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# On the $(q, h)$-Discretization of Ladder Operators

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**Abstract**

In this paper we shall study the problem of the $(q, h)$-discretization (generalization) of ladder operators. We shall present a few illustrative and important examples including the Weber, Bessel and Laguerre ladders and their $(q, h)$-analogues.

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**Keywords:** Weyl algebra, deformation, ladder theory, difference operators, factorization.

## 1 Introduction

The $q$- and $h$- generalizations of the classical Weyl algebra and the ladders in these algebras were studied in [5]. Further, the $(q, h)$-deformation of the Weyl algebra was

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introduced in [6] and a few examples of ladders in this algebra are given therein. In general, ladders give a factorization of certain $q$- and $h$-difference operators of second order, and, therefore, they are very important in applications. Moreover, basic information about special functions is encoded in ladder theory.

In this paper we shall first recall the abstract theory of ladder operators following [5]. Next we shall recall the $(q, h)$-deformation of the Weyl algebra with one of its representations following [6]. Finally, we shall present new results concerning the Laguerre, Weber and Bessel ladders and their $(q, h)$-analogues and discuss the problems arising in the discretization process. Moreover, we shall discuss, as an example, the $q$-case of the $(q, h)$-Bessel function and show how it appears in the existing literature.

# 2 The Ladder Theory

Ladders appear abundantly in physics (e.g., in quantum mechanics and quantum field theory). There is also an extensive literature on the $q$-theory of ladder operators, equations and special functions.

Recall that according to [5] ladders are the (finite or infinite) sequence of real or complex vector spaces $V_n$, $n \in \mathbb{Z}$, and operators $A_n^+, A_n^-$ acting between them as follows:

$$\cdots \quad V_{n-1} \quad \underset{A_n^-}{\overset{A_n^+}{\rightleftarrows}} \quad V_n \quad \underset{A_{n+1}^-}{\overset{A_{n+1}^+}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{2.1}$$

See [5] for various examples. In most cases the operators are elements of a fixed (classical or deformed) Weyl algebra $\mathcal{W}$, the vector spaces $V_n$ are fixed function spaces $V_n = V$, where $\rho : \mathcal{W} \to \mathrm{End}(V)$ constitutes a representation of this algebra. A subladder is a collection of subspaces of the space of the original ladder, such that the operators acting between them are still well defined.

Assume that we are also given a sequence of complex numbers $(\alpha_n)$ such that each $\alpha_n$ is thought to be attached to a pair of operators $A_n^+$, $A_n^-$. The following theorem holds.

**Theorem 2.1** (See [5]). *Let the commutator of the ladder coincide with the forward difference of the sequence, i.e.,*

$$A_{n+1}^- A_{n+1}^+ - A_n^+ A_n^- = \alpha_{n+1} - \alpha_n, \qquad n \in \mathbb{Z}. \tag{2.2}$$

*(i) Let the vector spaces $V_n'$ be the eigenspaces of the loop operators*

$$\begin{aligned} V_n' \quad &:= \quad \mathrm{eig}(A_{n+1}^- A_{n+1}^+, \alpha_{n+1}) = \mathrm{eig}(A_n^+ A_n^-, \alpha_n) \qquad \textit{(loop operators)} \\ &= \quad \mathrm{eig}(A_{n+1}^- A_{n+1}^+ + A_n^+ A_n^-, \alpha_{n+1} + \alpha_n) \qquad \textit{(anti-commutator).} \end{aligned}$$

*Then the following ladder is a well defined subladder of (2.1)*

$$A^+_{n+1} : V'_n \to V'_{n+1}, \qquad A^-_{n+1} : V'_{n+1} \to V'_n. \tag{2.3}$$

*(ii) If $\alpha_N = 0$ for some fixed $N \in \mathbb{Z}$, then the ground state space $V''_N$ generates an ascending subladder $V''_n \subseteq V'_n$ in the following recursive way:*

$$V''_N \quad := \quad \ker A^-_N, \qquad\qquad V''_{n+1} := A^+_{n+1}(V''_n) \quad \textit{for } n \geq N.$$

