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Approximative Full Solutions of Difference Equations

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Abstract

In the study of asymptotic properties of solutions of difference equations, so called generalized solutions are often considered, that is, sequences for which a given equation is satisfied from some starting point. In this paper we establish conditions which allow us to control the starting point. Moreover, we establish conditions which allow us to change some finite terms of generalized solutions so that a full solution is obtained.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of positive integers, the set of all integers and the set of real numbers, respectively. For $p, k \in \mathbb{Z}$ let

$$\mathbb{N}(p) = \{p, p+1, \dots\}, \qquad \mathbb{N}(p,k) = \{p, p+1, \dots, k\}.$$

Let $m \in \mathbb{N}(1)$. In this paper we consider difference equations of the form

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n + c_n, \tag{E}$$

$$\Delta^m x_n = a_n \varphi(x_{\sigma(n)}) + b_n + c_n, \tag{AE}$$

where

$$a_n, b_n, c_n \in \mathbb{R}, \quad f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \quad \varphi : \mathbb{R} \to \mathbb{R}, \quad \sigma : \mathbb{N} \to \mathbb{Z}, \quad \lim \sigma(n) = \infty.$$

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We assume f and φ are continuous, $s \in (-\infty, 0]$ and

$$\sum_{n=1}^{\infty} n^{m-1-s}|a_n| < \infty, \qquad \sum_{n=1}^{\infty} n^{m-1-s}|c_n| < \infty. \tag{1.1}$$

Moreover, we introduce an index n_0 by

$$n_0 = \min\{n \in \mathbb{N} : \sigma(n) \geq 1\}.$$

Let $p \in \mathbb{N}(n_0)$. We say that a sequence $x : \mathbb{N} \to \mathbb{R}$ is a p-solution of equation (E) if equality (E) is satisfied for any $n \geq p$. We say that x is a solution if it is a p-solution for certain $p \geq n_0$. By a full solution we mean an n_0-solution.

Note that, by (1.1), the sequences a and c are "small". On the other hand b is an arbitrary sequence of real numbers. In this paper we are interested in solutions of (E) and (AE) which are asymptotically equivalent to a given solution y of the equation $\Delta^m y = b$. More precisely, in Theorems 3.1 and 4.1, using Schauder's fixed point theorem, we establish conditions under which for a given index p there exists a p-solution x such that $x = y + \mathrm{o}(n^s)$. For $b = 0$ we obtain asymptotically polynomial solutions. On the other hand, taking $p = n_0$, we obtain full solutions. Unfortunately, in many cases, the assumptions of Theorem 3.1 or Theorem 4.1 are satisfied only for large p. In Section 5 we establish conditions which allow us to change some finite terms of generalized solution so that a full solution is obtained. Note that an autonomous equation (AE) is a special case of (E) but Theorem 3.1 is not a special case of Theorem 4.1.

This paper is a continuation of the papers [2, 3] and [4]. The results obtained in Section 5 are inspired by [1, Remark 2]. In Section 2, we introduce the space of m-times summable sequences $\mathrm{S}(m)$ and the iterated remainder operator $r : \mathrm{S}(m) \to \mathrm{S}(0)$. In Lemma 2.1, we establish some fundamental properties of the operator r^m. For more details see [3]. The operator r^m is used in the proofs of Theorems 3.1 and 4.1, which are the main results of this paper.

2 Notation and Terminology

The space of all sequences $x : \mathbb{N} \to \mathbb{R}$ we denote by SQ. Assume X is a metric space. For a subset A of X and $\varepsilon > 0$ we define an ε-framed interior of A by

$$\mathrm{Int}(A, \varepsilon) = \{x \in X : \mathrm{B}(x, \varepsilon) \subset A\},$$

where $\mathrm{B}(x, \varepsilon)$ denotes an open ball of radius ε about x. We say that a subset U of X is an uniform neighborhood of a subset Z of X if there exists a positive number ε such that $Z \subset \mathrm{Int}(U, \varepsilon)$.

