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Some Problems of Second-Order Rational Difference Equations with Quadratic Terms

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Abstract

We consider some special cases of second-order quadratic rational difference equations with nonnegative parameters and with arbitrary nonnegative initial conditions such that the denominator is always positive. The main goal is to confirm some conjectures and solve some open problems stated by A. M. Amleh, E. Camouzis and G. Ladas in the papers *"On the Dynamics of a Rational Difference Equations, Part 1"* (Int. J. Difference Equ., 3(1), pp. 1–35, 2008), and *"On the Dynamics of a Rational Difference Equations, Part 2"* (Int. J. Difference Equ., 3(2), pp. 195–225, 2008) about the global stability character, the periodic nature, and the boundedness of solutions of second-order quadratic rational difference equations.

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1 Introduction and Preliminaries

We consider some special cases of a second-order quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

with nonnegative parameters and with arbitrary nonnegative initial conditions such that the denominator is always positive. Equation (1.1) arises from the system of rational difference equations in the plane:

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1 + \gamma_1 y_n}{y_n}, \\ y_{n+1} = \dfrac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n}, \end{cases} \quad n = 0, 1, \ldots \tag{1.2}$$

Results have been obtained about 30 special cases of (1.1). Open problems and conjectures about this class of difference equations can be found in papers [1, 2]. In this paper the authors consider two equations which are denoted in the paper [2] as $\#24$ and $\#25$, pose ideas how to confirm some conjectures and solve given open problems about the local and global stability character, the periodic nature, and the boundedness of difference equation (1.1).

Behavior of different types of rational difference equations with quadratic terms were discussed in [1–3, 6, 7, 15–17, 19]. For related work on other nonlinear difference equations, see [4, 5, 8, 9, 11–13, 18].

Now we recall some well-known results, which will be useful in the investigation of (1.1) and which are given in [1, 2, 5, 12, 14]. We consider a difference equation defined by the formula

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots. \tag{1.3}$$

Let I be some interval of real numbers and let

$$f : I \times I \to I$$

be a continuously differentiable function. Then, for every set of initial conditions x_{-1}, $x_0 \in I$, the difference equation (1.3) has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of (1.3) if

$$\overline{x} = f(\overline{x}, \overline{x}),$$

that is,

$$x_n = \overline{x}, \quad \forall n \geq 0$$

is a solution of (1.3) or, equivalently, $\overline{x}$ is a fixed point of f.

Let $p = \dfrac{\partial f}{\partial u}(\overline{x}, \overline{x})$ and $q = \dfrac{\partial f}{\partial v}(\overline{x}, \overline{x})$ denote the partial derivatives of $f(u, v)$ evaluated at the equilibrium $\overline{x}$ of (1.3). Then the equation

$$y_{n+1} = p y_n + q y_{n-1}, \quad n = 0, 1, \ldots \tag{1.4}$$

is called the linearized equation associated with (1.3) about the equilibrium point $\overline{x}$ and the equation

$$\lambda^2 - p\lambda - q = 0 \tag{1.5}$$

is called a characteristic equation of (1.4) about $\overline{x}$.

The following two theorems provide criteria for stability/unstability of second-order difference equations.

Theorem 1.1 (See [12]). *1. If both roots of quadratic equation* (1.5) *lie in the open unit disk* $|\lambda| < 1$*, then the equilibrium* $\overline{x}$ *of* (1.3) *is locally asymptotically stable.*

2. *If at least one of the roots of* (1.5) *has absolute value greater than one, then the equilibrium* $\overline{x}$ *of* (1.3) *is unstable.*

3. *A necessary and sufficient condition for both roots of* (1.5) *to lie in the open unit disk* $|\lambda| < 1$*, is*

$$|p| < 1 - q < 2. \tag{1.6}$$

In the next theorem we use the following notation associated with a function $f(z_1, z_2)$, which is monotonic in both arguments. For each pair of numbers (m, M) and for each $i \in \{1, 2\}$, we define

$$M_i(m, M) = \begin{cases} M, & \text{if } f \text{ is increasing in } z_i \\ m, & \text{if } f \text{ is decreasing in } z_i \end{cases}$$

and

$$m_i(m, M) = M_i(M, m).$$

Theorem 1.2 ("M & m" Theorem [5, 12]). *Let* $[a, b]$ *be a closed and bounded interval of real numbers and let* $f \in C([a, b]^2, [a, b])$ *satisfy the following conditions:*

1. $f(z_1, z_2)$ *is monotonic in each of its arguments.*

2. *If (m,M) is a solution of the system*

$$\begin{cases} M = f(M_1(m, M), M_2(m, M)) \\ m = f(m_1(m, M), m_2(m, M)) \end{cases},$$

then $M = m$.