In other words, if the commutator (2.2) is a scalar and $\alpha_N = 0$, then the functions $g_n \in V''_n$ in the so called ground state ladder defined by

$$g_N \in \ker A^-_N, \qquad g_{n+1} = A^+_{n+1} \cdots A^+_{N+2} A^+_{N+1} g_N, \quad n \geq N, \tag{2.4}$$

appear as eigenfunctions of the right and left loop operators

$$A^-_{n+1} A^+_{n+1}\, g_n = \alpha_{n+1}\, g_n, \tag{2.5}$$

$$A^+_n A^-_n\, g_n = \alpha_n\, g_n. \tag{2.6}$$

The raising and lowering ladder operators act between these functions by

$$A^+_{n+1}\, g_n = g_{n+1}, \tag{2.7}$$

$$A^-_n\, g_n = \alpha_n\, g_{n-1}. \tag{2.8}$$

# 3 The $(q,h)$-Deformation of the Weyl Algebra and its Representation

In this section we recall the deformed $(q,h)$-Weyl algebra $\mathcal{W}$ from [6]. In the limit $q \to 1$ and $h \to 0$ it degenerates to the classical Weyl algebra. The classical Weyl algebra is generated by two operators $X$ and $D$ with a single relation $DX - XD = 1$. In the standard representation on the space of smooth functions of a real (or complex) variable the operator $X$ acts by multiplying the function by this variable $x$ and $D$ is the usual derivative.

Let $q \in \mathbb{R}^\times \backslash \{1\}$ and $h \in \mathbb{R}^\times$ be fixed. We consider the noncommutative, associative and unital algebra over $\mathbb{C}$, which is generated by two elements $S$, $X$ subject to the fundamental relation

$$SX = (qX + h)S. \tag{3.1}$$

We also assume throughout that the operator $S$, which we call the shift operator, and the graininess operator

$$\mu(X) = (q-1)X + h \tag{3.2}$$

are invertible. The entire algebra with these relations will be called the $(q, h)$-deformed Weyl algebra or, for short, the $(q, h)$-Weyl algebra.

There are several important representations of the $(q, h)$-deformed Weyl algebra. One of them is on the space $\mathcal{F}(\mathbb{T}_{(q,h,x_0)}, \mathbb{C})$ of functions defined on the $(q, h)$-grid

$$\mathbb{T}_{(q,h,x_0)} = \{\sigma^j(x_0) \,|\, j \in \mathbb{Z}\} \subseteq \mathbb{R}, \quad x_0 \neq \frac{-h}{q-1}. \tag{3.3}$$

where the jump operator $\sigma$ and its iterates are defined by

$$\sigma(x) \;=\; qx + h, \qquad \sigma^j(x) = q^j x + [j]_q h. \tag{3.4}$$

Here we use the following notation for the $q$-number:

$$[j]_q = \frac{q^j - 1}{q - 1}.$$

The operators are represented as follows

$$\begin{aligned} Xf(x) \;&:=\; x\,f(x), \\ Sf(x) \;&:=\; f(qx + h), \\ S^{-1}f(x) \;&:=\; f\left((x - h)/q\right), \\ \mu(X)f(x) \;&:=\; ((q-1)x + h) \cdot f(x), \\ \mu(X)^{-1}f(x) \;&:=\; ((q-1)x + h)^{-1} \cdot f(x). \end{aligned} \tag{3.5}$$

It is obvious in this representation that the relation (3.1) is true, i.e.,

$$SXf(x) \;=\; (qx + h)f(qx + h) = (qX + h)Sf(x).$$

We can further define the operator $D := \mu(X)^{-1}(S - 1)$, which turns out to be the $(q, h)$-forward difference operator in this representation

$$Df(x) = \frac{f(qx + h) - f(x)}{(q - 1)x + h}.$$

Note that we are in the context of the $(q, h)$-Weyl algebra. The letter $D$ should not be confused with the differentiation operator in the usual Weyl algebra.

In [6] several examples of ladders in a fixed representation of the $(q, h)$-Weyl algebra (spaces $V_n = V$) are presented. In this paper we present the so-called Weber, Bessel and Laguerre ladders and their (*q, h*)-analogues.

In the following sections we keep the representations arbitrary unless otherwise stated.

# 4 The ($q$, $h$)-Weber Ladder

Let us consider the ladder, which we call the Weber ladder, in the usual Weyl algebra with constant operators

$$\cdots \quad V_{n-1} \quad \underset{D+\frac{1}{2}X}{\overset{D-\frac{1}{2}X}{\rightleftarrows}} \quad V_n \quad \underset{D+\frac{1}{2}X}{\overset{D-\frac{1}{2}X}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{4.1}$$

with

$$A_n^+ = D - \frac{1}{2}X, \quad A_n^- = D + \frac{1}{2}X,$$

where $X$ and $D$ are generators of the usual Weyl algebra (in the standard representation, $X$ is the multiplication by $x$ and $D$ is the usual differentiation). We can easily check that condition (2.2) is satisfied provided $\alpha_n = -n$. We can also check that the parabolic cylinder functions appear as eigenfunctions of the loop operators. The ground state ladder with respect to the standard representation is generated by

$$\ker A_0^- = \langle e^{-x^2/4} \rangle.$$

The Weber ladder is similar to the ladder of the classical Dirac Harmonic Oscillator with $A_n^+ = D - X, \quad A_n^- = D + X$. The $(q,h)$-analogue of the Harmonic Oscillator Ladder is given in [6]. By modifying this ladder, we get the following discretization of the Weber ladder (4.1).