A function $g : X \to \mathbb{R}$ is called locally bounded if for any $x \in X$ there exists a neighborhood U of x such that $g|U$ is bounded. Note that if X is a closed subset of

$\mathbb{R}$, then a function $g : X \to \mathbb{R}$ is locally bounded if and only if it is bounded on every bounded subset of X.

If $g : X \to \mathbb{R}$ and $M \in \mathbb{R}$, then by $|g \le M|$ we denote the set

$$|g \le M| = \{x \in X : |g(x)| \le M\}.$$

Let

$$\mathrm{S}(0) = \{x \in \mathrm{SQ} : x = \mathrm{o}(1)\}, \quad \mathrm{S}(1) = \left\{ x \in \mathrm{SQ} : \text{ the series } \sum_{n=1}^{\infty} x_n \text{ is convergent} \right\}.$$

For $x \in \mathrm{S}(1)$, we define the sequence $r(x)$ by the formula

$$r(x)(n) = \sum_{j=n}^{\infty} x_j.$$

Then $r(x) \in \mathrm{S}(0)$ and we obtain the remainder operator $r : \mathrm{S}(1) \to \mathrm{S}(0)$. If $m \in \mathbb{N}(1)$ then, by induction, we define the space $\mathrm{S}(m+1)$ and the operator $r^{m+1} : \mathrm{S}(m+1) \to \mathrm{S}(0)$ by

$$\mathrm{S}(m+1) = \{x \in \mathrm{S}(m) : r^m(x) \in \mathrm{S}(1)\}, \qquad r^{m+1}(x) = r(r^m(x)).$$

The value $r^m(x)(n)$ we denote also by $r_n^m(x)$ or simply $r_n^m x$. Note that

$$r_n^m x = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m}.$$

Lemma 2.1. *Assume $x, y \in \mathrm{SQ}$, $m \in \mathbb{N}(1)$, $p \in \mathbb{N}(0)$ and $s \in (-\infty, 0]$. Then*

(a) *if $|x| \in \mathrm{S}(m)$, then $x \in \mathrm{S}(m)$ and $|r^m x| \le r^m |x|$,*

(b) *$|x| \in \mathrm{S}(m)$ if and only if $\sum_{n=1}^{\infty} n^{m-1}|x_n| < \infty$,*

(c) *if $|x| \in \mathrm{S}(m)$, then $r_k^m |x| \le \sum_{n=k}^{\infty} n^{m-1}|x_n|$ for any $k \in \mathbb{N}(1)$,*

(d) *if $x \in \mathrm{S}(m)$, then $\Delta^m r^m x = (-1)^m x$,*

(e) *if $x, y \in \mathrm{S}(m)$ and $x_n \le y_n$ for $n \ge p$, then $r_n^m x \le r_n^m y$ for $n \ge p$,*

(f) *if $\sum_{n=1}^{\infty} n^{m-1-s}|x_n| < \infty$, then $x \in \mathrm{S}(m)$ and $r^m x = \mathrm{o}(n^s)$.*

Proof. The assertion (a) is proved in [3, Lemma 1]. (b) is proved in [3, Lemma 3]. (c) follows from [3, Lemma 2] and from the proof of [3, Lemma 3]. (d) is proved in [3, Lemma 5]. (e) is obvious for $m = 1$. For $m > 1$ it can be easily proved by induction. Assume $\sum_{n=1}^{\infty} n^{m-1-s}|x_n| < \infty$. By (a) we have $x \in \mathrm{S}(m)$. Let $u_n = n^{-s}$. By [4, Lemma 2.3] there exists a sequence $w = \mathrm{o}(n^s)$ such that $\Delta^m w = x$. By [4, Lemma 2.2] we have $r^m x = (-1)^m w$. Hence $r^m x = \mathrm{o}(n^s)$. □

3 Autonomous Case

In this section we obtain conditions under which for a given index p there exists a p-solution x of autonomous equation

$$\Delta^m x_n = a_n \varphi(x_{\sigma(n)}) + b_n + c_n \tag{AE}$$

which is asymptotic to a given solution y of $\Delta^m y = b$ with "order" of approximation $\mathrm{o}(n^s)$. In the main condition (3.1) we use a framed interior of the set

$$|\varphi \le M| = \{t \in \mathbb{R} : |\varphi(t)| \le M\},$$

where M is a given positive constant.

