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The q-deformation of Hyperbolic and Trigonometric Potentials

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Abstract

We present solutions of q-deformed Schrödinger equation for the hyperbolic and trigonometric potentials given by a factorization method. Their various properties including the correspondence $q \to 1$ to the non–deformed Schrödinger operators are discussed.

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1 Introduction

The paper deals with a second order q-difference equation such that in the limit $q \to 1$ we get a differential equation related to the stationary Schrödinger equation with the Scarf potentials [14].

We shall consider eigenproblems

$$\mathbf{H}_k \psi_k = \lambda_k \psi_k, \tag{1.1}$$

for the discrete family $k \in \mathbb{N} \cup \{0\}$ of the second order q-difference operators

$$\mathbf{H}_k = Z_k(x)\partial_q Q^{-1}\partial_q + W_k(x)\partial_q + V_k(x), \qquad k \in \mathbb{N} \cup \{0\} \tag{1.2}$$

acting in the Hilbert spaces $\mathcal{H}_k$, where $0 < q < 1$ and ∂_q is the q-derivative (see [9])

$$\partial_q \psi(x) = \frac{\psi(x) - \psi(qx)}{(1-q)x}. \tag{1.3}$$

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By definition, $\mathcal{H}_k$ consists of the complex valued functions $\psi : [a,b]_q \to \mathbb{C}$ defined on the q-interval $[a,b]_q := \{q^n a : n \in \mathbb{N}\cup\{0\}\} \cup \{q^n b : n \in \mathbb{N}\cup\{0\}\}$, $a, b \in \mathbb{R}$, $a < b$, $a \neq q^n b$, which are square–integrable with respect to the scalar products

$$\langle\psi|\varphi\rangle_k := \int_a^b \overline{\psi(x)}\varphi(x)\varrho_k(x)d_qx. \tag{1.4}$$

Let us recall that the q-integral is given by

$$\int_a^b \psi(x)d_qx := \sum_{n=0}^{\infty}(1-q)q^n\left(b\psi(q^nb) - a\psi(q^na)\right), \tag{1.5}$$

$$\int_{-\infty}^{\infty} \psi(x)d_qx = \sum_{n=-\infty}^{\infty}(1-q)q^n\psi(q^n) + \sum_{n=-\infty}^{\infty}(1-q)q^n\psi(-q^n)$$

and the shift operators Q, Q^{-1} are given by

$$Q\psi(x) = \psi(qx), \tag{1.6}$$

$$Q^{-1}\psi(x) = \psi(q^{-1}x). \tag{1.7}$$

We postulate some conditions which must be satisfied by weight functions,

$$\varrho_{k-1} = Q\left(B_k\varrho_k\right), \tag{1.8}$$

$$\partial_q\left(B_k\varrho_k\right) = A_k\varrho_k, \tag{1.9}$$

where A_k, B_k are real valued functions on $[a,b]_q$. The equation (1.9) corresponds in the limit $q \to 1$ to the Pearson equation, which is important for the theory of classical orthogonal polynomials. Additionally we impose the boundary conditions

$$B_k(a)\varrho_k(a) = B_k(b)\varrho_k(b) = 0. \tag{1.10}$$

The basic idea of the factorization method is well known, see [10–12]. Some results concerning the factorization method for the second order q-difference equations were presented for example in papers [3, 13]. In this work, we apply our previous results obtained in the paper [6]. We will factorize the chain of second order q-difference operators (1.2) using the first order q-difference operators of the form

$$\mathbf{A}_k = \partial_q + f_k, \tag{1.11}$$

$$\mathbf{A}_k^* = \left(\partial_q + f_k\right)^* = B_k\left(-\partial_q Q^{-1} + f_k\right) - A_k\left(1 + (1-q)xf_k\right). \tag{1.12}$$

The operator $\mathbf{A}_k^* : \mathcal{H}_{k-1} \to \mathcal{H}_k$ is adjoint to $\mathbf{A}_k : \mathcal{H}_k \to \mathcal{H}_{k-1}$ with respect to the scalar products (1.4). We say that the operators $\mathbf{H}_k$ admit a factorization if there exist sequences of operators $\mathbf{A}_k$, $\mathbf{A^*}_k$ and constants a_k such that

$$\mathbf{H}_k = \mathbf{A}_k^*\mathbf{A}_k + a_k = \mathbf{A}_{k+1}\mathbf{A}_{k+1}^* + a_{k+1} \qquad \text{for} \qquad k \in \mathbb{N}\cup\{0\}. \tag{1.13}$$

In papers [5, 7] we considered the q-deformation of the Schrödinger operator with the following potentials: shifted oscillator, isotropic oscillator, Rosen–Morse II, Eckart II, Poschl–Teller I and II. Next in the paper [8] we considered the case of q-deformed Schrödinger equation for the Morse potential. In this work we will study the case of q-deformed Schrödinger equation for the q-hyperbolic and q-trigonometric Scarf potentials.

