 33-36, 1619–1630.

[20] M. S. Stanković, P. M. Rajković and S. D. Marinković, On fractional q-integrals, q-derivatives and corresponding equations, in *Application of mathematics in technical and natural sciences*, 300–308, AIP Conf. Proc., 1186 Amer. Inst. Phys., Melville, NY.

[21] C. Yu and J. Wang, Positive solutions of nonlocal boundary value problem for high-order nonlinear fractional q-difference equations, Abstr. Appl. Anal. **2013**, Art. ID 928147, 9 pp.