Theorem 3.1. *Assume* $p \in \mathbb{N}(n_0)$,

$$M > 0, \quad y \in \mathrm{SQ}, \quad \Delta^m y = b$$

and let R *be a sequence defined by* $R = M r^m |a| + r^m |c|$. *If*

$$(y \circ \sigma)(\mathbb{N}(p)) \subset \mathrm{Int}(|\varphi \le M|, R_p), \tag{3.1}$$

then there exists a p*-solution* x *of* (AE) *such that*

$$x = y + \mathrm{o}(n^s).$$

Proof. For $x \in \mathrm{SQ}$ let x^* be defined by $x^*_n = \varphi(x_{\sigma(n)})$ for $n \ge n_0$ and $x^*_n = 0$ for $n < n_0$. Let $\rho_n = R_n$ for $n \ge p$ and $\rho_n = 0$ for $n < p$. Moreover, let

$$T = \{x \in \mathrm{SQ} : |x - y| \le R_p\}, \qquad S = \{x \in \mathrm{SQ} : |x - y| \le \rho\}.$$

Obviously $S \subset T$. Let $x \in S$. If $n \ge p$, then $|x_{\sigma(n)} - y_{\sigma(n)}| \le R_p$. Hence

$$x_{\sigma(n)} \in \mathrm{B}(y_{\sigma(n)}, R_p) \subset |\varphi \le M|.$$

Therefore for $x \in S$ we have $|x^*| \le M$ and, by (1.1), $ax^* + c \in \mathrm{S}(m)$. Let

$$A : S \to \mathrm{SQ}, \qquad A(x)(n) = \begin{cases} y_n & \text{for} \quad n < p \\ y_n + (-1)^m r^m_n (ax^* + c) & \text{for} \quad n \ge p. \end{cases} \tag{3.2}$$

If $x \in S$ and $n \geq p$, then

$$|A(x)(n) - y_n| = |r_n^m(ax^* + c)| \leq r_n^m|ax^* + c| \leq Mr_n^m|a| + r_n^m|c| = R_n = \rho_n.$$

Hence $A(S) \subset S$. As in the proof of [3, Theorem 1] one can show that A is continuous and there exists a sequence $x \in S$ such that $Ax = x$. Now using Lemma 2.1 we obtain the result. □

Corollary 3.2. *Assume* $y \in \mathrm{SQ}$, $\Delta^m y = b$ *and* φ *is bounded on certain uniform neighborhood of the set* $(y \circ \sigma)(\mathbb{N})$. *Then there exists a solution* x *of* (AE) *such that* $x = y + \mathrm{o}(n^s)$.

Proof. There exist a positive M and $\varepsilon > 0$ such that $(y \circ \sigma)(\mathbb{N}) \subset \mathrm{Int}(|\varphi \leq M|, \varepsilon)$. Let R be defined by (3.1). Then $R = \mathrm{o}(1)$ and $R_p < \varepsilon$ for certain p. Now the assertion follows from Theorem 3.1. □

Corollary 3.3. *Assume* φ *is bounded. Then for every full solution* y *of the equation* $\Delta^m y = b$ *there exists a full solution* x *of* (AE) *such that* $x = y + \mathrm{o}(n^s)$.

Proof. Choose M such that $|\varphi(t)| \leq M$ for any $t \in \mathbb{R}$. Then $|\varphi \leq M| = \mathbb{R}$. Hence $\mathrm{Int}(|\varphi \leq M|, \varepsilon) = \mathbb{R}$ for any positive ε and we may take $p = n_0$ in Theorem 3.1. □

4 Nonautonomous Case

In this section we assume $g : [0, \infty) \to [0, \infty)$, $w \in \mathrm{SQ}$, w is positive and bounded and

$$|f(n, t)| \leq g(|t|w_n), \quad (n, t) \in \mathbb{N} \times \mathbb{R}. \tag{4.1}$$

In Theorem 4.1, which is analogous to Theorem 3.1, we establish conditions under which for a given index p there exists a p-solution x of nonautonomous equation

$$\Delta^m x_n = a_n f(n, x_{\sigma(n)}) + b_n + c_n \tag{E}$$

which is asymptotic to a given solution y of $\Delta^m y = b$ with "order" of approximation $\mathrm{o}(n^s)$. We replace a metric condition (3.1) by an analytic condition (4.2).

