

On the Generating Functions of a Periodic Infinite Order Linear Recurrence

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Abstract

Let \mathcal{G} be the space of generating functions of a periodic infinite order linear recurrence. In this paper we provide an explicit procedure for computing a basis of \mathcal{G} .

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1 Introduction and Statement of the Main Result

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of the natural numbers and p a positive integer. In this paper we study a particular class of vectorial homogeneous recurrences of the type

$$x_{n+1} = \sum_{i=0}^{+\infty} A_i x_{n-i}, \text{ for all } n \in \mathbb{N}, \quad (1.1)$$

where $x_n \in \mathbb{C}^p$ and $\{A_n\}_{n \in \mathbb{N}}$ is an infinite sequence of $p \times p$ matrices with complex entries.

As in [3], we call this type of infinite order linear recurrences *generalized Fibonacci recurrences*, or Fib_p^∞ recurrences for short. A Fib_p^∞ recurrence is said to be an s -periodic (with $s \in \mathbb{Z}^+$) if $A_{n+s} = A_n$ for all $n \in \mathbb{N}$.

The main result of this work concerns the study of the solutions of an s -periodic Fib_p^∞ recurrence.

Before stating the main result of this work, we introduce some notation.

Throughout the paper, $\mathbb{C}[[z]]$ denotes the commutative ring of formal power series with complex coefficients. Matrices with entries in \mathbb{C} and $\mathbb{C}[[z]]$ will be denoted respectively as elements of $\mathbb{C}^{m \times n}$ and $\mathbb{C}[[z]]^{m \times n}$. As usual, the entry (i, j) of a matrix \mathbf{A} will be denoted by $\mathbf{A}(i, j)$.

The infinite-dimensional vector spaces over \mathbb{C}

$$U = \bigoplus_{n \in \mathbb{N}} \mathbb{C}^p \text{ and } V = \prod_{n \in \mathbb{Z}} \mathbb{C}^p,$$

will play an important role in this discussion. We write \mathbf{u} and \mathbf{v} to denote the vectors of U and V , with components $u_n \in \mathbb{C}^p$ and $v_n \in \mathbb{C}^p$, i.e.,

$$\mathbf{u} = (u_n)_{n \in \mathbb{N}} = (u_0, u_1, \dots), \text{ with } u_n \in \mathbb{C}^p, u_n = 0 \text{ for almost all } n,$$

and

$$\mathbf{v} = (v_n)_{n \in \mathbb{N}} = (\dots, v_{-1}, v_0, v_1, \dots), \text{ with } v_n \in \mathbb{C}^p.$$

We write $\mathbf{e}_1, \dots, \mathbf{e}_p$ to denote the vectors of the standard basis of \mathbb{C}^p . The symbols \mathbf{e}_β , with $\beta \in \mathbb{Z}^+$, denote the vectors of the standard basis of U :

$$\begin{aligned} \mathbf{e}_1 &= (e_1, 0, 0, \dots), \mathbf{e}_2 = (e_2, 0, 0, \dots), \dots, \mathbf{e}_p = (e_p, 0, 0, \dots), \\ \mathbf{e}_{p+1} &= (0, e_1, 0, \dots), \mathbf{e}_{p+2} = (0, e_2, 0, \dots), \dots, \mathbf{e}_{2p} = (0, e_p, 0, \dots), \\ &\dots \end{aligned}$$

where 0 denotes the zero vector of \mathbb{C}^p .

A vector $\mathbf{v} = (v_n)_{n \in \mathbb{N}} \in V$ is said to be a solution of the Fib_p^∞ recurrence (1.1), if the set $\{n \leq 0 : v_n \neq 0\}$ is finite and

$$v_{n+1} = \sum_{i=0}^{+\infty} \mathbf{A}_i v_{n-i}, \text{ for all } n \in \mathbb{N}.$$

The subspace of V whose vectors are the solutions of the Fib_p^∞ recurrence is denoted by S .

Naturally, there exists an isomorphism $\Theta : U \mapsto S$, where

$$\Theta(\mathbf{u}) = (v_n)_{n \in \mathbb{N}},$$

denotes the unique vector of S satisfying

$$v_{-n} = u_n \text{ for all } n \in \mathbb{N}.$$

The vector $\Theta(\mathbf{u}) \in S$ is called the solution of the Fib_p^∞ recurrence for the initial condition $\mathbf{u} \in U$. The vector space U is called space of initial conditions.