Then the difference equation (1.3) *has a unique equilibrium point* $\overline{x} \in [a, b]$ *and every solution of* (1.3)*, with initial conditions in* $[a, b]$*, converges to* $\overline{x}$.

A solution $\{x_n\}_{n=-1}^{\infty}$ of (1.3) is said to be periodic with period p if

$$x_{n+p} = x_n \quad \text{for all} \quad n \geq -1. \tag{1.7}$$

A solution $\{x_n\}_{n=-1}^{\infty}$ of (1.3) is said to be periodic with prime period p, or a p-cycle, if it is periodic with period p and p is the least positive integer for which (1.7) holds.

In reference [14] the following result is proved.

Theorem 1.3 (See [14]). *Let $[a, b]$ be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \to [a, b]$$

is a continuous function satisfying the following properties:

1. *$f(x, y)$ is nonincreasing in $x \in [a, b]$ for each $y \in [a, b]$, $f(x, y)$ is nondecreasing in $y \in [a, b]$ for each $x \in [a, b]$.*
2. *The difference equation* (1.3) *has no solutions of prime period two in $[a, b]$.*

Then (1.3) *has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of* (1.3) *converges to $\overline{x}$.*

2 Equation $\#25$

One special case of the second-order rational quadratic difference equation (1.1) is

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{1 + Bx_n x_{n-1} + x_{n-1}}, n = 0, 1, 2, \dots . \tag{2.1}$$

Equation (2.1) has a unique equilibrium $\overline{x}$, where $\overline{x}$ is a positive solution of the cubic equation

$$B\overline{x}^3 + \overline{x}^2 + (1 - \gamma)\overline{x} - \alpha = 0. \tag{2.2}$$

It is easy to show that the following lemma holds for the difference equation (2.1).

Lemma 2.1.

1. *Difference equation* (2.1) *has no positive solutions of prime period two.*
2. *Every positive solution of* (2.1) *is bounded above and below by positive constants*

$$\frac{\alpha}{1 + B(\alpha + \gamma)^2 + (\alpha + \gamma)} \le x_{n+1} < \alpha + \gamma, \quad n = 3, 4, \dots . \tag{2.3}$$

3. *The function*

$$f(z_1, z_2) = \frac{\alpha + \gamma z_2}{1 + Bz_1 z_2 + z_2}, \tag{2.4}$$

which is associated with (2.1)*, is decreasing in z_1 and decreasing in z_2 when*

$$\gamma \le \alpha. \tag{2.5}$$

Lemma 2.1 will be useful for the investigation in this section.

In paper [2], a conjecture about the global asymptotically stability is posed:

Conjecture 2.2 (See [12]). Every positive solution of (2.1) has a finite limit.

According to Lemma 2.1, Theorem 1.3 does not solve Conjecture 2.2.

Some results about the stability can be obtained by using Theorem 1.2, that is, the fact that if (m, M) is a solution of system, namely,

$$\begin{cases} M = \dfrac{\alpha + \gamma m}{1 + Bm^2 + m} \\ m = \dfrac{\alpha + \gamma M}{1 + BM^2 + M} \end{cases}, \tag{2.6}$$

then $m = M$ when

$$(\alpha B - 1 - \gamma)^2 - 4B(1 + \gamma) < 0. \tag{2.7}$$

Summarizing the above mentioned conditions, we have the following result:

Theorem 2.3. *Assume that the inequalities $\gamma \leq \alpha$ and $(\alpha B - 1 - \gamma)^2 - 4B(1+\gamma) < 0$ hold. Then the positive solution of* (2.1) *has a finite limit.*

Proof. This proof is based on Theorem 1.2. Every solution of (2.1) is bounded (follows from Lemma 2.1) and

$$[a, b] := \left[\frac{\alpha}{1 + B(\alpha + \gamma)^2 + (\alpha + \gamma)}, \alpha + \gamma\right]$$

$\forall\ z_1, z_2 \in [a, b]$: the function $f(z_1, z_2) = \dfrac{\alpha + \gamma z_2}{1 + Bz_1 z_2 + z_2}$ is monotonically decreasing in z_1 and in z_2, if $\gamma \leq \alpha$. (m, M) is a solution of system (2.6) if (2.7) holds. Since $\forall$ $x_{-1}, x_0 \in [0, +\infty[$ inequality (2.3) holds, the solution of (2.1) has a finite limit for every pair of positive initial conditions. This concludes the proof. □

3 Equation $\#24$

Let us consider a difference equation in the form

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1}}{1 + Bx_n x_{n-1} + x_{n-1}}, n = 0, 1, 2, \ldots. \tag{3.1}$$

In the above mentioned articles, no results about equation (3.1) are given. However, the following conjecture is posed.