Theorem 4.1. *Assume* $p \in \mathbb{N}(n_0)$,

$$L, M > 0, \quad g([0, L]) \subset [0, M], \quad y \in \mathrm{SQ}, \quad \Delta^m y = b,$$

and let R *be a sequence defined by* $R = Mr^m|a| + r^m|c|$. *If*

$$|y_{\sigma(n)}| \leq Lw_n^{-1} - R_p \quad \text{for} \quad n \geq p, \tag{4.2}$$

then there exists a p*-solution* x *of* (E) *such that*

$$x = y + \mathrm{o}(n^s).$$

Proof. For $x \in \mathrm{SQ}$ let x^* be defined by $x^*_n = f(n, x_{\sigma(n)})$ for $n \geq n_0$ and $x^*_n = 0$ for $n < n_0$. Define ρ, T and S as in the proof of Theorem 3.1. Obviously $S \subset T$. Let $x \in S$. If $n \geq p$, then

$$|x_{\sigma(n)}w_n| = |x_{\sigma(n)}w_n - y_{\sigma(n)}w_n + y_{\sigma(n)}w_n| \leq |x_{\sigma(n)} - y_{\sigma(n)}|w_n + |y_{\sigma(n)}w_n|$$

$$\leq (|x_{\sigma(n)} - y_{\sigma(n)}| + |y_{\sigma(n)}|)w_n \leq (R_p + |y_{\sigma(n)}|)w_n \leq L.$$

Hence by (4.1) we have

$$|x^*_n| = |f(n, x_{\sigma(n)})| \leq g(|x_{\sigma(n)}w_n|) \leq M.$$

Therefore for $x \in S$ we have $|x^*| \leq M$ and $ax^* + c \in \mathrm{S}(m)$. Define $A : S \to \mathrm{SQ}$ by (3.2). If $x \in S$ and $n \geq p$, then

$$|A(x)(n) - y_n| = |r^m_n(ax^* + c)| \leq r^m_n|ax^* + c| \leq Mr^m_n|a| + r^m_n|c| = R_n = \rho_n$$

and $A(S) \subset S$. The rest of the proof is analogous to the last part of the proof of Theorem 3.1. □

Theorem 4.1 generalizes [4, Theorem 3.1].

Corollary 4.2. *Assume g is locally bounded, $y \in \mathrm{SQ}$, $\Delta^m y = b$ and $y \circ \sigma = \mathrm{O}(w_n^{-1})$. Then there exists a solution x of* (E) *such that $x = y + \mathrm{o}(n^s)$.*

Proof. Choose positive constants L and β such that $|y_{\sigma(n)}| \leq Lw_n^{-1} - \beta$ for $n \geq n_0$. Choose a constant M such that $g([0, L]) \subset [0, M]$. Let R be defined by (4.2). Then $R = \mathrm{o}(1)$ and $R_p < \beta$ for certain $p \geq n_0$. Now, we can apply Theorem 4.1. □

Corollary 4.3. *Assume g is bounded, $y \in \mathrm{SQ}$, $\Delta^m y = b$ and $y \circ \sigma = \mathrm{O}(w_n^{-1})$. Then there exists a full solution x of* (E) *such that $x = y + \mathrm{o}(n^s)$.*

Proof. Choose M such that $|g(t)| \leq M$ for any t. Define R as in Theorem 4.1. Choose a positive L such that $|y_{\sigma(n)}| \leq Lw_n^{-1} - R_{n_0}$ for $n \geq n_0$ and apply Theorem 4.1. □