Define

$$Y = -h\mu(X)^{-1}X,$$

$$\sigma^n(X) = q^n X + [n]_q\, h, \;\; \sigma^n(Y) = -q^n h\mu(X)^{-1}\sigma^{-n}(X), \;\; n \in \mathbb{Z}.$$

We note that in the limit $q \to 1, h \to 0$ we have $\sigma^n(X) = X, \; \sigma^n(Y) = -X$. Then the $(q,h)$-Weber ladder is given by

$$\cdots \quad V_{n-1} \quad \underset{h^{-1}(q^nS-1)+\sigma^n(X)/2}{\overset{\mu(X)^{-1}(S-1)+q^{-n}\sigma^n(Y)/2}{\rightleftarrows}} \quad V_n \quad \underset{h^{-1}(q^{n+1}S-1)+\sigma^{n+1}(X)/2}{\overset{\mu(X)^{-1}(S-1)+q^{-(n+1)}\sigma^{n+1}(Y)/2}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{4.2}$$

with

$$\begin{aligned} A_n^+ &= \mu(X)^{-1}(S-1) + q^{-n}\sigma^n(Y)/2, \\ A_n^- &= h^{-1}(q^nS-1) + \sigma^n(X)/2. \end{aligned}$$

The commutator condition (2.2) is satisfied provided

$$4\alpha_n = [n]_q \left( \frac{h^2}{q-1} - 2 \right) + [-n]_q \left( \frac{h^2}{q-1} + 2 \right).$$

Note that in the last expression the limit $q \to 1, h \to 0$ is $-n$, as in the Weber ladder (4.1).

We have $\alpha_0 = 0$ and the ground state ladder is generated by $\ker A_0^-$, which (with respect to the representation (3.5)) is given by

$$\frac{f(qx+h) - f(x)}{h} + \frac{1}{2} x f(x) = 0. \tag{4.3}$$

We can use (4.3) for computing the values of the function $f$ at the points (3.3). For instance, by fixing $f(0) = 1$ we have

$$f(h) = 1,\ f((q+1)h) = \frac{1}{2}(2 - h^2),\ f(-h/q) = \frac{2q}{h^2 + 2q}$$

and, further,

$$f(\sigma^n(h)) = \frac{(-1)^n}{2^n} \prod_{j=1}^{n} (-2 + h^2 [j]_q),\ n \in \mathbb{N},$$

$$f(\sigma^{-(n+1)}(h)) = (-1)^n 2^n \prod_{j=-n}^{-1} (-2 + [j]_q h^2)^{-1},\ n \in \mathbb{N}.$$

Numerically, the points $(x, f(x))$ for given $q$ and $h$ are distributed closely to the graph of the function $f(x) = e^{-x^2/4}$.

Clearly, the ladder

$$\cdots \quad V_{n-1} \quad \underset{cD + dX}{\overset{aD - bX}{\rightleftarrows}} \quad V_n \quad \underset{cD + dX}{\overset{aD - bX}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{4.4}$$

with

$$A_n^+ = aD - bX, \quad A_n^- = cD + dX,$$

constants $ad + bc \neq 0,\ d > 0,\ c > 0,\ \alpha_n = -(ad + bc)n,$ and $\ker A_0^- = \langle e^{-dx^2/(2c)} \rangle$ can be discretized in a similar manner. We have

$$\cdots \quad V_{n-1} \quad \underset{c\,h^{-1}(q^n S - 1) + d\,\sigma^n(X)}{\overset{a\,\mu(X)^{-1}(S-1) + b\,q^{-n}\sigma^n(Y)}{\rightleftarrows}} \quad V_n \quad \underset{c\,h^{-1}(q^{n+1} S - 1) + d\,\sigma^{n+1}(X)}{\overset{a\,\mu(X)^{-1}(S-1) + b\,q^{-(n+1)}\sigma^{n+1}(Y)}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{4.5}$$

with

$$\begin{aligned} A_n^+ &= a\,\mu(X)^{-1}(S-1) + b\,q^{-n}\sigma^n(Y), \\ A_n^- &= c\,h^{-1}(q^n S - 1) + d\,\sigma^n(X) \end{aligned}$$

and

$$\alpha_n = [n]_q \left( \frac{bdh^2}{q-1} - ad \right) + [-n]_q \left( \frac{bdh^2}{q-1} + bc \right).$$