In order to analyze the asymptotic behaviour of a solution $\Theta(\mathbf{u}) = (v_n)_{n \in \mathbb{Z}} \in S$, we define the generating function $G(\mathbf{u})$ as the formal power series with coefficients in \mathbb{C}^p

$$G(\mathbf{u}) = \sum_{n \geq 0} v_n z^n.$$

Alternatively, $G(\mathbf{u})$ can be defined as an element of the \mathbb{C} -vector space $\mathbb{C}[[z]]^p$ such that

$$G(\mathbf{u}) = (G_1(\mathbf{u}), \dots, G_p(\mathbf{u})),$$

with

$$G_\alpha(\mathbf{u}) = \sum_{n \geq 0} v_n^{(\alpha)} z^n \in \mathbb{C}[[z]],$$

where $v_n^{(\alpha)}$ denotes the α -component of v_n with respect to the standard base of \mathbb{C}^p .

Naturally, the map

$$\begin{aligned} G : U &\mapsto \mathbb{C}[[z]]^p \\ \mathbf{u} &\rightarrow G(\mathbf{u}), \end{aligned}$$

is linear. We call $\mathcal{G} = G(U)$ the space of generating functions of the Fib_p^∞ recurrence.

Notice that, from the linearity of the map G , we have

$$G(\mathbf{u}) = \sum_{\beta \geq 1} c_\beta G(\mathbf{e}_\beta), \quad (1.2)$$

where $(c_\beta)_{\beta \in \mathbb{Z}^+}$ denote the coordinates of \mathbf{u} with respect to the standard basis, $(\mathbf{e}_\beta)_{\beta \in \mathbb{Z}^+}$, of U . Therefore, the set

$$\{G(\mathbf{e}_\beta) : \beta \in \mathbb{Z}^+\},$$

spans \mathcal{G} .

Now, we can state the main result of this work concerning the important class of periodic Fib_p^∞ recurrences. As we will see, this result combined with the main theorem of [1] provides an explicit procedure for computing a basis of \mathcal{G} .

Theorem 1.1. *If a Fib_p^∞ recurrence is s -periodic, then all generating functions $G(\mathbf{u}) \in \mathcal{G}$ are rational function of z . Moreover, the generating functions $G(\mathbf{e}_1), \dots, G(\mathbf{e}_{(s+1)p})$ span the space \mathcal{G} . Hence, $\dim \mathcal{G} \leq (s+1)p$.*

2 Proof of Theorem 1.1

Proof of Theorem 1.1. The notions of kneading matrix and kneading determinant of a Fib_p^∞ recurrence, introduced in [1] (see also [2]), are the main ingredients of the proof of Theorem 1.1. In order to improve the readability of the proof of Theorem 1.1, we present a brief description of the main ideas of [1].

Let $\alpha = 1, \dots, p$ and β be a positive integer. Analogously to the kneading theory, stated by Milnor and Thurston in [4], we introduce the (α, β) -kneading increment of a Fib_p^∞ recurrence as the following formal power series

$$K(\alpha, \beta) = \begin{cases} \sum_{n \geq 0} \mathbf{A}_{n+q-1}(\alpha, p) z^n \in \mathbb{C}[[z]], & \text{if } p \text{ divides } \beta, \\ \sum_{n \geq 0} \mathbf{A}_{n+q}(\alpha, r) z^n \in \mathbb{C}[[z]], & \text{otherwise,} \end{cases}$$

where q and r denote, respectively, the quotient and the remainder of the integer division of β by p .

Notice that, in the particular case $\alpha, \beta = 1, \dots, p$ the kneading increment is given by

$$K(\alpha, \beta) = \sum_{n \geq 0} \mathbf{A}_n(\alpha, \beta) z^n,$$

and the $p \times p$ -matrix of formal power series

$$\mathbf{K} = \begin{pmatrix} K(1, 1) & \cdots & K(1, p) \\ \vdots & \ddots & \vdots \\ K(p, 1) & \cdots & K(p, p) \end{pmatrix} \in \mathbb{C}[[z]]^{p \times p},$$

receives the name of kneading matrix of the Fib_p^∞ recurrence.

Finally, we define kneading determinant of the Fib_p^∞ recurrence by

$$\Delta = \det(\mathbf{I} - z\mathbf{K}) \in \mathbb{C}[[z]],$$

where \mathbf{I} denotes the $p \times p$ -identity matrix.