5 Completing Solutions

In this section we establish conditions which allow us to change some finite terms of generalized solution so that a full solution is obtained. First we define the sets

$$I(\sigma), \quad \Gamma(\varphi), \quad \Gamma(f, n)$$

which we use in Lemmas 5.2 and 5.3. These lemmas and Corollaries 5.4 and 5.5 are the main results of this section. Combining these results with Theorems 3.1 and 4.1 we obtain a way to prove the existence of full solutions of equations (AE) and (E) which are asymptotic to a given solution y of the equation $\Delta^m = b$. Let

$$I(\sigma) = \{n \in \mathbb{N} : \sigma(n) = n\}.$$

For $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ let

$$\Gamma_\alpha(\varphi) : \mathbb{R} \to \mathbb{R}, \qquad \Gamma_\alpha(f, n) : \mathbb{R} \to \mathbb{R}$$

be defined by

$$\Gamma_\alpha(\varphi)(t) = t - \alpha\varphi(t), \qquad \Gamma_\alpha(f, n)(t) = t - \alpha f(n, t).$$

Moreover, let

$$\Gamma(\varphi) = \{\alpha \in \mathbb{R} : \Gamma_\alpha(\varphi)(\mathbb{R}) = \mathbb{R}\}, \qquad \Gamma(f, n) = \{\alpha \in \mathbb{R} : \Gamma_\alpha(f, n)(\mathbb{R}) = \mathbb{R}\}.$$

Note that a real number α belongs to $\Gamma(f, n)$ if and only if the equation $t = \alpha f(n, t) + s$ has a solution t for every real s.

Example 5.1. Let $a, b \in \mathbb{R}$, $a \neq 0$.

(a) If φ is continuous and bounded, then $\Gamma(\varphi) = \mathbb{R}$.

(b) If φ is continuous and $t\varphi(t) > 0$ for $t \neq 0$, then $(-\infty, 0] \subset \Gamma(\varphi)$.

(c) If φ is continuous and $t\varphi(t) < 0$ for $t \neq 0$, then $[0, \infty) \subset \Gamma(\varphi)$.

(d) $\Gamma(at + b) = (-\infty, a^{-1}) \cup (a^{-1}, \infty)$.

(e) If φ is a polynomial of odd degree $m \geq 3$, then $\Gamma(\varphi) = \mathbb{R}$.

(f) If φ is a polynomial of even degree $m \geq 2$, then $\Gamma(\varphi) = \{0\}$.

(g) $\Gamma(e^t) = (-\infty, 0]$, $\Gamma(|t|) = (-1, 1)$, $\Gamma(\operatorname{sgn} t) = [0, \infty)$.

Lemma 5.2. *Assume $p > n_0$, $P = \mathbb{N}(n_0, p-1)$, $\sigma(n) \geq n$ for $n \in P$ and*

$$(-1)^m a_n \in \Gamma(f, n) \qquad \textit{for} \qquad n \in P \cap I(\sigma).$$

Then in every p-solution x of the equation (E) *we can change the terms*

$$x_{p-1}, x_{p-2}, \ldots, x_{n_0}$$

so that a full solution is obtained.

Proof. For any sequence x and any index n we have

$$\Delta^m x_n = \sum_{k=0}^{m} (-1)^{m+k} \binom{m}{k} x_{n+k}.$$

Hence, if x is a p-solution of (E), then

$$x_n = (-1)^m a_n f(n, x_{\sigma(n)}) + (-1)^m (b_n + c_n) - \sum_{k=1}^{m} (-1)^k \binom{m}{k} x_{n+k} \tag{5.1}$$

for $n \geq p$. If $\sigma(p-1) > p-1$, then, of course, we can change the value x_{p-1} so that the equality (5.1) is fulfilled for $n = p-1$. In the case $\sigma(p-1) = p-1$ a similar possibility follows from the assumption $(-1)^m a_{p-1} \in \Gamma(f, p-1)$. We obtain a $(p-1)$-solution, etc. After finite amount of steps a full solution is obtained. □