## 5 The (*q, h*)-Bessel Ladder

The classical Bessel ladder, in the usual Weyl algebra, is given by

$$\cdots \quad V_{n-1} \quad \underset{nX^{-1}+D}{\overset{(n-1)X^{-1}-D}{\rightleftarrows}} \quad V_n \quad \underset{(n+1)X^{-1}+D}{\overset{nX^{-1}-D}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{5.1}$$

Here

$$A_n^+ = (n-1)X^{-1} - D, \quad A_n^- = nX^{-1} + D$$

and we additionally assume that the operator $X$ is invertible. We can easily check that the commutator condition (2.2) is zero, and, hence, the sequence $\alpha_n$ is constant. We can also check that the Bessel functions [1, Sect. 6] appear as eigenfunctions of the loop operators. In comparison with the Weber ladder, the terms with $X^{-1}$ are not easily discretized and some methods (to be explained elsewhere) from [2] are needed to get the right discretization:

$$\cdots \quad V_{n-1} \quad \underset{\mu(X)^{-1}(S-q^{-n})}{\overset{\mu(X)^{-1}(q^{2n}S^{-1}-q^{n+1})}{\rightleftarrows}} \quad V_n \quad \underset{\mu(X)^{-1}(S-q^{-(n+1)})}{\overset{\mu(X)^{-1}(q^{2(n+1)}S^{-1}-q^{n+2})}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{5.2}$$

We can check that the commutator (2.2) is zero. In the limit $h = 0$ , then $q \to 1$, the operators are reduced to the operators in the classical Bessel ladder, as can be seen by rearranging the expressions for the operators:

$$\begin{aligned} A_n^+ &= \mu(X)^{-1}(q^{2n}S^{-1} - q^{n+1}) = \mu(X)^{-1}(q^{2n} - q^{n+1}) - q^{2n}\mu(X)^{-1}(1 - S^{-1}), \\ A_n^- &= \mu(X)^{-1}(S - q^{-n}) \ = \ \mu(X)^{-1}(1 - q^{-n}) + \mu(X)^{-1}(S - 1). \end{aligned}$$

From (5.2) and (2.6) we have that in our case the $(q, h)$-Bessel function satisfies the following $(q, h)$-difference equation

$$Sg_n(x) + S^{-1}g_n(x) - \left(q^{-n} + q^n\right) g_n(x) = -\alpha q^{-(n+1)} \left((q-1)x + h\right)^2 g_n(x). \tag{5.3}$$

If in (5.3) we put $h = 0$ we obtain the $q$-difference equation

$$g_n(q^2x) + q^{-n} \left((1-q)^2 q\alpha x^2 - 1 - q^{2n}\right) g_n(qx) + g_n(x) = 0. \tag{5.4}$$

Next replacing $q^2$ in (5.4) by $\tilde{q}$ and transforming the variable $x = \sqrt{\dfrac{q}{\alpha}}\dfrac{y}{1-q}$, we find that equation (5.4) can be written in the form

$$J_n(\tilde{q}y; \tilde{q}) + \tilde{q}^{-\frac{n}{2}} \left(\tilde{q}y^2 - 1 - \tilde{q}^n\right) J_n(\tilde{q}^{\frac{1}{2}}y; \tilde{q}) + J_n(y; \tilde{q}) = 0, \tag{5.5}$$

where $J_n(y;\tilde{q}) = g_n\left(\sqrt{\frac{q}{\alpha}}\frac{y}{1-q}\right)$. Using the papers [7, 8] we recognize the equation for the Hahn-Exton $q$-Bessel function, which is defined by

$$J_\nu(y;\tilde{q}) = y^\nu \frac{(\tilde{q}^{\nu+1};\tilde{q})_\infty}{(\tilde{q};\tilde{q})_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n \tilde{q}^{\frac{n(n+1)}{2}} y^{2n}}{(\tilde{q}^{\nu+1};\tilde{q})_n (\tilde{q};\tilde{q})_n}. \tag{5.6}$$