Conjecture 3.1 (See [12]). Every positive solution of (3.1) has a finite limit.

We begin by proving the following lemma.

Lemma 3.2. *1. Difference equation* (3.1) *has a unique positive equilibrium point $\overline{x}$.*

2. If $\beta > 1$, then

$$\alpha < \overline{x} < \frac{\beta - 1}{B} \text{ or } \frac{\beta - 1}{B} < \overline{x} < \alpha \text{ or } \alpha = \overline{x} = \frac{\beta - 1}{B}.$$

3. *If* $0 < \beta < 1$, *then* $\overline{x} < \alpha$.

4. 0 *is not an equilibrium.*

Proof. The equilibrium equation of (3.1),

$$B\overline{x}^3 + (1-\beta)\overline{x}^2 + \overline{x} - \alpha = 0, \tag{3.2}$$

can be considered as two curves with behavior

$$\underbrace{B\overline{x}^2 + (1-\beta)\overline{x}}_{\text{a parabola}} = \underbrace{\frac{\alpha}{\overline{x}} - 1}_{\text{a hyperbola}}. \tag{3.3}$$

Equation (3.1) has a unique positive equilibrium point $\overline{x}$, which can be obtained as an intersection point of these two curves (see Figures 3.1 and 3.2). From Figures 3.1 and

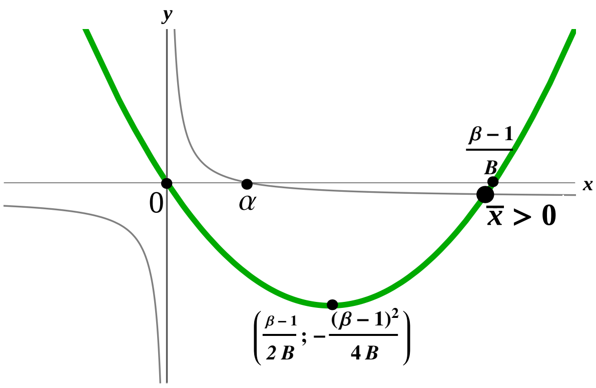

Figure 3.1: The equilibrium of (3.1), $\beta > 1$

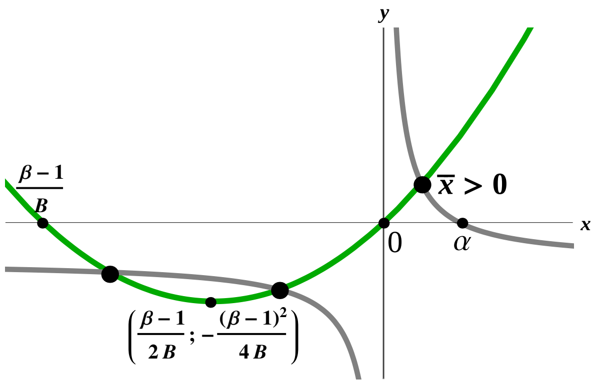

Figure 3.2: The equilibrium of (3.1), $0 < \beta < 1$

3.2, we obtain the required conclusion. □

Next lemma describes the behavior of the solutions of the difference equation (3.1).

Lemma 3.3. *1. Difference equation* (3.1) *has no positive solutions of prime period two.*

2. *Every positive solution of* (3.1) *is bounded above and below by positive constants:*

$$\frac{\alpha B}{B+(\alpha B+\beta)^2+(\alpha B+\beta)} \leq x_{n+1} \leq \frac{\alpha B+\beta}{B}, n=3,4,5,\ldots. \tag{3.4}$$

3. *The function*

$$f(z_1,z_2)=\frac{\alpha+\beta z_1 z_2}{1+Bz_1 z_2+z_2}, \tag{3.5}$$

which is associated with (3.1)*, is decreasing in* z_1 *when*

$$\beta(B+\beta)+\alpha B(\beta-B)<0 \tag{3.6}$$

and decreasing in z_2 *when*

$$\beta \leq \alpha B. \tag{3.7}$$

According to Lemma 3.3, Theorem 1.3 does not solve Conjecture 3.1. By using the "M & m" theorem, that is, the fact that if (m,M) is a solution of system, namely,

$$\begin{cases} M=\dfrac{\alpha+\beta m^2}{1+Bm^2+m} \\ m=\dfrac{\alpha+\beta M^2}{1+BM^2+M} \end{cases}, \tag{3.8}$$

then $m=M$ when

$$(\alpha B-1-\beta)^2-4(B+\beta+\beta^2)(1+\alpha\beta)<0, \tag{3.9}$$

we can finally state the main result of this section for the difference equation (3.1):