2 The q-hyperbolic and q-trigonometric Potentials

We obtain the case of the q-deformation of hyperbolic and trigonometric potentials if we consider the operators $\mathbf{H}_k$ in the following forms:

$$\begin{aligned}
\mathbf{H}_k &= -q^k\left(b_2x^2+b_0\right)\sqrt{\frac{\left(qb_2+\frac{(a_0-a_1)}{(1-q)^{-1}}\right)q^{-2k-1}x^2+(1-q)hq^{-k-1}x+b_0}{b_2q^{-2k-2}x^2+b_0}}\partial_q Q^{-1}\partial_q \\
&+\frac{q^k}{(1-q)x} \\
&\times\left(\sqrt{\left(\left(qb_2+\frac{(a_0-a_1)}{(1-q)^{-1}}\right)q^{-2k+1}x^2+(1-q)hq^{-k}x+b_0\right)(b_2q^{-2k}x^2+b_0)}\right. \\
&\left.-\left(b_2x^2+b_0\right)\sqrt{\frac{\left(qb_2+\frac{(a_0-a_1)}{(1-q)^{-1}}\right)q^{-2k-1}x^2+(1-q)hq^{-k-1}x+b_0}{b_2q^{-2k-2}x^2+b_0}}\right)\partial_q \qquad (2.1)\\
&+\frac{q^{k+1}(b_2x^2+b_0)}{(1-q)^2x^2} \\
&\times\sqrt{\frac{\left(qb_2+\frac{(a_0-a_1)}{(1-q)^{-1}}\right)q^{-2k-1}x^2+(1-q)hq^{-k-1}x+b_0}{b_2q^{-2k-2}x^2+b_0}}- \\
&-\frac{q^k}{(1-q)^2x^2} \\
&\times\sqrt{\left(\left(qb_2+\frac{(a_0-a_1)}{(1-q)^{-1}}\right)q^{-2k+1}x^2+(1-q)hq^{-k}x+b_0\right)(b_2q^{-2k}x^2+b_0)} \\
&+\frac{q^k}{(1-q)^2x^2} \\
&\times\left(\left(qb_2+\frac{(a_0-a_1)}{(1-q)^{-1}}\right)q^{-2k+1}x^2+(1-q)hq^{-k}x+b_0\right) \\
&+\frac{q^{k+1}(b_2x^2+b_0)}{(1-q)^2x^2}+q^{1-k}\left(a_1[k]_q-a_0[k-1]_q-qb_2[k-1]_q[k]_q\right),
\end{aligned}$$

where

$$[k]_q = \frac{1-q^k}{1-q}.$$

This chain is factorized using first order q-difference operators

$$\mathbf{A}_k = -\frac{1}{(1-q)x}Q + \sqrt{\frac{\left(qb_2 + \frac{(a_0-a_1)}{(1-q)^{-1}}\right) q^{-2k+1}x^2 + (1-q)hq^{-k}x + b_0}{(1-q)^2x^2(b_2q^{-2k}x^2+b_0)}}, \tag{2.2}$$

$$\mathbf{A}_k^* = -\frac{q^k(b_2x^2+b_0)}{(1-q)x}Q^{-1} - \frac{q^k}{(1-q)x}\sqrt{b_2q^{-2k}x^2+b_0} \tag{2.3}$$

$$\times \sqrt{\left(qb_2 + \frac{(a_0-a_1)}{(1-q)^{-1}}\right) q^{-2k+1}x^2 + (1-q)hq^{-k}x + b_0}.$$

The constants a_k in the factorization relations (1.13) are given by the following formula

$$a_k = q^{1-k}\left(a_1[k]_q - a_0[k-1]_q - qb_2[k-1]_q[k]_q\right). \tag{2.4}$$