Similarly, for each $\alpha = 1, \dots, p$ and $\beta \in \mathbb{Z}^+$ we define the extended kneading matrix $\mathbf{K}_\alpha(\beta)$ and the extended kneading determinant $\Delta_\alpha(\beta)$ of a Fib_p^∞ recurrence by setting

$$\mathbf{K}_\alpha(\beta) = \begin{pmatrix} K(1, 1) & \cdots & K(1, p) & K(1, \beta) \\ \vdots & \ddots & \vdots & \vdots \\ K(p, 1) & \cdots & K(p, p) & K(p, \beta) \\ \delta(\alpha, 1) & \cdots & \delta(\alpha, p) & \delta(\alpha, \beta) \end{pmatrix} \in \mathbb{C}[[z]]^{(p+1) \times (p+1)}, \quad (2.1)$$

where $\delta(i, j)$ is the usual Kronecker delta function and

$$\Delta_\alpha(\beta) = \det(\mathbf{I} - z\mathbf{K}_\alpha(\beta)) \in \mathbb{C}[[z]],$$

where \mathbf{I} denotes the $(p+1) \times (p+1)$ identity matrix.

Next, we recall the main result of [1], which establishes a main relationship between the generating functions $G_\alpha(\mathbf{e}_\beta)$ and the kneading determinants Δ and $\Delta_\alpha(\beta)$ of a Fib_p^∞ recurrence.

Theorem 2.1. For every $\alpha = 1, \dots, p$ and every vector \mathbf{e}_β of the standard basis of U , the generating function $G_\alpha(\mathbf{e}_\beta)$ of a Fib_p^∞ recurrence satisfies the following equation in $\mathbb{C}[[z]]$

$$zG_\alpha(\mathbf{e}_\beta) = 1 - \Delta^{-1}\Delta_\alpha(\beta).$$

We have all the ingredients needed to prove Theorem 1.1.

First, observe that for any $\alpha = 1, \dots, p$ and $\beta \in \mathbb{Z}^+$ the kneading increment $K(\alpha, \beta)$ of an s -periodic Fib_p^∞ recurrence is a rational function of z . Indeed, as $\mathbf{A}_{n+s} = \mathbf{A}_n$ for all $n \in \mathbb{N}$, one gets

$$\begin{aligned} K(\alpha, \beta) &= \begin{cases} \sum_{n \geq 0} \mathbf{A}_{n+q-1}(\alpha, p) z^n, & \text{if } p \text{ divides } \beta, \\ \sum_{n \geq 0} \mathbf{A}_{n+q}(\alpha, r) z^n, & \text{otherwise,} \end{cases} & (2.2) \\ &= \begin{cases} \sum_{m \geq 0} \left(\sum_{k=0}^{s-1} \mathbf{A}_{k+q-1}(\alpha, p) z^k \right) z^{sm}, & \text{if } p \text{ divides } \beta, \\ \sum_{m \geq 0} \left(\sum_{k=0}^{s-1} \mathbf{A}_{k+q}(\alpha, r) z^k \right) z^{sm}, & \text{otherwise,} \end{cases} \\ &= \begin{cases} \left(\sum_{k=0}^{s-1} \mathbf{A}_{k+q-1}(\alpha, p) z^k \right) \sum_{m \geq 0} z^{sm}, & \text{if } p \text{ divides } \beta, \\ \left(\sum_{k=0}^{s-1} \mathbf{A}_{k+q}(\alpha, r) z^k \right) \sum_{m \geq 0} z^{sm}, & \text{otherwise,} \end{cases} \\ &= \begin{cases} \frac{\sum_{k=0}^{s-1} \mathbf{A}_{k+q-1}(\alpha, p) z^k}{1 - z^s}, & \text{if } p \text{ divides } \beta, \\ \frac{\sum_{k=0}^{s-1} \mathbf{A}_{k+q}(\alpha, r) z^k}{1 - z^s}, & \text{otherwise,} \end{cases} \end{aligned}$$

where q and r denote, respectively, the quotient and the remainder of the integer division of β by p . So, the kneading determinant Δ , as well any extended kneading determinant $\Delta_\alpha(\beta)$ of an s -periodic Fib_p^∞ recurrence are rational functions of z . But, by Theorem 2.1, this means that any generation function $G_\alpha(\mathbf{e}_\beta)$ is rational too. Combining this with (1.2) we finally conclude that any generating function $G(\mathbf{u})$ of an s -periodic Fib_p^∞ recurrence is a rational function of z . This is precisely what is stated in the first part of Theorem 1.1.