This function was introduced by Hahn [3] and generalized by Exton [4]. The Hahn-Exton $q$-Bessel function is the same as the Jackson $q$-Bessel function of the third kind. Moreover, we can slightly generalize (5.2) and consider the ladder

$$\cdots \quad V_{n-1} \quad \underset{a\,\mu(X)^{-1}\left(S-q^{-n}\right)}{\overset{-\mu(X)^{-1}(b\,q^n+c\,q^{n+1}-d\,q^{2n}S^{-1})}{\rightleftarrows}} \quad V_n \quad \underset{a\,\mu(X)^{-1}\left(S-q^{-(n+1)}\right)}{\overset{-\mu(X)^{-1}(b\,q^{n+1}+c\,q^{n+2}-d\,q^{2n+2}S^{-1})}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{5.7}$$

with

$$A_n^+ = -\mu(X)^{-1}(b\,q^n + c\,q^{n+1} - d\,q^{2n}S^{-1}), \quad A_n^- = a\,\mu(X)^{-1}\left(S - q^{-n}\right)$$

and an arbitrary parameter $b$, non-zero parameters $a$ and $c$, and $d > 0$. Clearly, when $a = c = d = 1,\ b = 0$ we have (5.2). The commutator (2.2) is zero as well. If $d - c - b = 0$, we can find a continuous limit of this ladder.

# 6 The (*q*, *h*)-Laguerre Ladder

The classical Laguerre ladder [1] is given by

$$\cdots \quad V_{n-1} \quad \underset{n-XD}{\overset{n+XD-X+k}{\rightleftarrows}} \quad V_n \quad \underset{n+1-XD}{\overset{n+1+XD-X+k}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{6.1}$$

with $k$ a fixed real number. Here we have

$$A_n^+ = n + XD - X + k, \quad A_n^- = n - XD.$$

We can easily calculate that the commutator condition (2.2) is fulfilled with $\alpha_n = n(k+n)$. Consider the $(q,h)$-analogue of the Laguerre ladder:

$$\cdots \quad V_{n-1} \quad \underset{[n]_q-XD}{\overset{[n]_q+XD-\sigma^{-n}(X)+k}{\rightleftarrows}} \quad V_n \quad \underset{[n+1]_q-XD}{\overset{[n+1]_q+XD-\sigma^{-(n+1)}(X)+k}{\rightleftarrows}} \quad V_{n+1} \quad \cdots \tag{6.2}$$

(see (3.4)) with

$$A_n^+ = [n]_q + XD - \sigma^{-n}(X) + k, \quad A_n^- = [n]_q - XD.$$

The commutator condition (2.2) is fulfilled provided

$$\alpha_n = [n]_q([n]_q + k) - [n]_q[-n]_q h.$$

We can check straightforwardly that in the limit $q \to 1,\ h \to 0$ we get the classical Laguerre ladder. We note that applying the $(q,h)$-operator $A_n^+$ consequently to $f(x) = 1$ with $n \geq 1$ we shall get polynomials in $x$ of degree $n$. They can be regarded as $(q,h)$-analogues of the classical Laguerre polynomials.

One can further generalize the ladder by putting arbitrary parameters in front of the operators, as in the previous cases.

# 7 Discussion

In general, it is difficult to find a $(q,h)$-analogue of a ladder in the usual Weyl algebra. Given a sequence of operators in the classical Weyl algebra such that the commutator is a scalar, how to find the $(q,h)$-analogue? Clearly, for the right discrete analogue the operators in the ladder will have a limit to the corresponding operators in the continuous case as $q \to 1,\ h \to 0$. One can also impose a stronger additional condition by requiring that the sequence of numbers attached to the ladders (defined up to a constant) has the continuous limit to the corresponding sequence of numbers attached to the continuous ladder.

As we have seen in the examples of the Weber, Bessel and Laguerre ladder, there are several possibilities to discretize the differentiation operator $D$ in the classical Weyl algebra. One of the ways to search for the discrete analogue is by using computer algebra: by inputting $A_n^{\pm}$ with the unknown coefficients we can search for the cases when the commutator is constant and then find the continuous limit. In general, one can find a lot of ladders in the $(q,h)$-Weyl algebra, e.g.,

$$A_n^+ = \mu(x)^{-1}(q^n S^{-1} - q^{a+n}), \;\; A_n^- = \mu(X)^{-1}(S - q^{-a}),$$

where the commutator (2.2) is zero. We have also seen in the example of the Weber and Bessel ladder that changing the coefficients $X$ to $X^{-1}$ changes the $(q,h)$-analogue significantly.

The algebraic approach in the current papers and in [5, 6] sheds a new light onto the theory of special functions. In the literature one can find many ways of finding discrete analogues of special functions. We argue that the algebraic approach of the ladder theory provides a unified way to study special functions.

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