In this case the functions B_k, A_k and f_k are given by

$$B_k(x) = q^k\left(b_2x^2+b_0\right), \tag{2.5}$$

$$A_k(x) = q^k[-2k]_qb_2x, \tag{2.6}$$

$$f_k(x) = \sqrt{\frac{\left(qb_2 + \frac{(a_0-a_1)}{(1-q)^{-1}}\right) q^{-2k+1}x^2 + (1-q)hq^{-k}x + b_0}{(1-q)^2x^2(b_2q^{-2k}x^2+b_0)}} - \frac{1}{(1-q)x}. \tag{2.7}$$

Solving the equation (1.9) we obtain the weight function

$$\varrho_k(x) = \frac{1}{\left(-q^{-2k}x^2\frac{b_2}{b_0};q^2\right)_{k+1}}, \tag{2.8}$$

where

$$\left(x;q^2\right)_{k+1} = (1-x)(1-q^2x)\dots(1-q^{2k}x).$$

Substituting (2.5) and (2.8) into the boundary condition (1.10) we choose as the q-interval $[a,b]_q = [-\infty,\infty]_q$.

It is easy to see that any solution of the equation

$$\mathbf{A}_k\psi_k^0 = 0 \tag{2.9}$$

is automatically a solution of equation (1.1) with the eigenvalue $\lambda_k = a_k$. Solving the equation (2.9) we obtain

$$\psi_k^0(x) = C\sqrt{\prod_{i=-k}^{\infty}\frac{b_2q^{2i}x^2+b_0}{\left(qb_2 + \frac{(a_0-a_1)}{(1-q)^{-1}}\right) q^{2i+1}x^2 + (1-q)hq^ix + b_0}}, \tag{2.10}$$

where $C \in \mathbb{R}$. The solutions (2.10) can be used to construct solutions

$$\psi_{k+n} = \mathbf{A}^*_{k+n} \dots \mathbf{A}^*_{k+1} \psi^0_k$$

of the eigenvalue problems for the operators $\mathbf{H}_{k+n}$ given by (2.1).

3 Limit Case

The second order q-difference operator (2.1) in the limit case as $q \to 1$ becomes the second order differential operator

$$\begin{aligned} \mathbf{H}_k = &-(b_2x^2+b_0)\frac{d^2}{dx^2} + 2kb_2x\frac{d}{dx} + \frac{-2kb_2^2x^2 + kb_2(a_0-a_1)x^2 + kb_2hx}{b_2x^2+b_0} \\ &+ \frac{(a_0-a_1)x^2 + 2h(a_0-a_1)x + h^2 + 4b_2b_0}{4(b_2x^2+b_0)} - \frac{1}{2}(a_0-a_1) \\ &+ ka_1 - (k-1)a_0 - k(k-1)b_2. \end{aligned} \tag{3.1}$$

The chains of operators (1.11) and (1.12) in the limit are given by

$$\mathbf{A}_k = \frac{d}{dx} + \frac{-2b_2x + (a_0-a_1)x + h}{2\,(b_2x^2+b_0)}, \tag{3.2}$$

$$\mathbf{A}^*_k = -(b_2x^2+b_0)\frac{d}{dx} + (2k-1)b_2x + \frac{a_0-a_1}{2}x + \frac{h}{2}. \tag{3.3}$$

The constants a_k (2.4) are transformed into

$$a_k = ka_1 - (k-1)a_0 - k(k-1)b_2. \tag{3.4}$$

The ground state (2.10) tends to

$$\psi^0_k(x) = |b_2x^2+b_0|^{\frac{2b_2+a_1-a_0}{4b_2}} \exp\left(-\frac{h}{2b_0}\int \frac{dx}{1+\frac{b_2}{b_0}x^2}\right)$$

and the weight function (2.8) to

$$\varrho_k(x) = |b_2x^2+b_0|^{-(k+1)}.$$

In order to systematize the class of the potentials given by q-deformation we shall transform the considered differential operator (3.1) into the standard form

$$\tilde{\mathbf{H}}_k = -\frac{d^2}{dy^2} + V_k(x(y)), \tag{3.5}$$

where

$$V_k(x) = \frac{d_2x + d_1}{b_2x^2+b_0} + d_0 \tag{3.6}$$

and

$$d_2 = h\left(kb_2 + \frac{a_0 - a_1}{2}\right), \tag{3.7}$$

$$d_1 = \frac{h^2}{4} + \frac{1}{4}b_2 b_0 - \left(\left(k^2 + \frac{3}{4}\right)b_2 + (a_0 - a_1)\left(k + \frac{1}{4b_2}\right)\right)b_0, \tag{3.8}$$

$$d_0 = \frac{a_0 - a_1}{4b_2} + \frac{a_1 + a_0}{2} + \frac{1}{4}b_2. \tag{3.9}$$