Next, we prove the second part of Theorem 1.1.

Evidently, if $q \in \mathbb{N}$ and $r \in \mathbb{N}$ denote, respectively, the quotient and the remainder of the division of $\beta \in \mathbb{Z}^+$ by p , then the quotient and the remainder of the division of $\beta + sp$ by p are, respectively, $q + s$ and r . So, since the Fib_p^∞ recurrence is s -periodic,

one gets by (2.2)

$$\begin{aligned}
K(\alpha, \beta + sp) &= \begin{cases} \frac{\sum_{k=0}^{s-1} \mathbf{A}_{k+q+s-1}(\alpha, p) z^k}{1 - z^s}, & \text{if } p \text{ divides } \beta + sp, \\ \frac{\sum_{k=0}^{s-1} \mathbf{A}_{k+q+s}(\alpha, r) z^k}{1 - z^s}, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{\sum_{k=0}^{s-1} \mathbf{A}_{k+q-1}(\alpha, p) z^k}{1 - z^s}, & \text{if } p \text{ divides } \beta + sp, \\ \frac{\sum_{k=0}^{s-1} \mathbf{A}_{k+q}(\alpha, r) z^k}{1 - z^s}, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{\sum_{k=0}^{s-1} \mathbf{A}_{k+q-1}(\alpha, p) z^k}{1 - z^s}, & \text{if } p \text{ divides } \beta, \\ \frac{\sum_{k=0}^{s-1} \mathbf{A}_{k+q}(\alpha, r) z^k}{1 - z^s}, & \text{otherwise,} \end{cases} \\
&= K(\alpha, \beta),
\end{aligned}$$

for all $\alpha = 1, \dots, p$ and $\beta \in \mathbb{Z}^+$. Combining this with (2.1), one obtains

$$\mathbf{K}_\alpha(\beta) = \mathbf{K}_\alpha(\beta + sp), \text{ for all } \alpha = 1, \dots, p \text{ and } \beta > p.$$

Therefore,

$$\Delta_\alpha(\beta) = \det(\mathbf{I} - z\mathbf{K}_\alpha(\beta)) = \det(\mathbf{I} - z\mathbf{K}_\alpha(\beta + sp)) = \Delta_\alpha(\beta + sp),$$

and, by Theorem 2.1, we can write

$$G_\alpha(\mathbf{e}_\beta) = G_\alpha(\mathbf{e}_{\beta+sp}), \text{ for all } \alpha = 1, \dots, p \text{ and } \beta > p.$$

This proves that $G(\mathbf{e}_{\beta+sp}) = G(\mathbf{e}_\beta)$ for all $\beta > p$. Consequently,

$$\{G(\mathbf{e}_\beta) : \beta \in \mathbb{Z}^+\} = \{G(\mathbf{e}_\beta) : \beta = 1, \dots, p(1+s)\},$$

which proves that $G(\mathbf{e}_1), G(\mathbf{e}_2), \dots, G(\mathbf{e}_{p(1+s)})$ span \mathcal{G} .

Hence, $\dim \mathcal{G} \leq p(1+s)$. This is precisely what is stated in the second part of Theorem 1.1. \square

We finish the paper with two simple examples to illustrate the role played by Theorems 1.1 and 2.1 in the explicit computation of a basis of \mathcal{G} .

Example 2.2. Consider the 1-periodic Fib_2^∞ recurrence

$$x_{n+1} = \sum_{i=0}^{+\infty} \mathbf{A}_i x_{n-i}, \text{ for all } n \in \mathbb{N},$$

with

$$\mathbf{A}_n = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \text{ for all } n \in \mathbb{N}.$$

By (2.2), the kneading increments are

$$K(1, \beta) = \begin{cases} \frac{2}{1-z}, & \text{if } \beta \text{ is even,} \\ \frac{1}{1-z}, & \text{if } \beta \text{ is odd,} \end{cases} \text{ and } K(2, \beta) = \begin{cases} \frac{4}{1-z}, & \text{if } \beta \text{ is even,} \\ \frac{3}{1-z}, & \text{if } \beta \text{ is odd,} \end{cases}$$

for all $\beta \in \mathbb{Z}^+$. Therefore,

$$\mathbf{K} = \begin{pmatrix} 1 & 2 \\ \frac{1}{3}z & \frac{1}{4}z \\ \frac{1}{1-z} & \frac{1}{1-z} \end{pmatrix}, \Delta = \det(\mathbf{I} - z\mathbf{K}) = \frac{4z^2 - 7z + 1}{(1-z)^2}. \quad (2.3)$$