Theorem 3.4. *Assume that the following inequalities hold:*

$$\begin{gathered} \beta(B+\beta)+\alpha B(\beta-B)<0, \\ \beta \leq \alpha B, \\ (\alpha B-1-\beta)^2-4(B+\beta+\beta^2)(1+\alpha\beta)<0. \end{gathered}$$

Then the positive solution of (3.1) *is globally asymptotically stable.*

Conditions of Theorem 3.4 do not form an empty set of parameters. For example, $\beta=2$, $\alpha=2$, and $B=4$ satisfy Theorem 3.4 (see Figure 3.3).

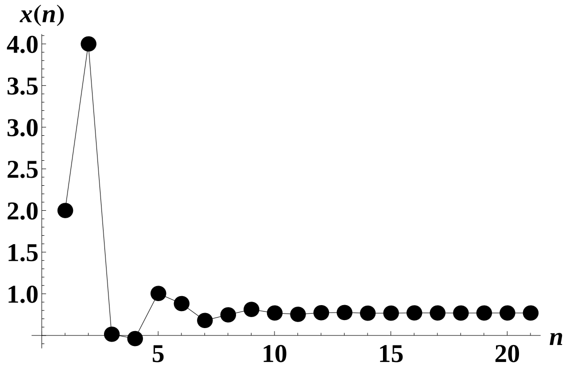

Figure 3.3: Solution of (3.1) converges to the unique positive equilibrium $\overline{x} \approx 0.77$ for values $\beta = 2$, $\alpha = 2$, $B = 4$

4 Linearization Technique

In [15] Kulenović and Mehuljić proved that the unique positive equilibrium is globally asymptotically stable for five special cases of (1.1). Namely,

$$x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{\alpha + x_n x_{n-1}}{A + x_n x_{n-1}}, \quad x_{n+1} = \frac{\alpha + x_{n-1}}{A + x_n x_{n-1}},$$
$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}}, \quad x_{n+1} = \frac{\alpha}{(1 + x_n) x_{n-1}} \quad n = 0, 1, \ldots .$$

Theorem 4.1 (See [10, 15])**.** *Let I be some interval of real numbers and let $f \in C[I \times I, I]$. Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{4.1}$$

where f is a real function in all variables. Let $j \in \{1, 2, \ldots\}$. Suppose that on some interval $I \subseteq \mathbb{R}$ equation (4.1) *has the linearization*

$$x_{n+j} = \sum_{i=1-j}^{m} g_i x_{n-i}, \tag{4.2}$$

where the nonnegative functions $g_i : I^{k+j} \to \mathbb{R}$ are such that $\sum_{i=j-1}^{m} g_i = 1$ is satisfied. Assume that there exists $A > 0$ such that

$$g_{1-j} \geq A, \quad n = 0, 1, \ldots .$$

If $x_{j-1}, \ldots, x_{-k} \in I$, then

$$\lim_{n \to \infty} x_n = L \in I.$$

The linearization for equation (2.1) can be written in the form

$$x_{n+2} = \underbrace{\frac{1+x_n}{1+Bx_{n+1}x_n+x_n}}_{g_1} x_{n+1} + \underbrace{\frac{\gamma - x_{n+1}}{1+Bx_{n+1}x_n+x_n}}_{g_0} x_n + \\ + \underbrace{\frac{Bx_{n+1}x_n + x_{n+1} - \gamma}{1+Bx_{n+1}x_n+x_n}}_{g_{-1}} x_{n-1}. \tag{4.3}$$

We can create a linearization such that $\sum_{i=j-1}^{m} g_i = g_{-1} + g_0 + g_1 = 1$, but we cannot show that there exists $A > 0$ such that $g_{-1}, g_0, g_1 \geq A$. We can write a similar linearization for (3.1) in the form

$$x_{n+2} = \underbrace{\frac{1+x_n+\beta x_{n-1}}{1+Bx_{n+1}x_n+x_n}}_{g_1} x_{n+1} + \underbrace{\frac{\beta x_{n+1} - x_{n+1} - \beta x_{n-1}}{1+Bx_{n+1}x_n+x_n}}_{g_0} x_n + \\ + \underbrace{\frac{Bx_{n+1}x_n + x_{n+1} - \beta x_{n+1}}{1+Bx_{n+1}x_n+x_n}}_{g_{-1}} x_{n-1}. \tag{4.4}$$

We cannot show that $g_0 \geq 0$. With the linearization technique, it can be obtained different expressions of g_{-1}, g_0, g_1. We have considered several cases, but they do not give a positive result.

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