The transformation connecting these operators and these eigenvalue problems are following

$$\psi_k(x) = \left(b_2 x^2 + b_0\right)^{\frac{2k+1}{4}} \varphi_k(y), \tag{3.10}$$

$$dy = \frac{dx}{\sqrt{b_2 x^2 + b_0}}, \tag{3.11}$$

where φ_k is eigenfunction of the operator (3.5). These functions belong to the standard Hilbert space $L^2(\mathbb{R}, dx)$.

For the different values of the parameters b_2, b_0 we have the following possibilities:

1. If $b_2, b_0 > 0$ from (3.11) we obtain

$$x = \sqrt{\frac{b_0}{b_2}} \sinh\left(\sqrt{b_2}(y - c)\right),$$

where c is constant. In this subcase the potential is given by

$$V_k(y) = \frac{d_1}{\cosh^2(y - c)} + \frac{d_2 \sinh(y - c)}{\cosh^2(y - c)} + d_0,$$

where we put $b_2 = b_0 = 1$. We see that in the limit we obtain the hyperbolic potential [1,2,4]. This potential was proposed and solved by F. Scarf. It is usually called the hyperbolic Scarf potential. The domain for this potential is $(-\infty, \infty)$.

2. If $b_2 < 0$ and $b_0 > 0$ from (3.11) we obtain

$$x = \sqrt{\frac{-b_0}{b_2}} \sin\sqrt{|b_2|}(y - c).$$

In this subcase the potential is given by

$$V_k(y) = \frac{d_1}{\cos^2(y - c)} + \frac{d_2 \sin(y - c)}{\cos^2(y - c)} + d_0,$$

where we put $b_2 = -b_0 = -1$. We see that in the limit we obtain the trigonometric potential [1,2,4]. In the literature it is know as trigonometric Scarf potential. The domain for this potential is $\left(-\frac{\pi}{2} + c, \frac{\pi}{2} + c\right)$.

References

[1] D.E. Alvarez-Castillo, Exactly solvable potentials and Romanovski polynomials in quantum mechanics, `arXiv:0808.1642`, 2009.

[2] B.K. Bagchi, *Supersymmetry in quantum and classical mechanics*, Chapman & Hall/CRC, 2001.

[3] G. Bangerezako and M.N. Hounkonnou, The factorization method for the general second order q-difference equation and the Laguerre-Hahn polynomials on the general q-lattice, J. Phys. A: Math. Gen., **36**, 765–773, 2003.

[4] J. Dereziski, M. Wrochna, Exactly solvable Schrödinger operators, Ann. Henri Poincaré, **12**, 397–418, 2011.

[5] A. Dobrogowska, T. Goliski, A. Odzijewicz, Change of variables in factorization method for second order functional equations, Czech. J. Phys., **54**, 1257–63, 2004.

[6] A. Dobrogowska, A. Odzijewicz, Second order q-difference equations solvable by factorization method, J. Comput. Appl. Math., **193**, 319–346, 2006.

[7] A. Dobrogowska, A. Odzijewicz, Solutions of the q-deformed Schrödinger equation for special potentials, J. Phys. A: Math. Theor., **40**, 2023–2036, 2007.

[8] A. Dobrogowska, The q-deformation of the Morse potential, Appl. Math. Lett., **7**, 769–773, 2013.

[9] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge: Cambridge University Press, 1990.

[10] L. Infeld, T.E. Hull, The Factorization Method, Rev. Mod. Phys., **23**, 21–68, 1951.

[11] B. Mielnik, L.M. Nieto, O. Rosas-Ortiz, The finite difference algorithm for higher order supersymmetry, Phys. Lett. A, **269**, 70–78, 2000.

[12] W. Miller, Jr., *Lie Theory and Special Functions*, Academic Press New York and London, 1968.

[13] W. Miller, Jr., Lie theory and q-difference equations, SIAM J. Math. Anal., **1**, 405–425, 1970.

[14] F.L. Scarf, New soluble energy band problem, Phys. Rev., **112**, 1137–1140, 1958.