On the other hand, as

$$\mathbf{K}_\alpha(\beta) = \begin{cases} \begin{pmatrix} 1 & 2 & 2 \\ \frac{1}{3}z & \frac{1}{4}z & \frac{1}{4}z \\ \delta(\alpha, 1) & \delta(\alpha, 2) & \delta(\alpha, \beta) \end{pmatrix}, & \text{if } \beta \text{ is even,} \\ \begin{pmatrix} 1 & 2 & 1 \\ \frac{1}{3}z & \frac{1}{4}z & \frac{1}{3}z \\ \delta(\alpha, 1) & \delta(\alpha, 2) & \delta(\alpha, \beta) \end{pmatrix}, & \text{if } \beta \text{ is odd,} \end{cases}$$

one has $\Delta_\alpha(\beta) = \det(\mathbf{I} - z\mathbf{K}_\alpha(\beta)) =$

$$\begin{cases} \frac{\delta(\alpha, \beta)(7z^2 - 4z^3 - z) - 2\delta(\alpha, 2)z^2 + 2z^3 + 2z^2 - 7z + 1}{(1-z)^2}, & \text{even } \beta, \\ \frac{\delta(\alpha, \beta)(-4z^3 + 7z^2 - z) + 2\delta(\alpha, 2)z^2(2z - 1) - z^3 + 3z^2 - 7z + 1}{(1-z)^2}, & \text{odd } \beta. \end{cases} \quad (2.4)$$

From Theorem 1.1, the generating functions $G(\mathbf{e}_1)$, $G(\mathbf{e}_2)$, $G(\mathbf{e}_3)$ and $G(\mathbf{e}_4)$ span \mathcal{G} . By (2.3), (2.4) and Theorem 2.1, the computation of the generating functions is straightforward

$$G(\mathbf{e}_1) = \left(\frac{5z^2 - 6z + 1}{4z^2 - 7z + 1}, \frac{-3z^2 + 3z}{4z^2 - 7z + 1} \right),$$

$$G(\mathbf{e}_2) = \left(\frac{-2z^2 + 2z}{4z^2 - 7z + 1}, \frac{2z^2 - 3z + 1}{4z^2 - 7z + 1} \right),$$

$$G(\mathbf{e}_3) = \left(\frac{z^2 + z}{4z^2 - 7z + 1}, \frac{-3z^2 + 3z}{4z^2 - 7z + 1} \right),$$

$$G(\mathbf{e}_4) = \left(\frac{-2z^2 + 2z}{4z^2 - 7z + 1}, \frac{-2z^2 + 4z}{4z^2 - 7z + 1} \right).$$

Thus, $G(\mathbf{e}_1)$, $G(\mathbf{e}_2)$, $G(\mathbf{e}_3)$, $G(\mathbf{e}_4)$ are linearly independent. Therefore, $\dim \mathcal{G} = 4$.

Example 2.3. Now, consider the 1-periodic Fib_2^∞ recurrence

$$x_{n+1} = \sum_{i=0}^{+\infty} \mathbf{A}_i x_{n-i}, \text{ for all } n \in \mathbb{N},$$

with

$$\mathbf{A}_n = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ for all } n \in \mathbb{N}.$$

As in the previous example, the generating functions $G(\mathbf{e}_1)$, $G(\mathbf{e}_2)$, $G(\mathbf{e}_3)$, $G(\mathbf{e}_4)$ span \mathcal{G} . Now, by Theorem 2.1, one gets

$$G(\mathbf{e}_1) = \left(\frac{2z - 1}{3z - 1}, \frac{-z}{3z - 1} \right), G(\mathbf{e}_2) = \left(\frac{-z}{3z - 1}, \frac{2z - 1}{3z - 1} \right),$$

and

$$G(\mathbf{e}_3) = G(\mathbf{e}_4) = \left(\frac{-z}{3z - 1}, \frac{-z}{3z - 1} \right).$$

Thus, $G(\mathbf{e}_1)$, $G(\mathbf{e}_2)$, $G(\mathbf{e}_3)$ are linearly independent and $\dim \mathcal{G} = 3$.

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