Lemma 5.3. *Assume $p > n_0$, $P = \mathbb{N}(n_0, p-1)$, $\sigma(n) < n$ for $n \in P$, the map $\sigma|P$ is injective and for $n \in P$ we have*

$$a_n \neq 0 \qquad \textit{and} \qquad f(\{n\} \times \mathbb{R}) = \mathbb{R}.$$

Then in every p-solution x of the equation (E) *we can change the terms of the finite sequence $\sigma|P$ so that a full solution is obtained.*

Proof. Let x be a p-solution of (E). If $a_n \neq 0$, then we can rewrite equality (E) in the form

$$f(n, x_{\sigma(n)}) = a_n^{-1}(\Delta^m x_n - b_n - c_n).$$

By assumption, we can change the value $x_{\sigma(p-1)}$ so that the equality

$$f(p-1, x_{\sigma(p-1)}) = a_{p-1}^{-1}(\Delta^m x_{p-1} - b_{p-1} - c_{p-1})$$

is fulfilled. Then the equality (E) is fulfilled for any $n \geq p-1$. Analogously, if $p-1 > n_0$, then we can change the value $x_{\sigma(p-2)}$ so that the equality

$$f(p-2, x_{\sigma(p-2)}) = a_{p-2}^{-1}(\Delta^m x_{p-2} - b_{p-2} - c_{p-2})$$

is fulfilled, etc. Using injectivity of the sequence

$$\sigma(n_0), \sigma(n_0+1), \ldots, \sigma(p-2), \sigma(p-1)$$

we can change successively the terms of finite sequence

$$x_{\sigma(p-1)}, x_{\sigma(p-2)}, \ldots, x_{\sigma(n_0)}$$

and we obtain a full solution. □

The following two corollaries are immediate consequences of Lemma 5.2 and Lemma 5.3 respectively.

Corollary 5.4. *Let one of the following conditions holds:*

(a) $(-1)^m a_n \in \Gamma(f, n)$ *for* $n \in I(\sigma)$ *and* $\sigma(n) \geq n$ *for* $n \geq n_0$,

(b) $f(\mathbb{N} \times \mathbb{R}) = \mathbb{R}$, $\sigma(n) < n$ *and* $a_n \neq 0$ *for* $n \geq n_0$.

Then from any solution x *of* (E) *we can obtain a full solution by changing a finite number of terms.*

Corollary 5.5. *Let one of the following conditions holds:*

(a) $(-1)^m a_n \in \Gamma(\varphi)$ *for* $n \in I(\sigma)$ *and* $\sigma(n) \geq n$ *for* $n \geq n_0$,

(b) $\varphi(\mathbb{R}) = \mathbb{R}$, $\sigma(n) < n$ *and* $a_n \neq 0$ *for* $n \geq n_0$.

Then from any solution x *of* (AE) *we can obtain a full solution by changing a finite number of terms.*

Now, we give some examples. In the first one we present an equation for which every constant $\lambda \in \mathbb{R}$ is the limit of certain full solution. But if $|\lambda| > 1$, then the condition (3.1) is not satisfied and Theorem 3.1 cannot be directly used.

Example 5.6. Let $\varphi(t) = t^3$, $b = 0$ and $c = 0$. Consider the equation

$$\Delta^m x_n = a_n x_{n-1}^3.$$

Assume $a_n \neq 0$ for any n and $r_p^m a = 1$ for certain p. Note that any constant sequence is a solution of the equation $\Delta^m y = b$. Moreover, by Corollary 3.2 and Corollary 5.5, for any real λ there exists a full solution x such that $x = \lambda + \mathrm{o}(n^s)$. That is, any real λ is the limit of certain full solution. Note that if $|\lambda| > 1$ and

$$\lambda \in |\varphi \leq M| = [-\sqrt[3]{M}, \sqrt[3]{M}],$$

then $M > 1$. Moreover, $R_p = M$ and we have

$$\mathrm{Int}(|\varphi \leq M|, R_p) = \mathrm{Int}(|\varphi \leq M|, M) = \emptyset.$$

Hence the condition (3.1) is not satisfied and Theorem 3.1 is not applicable.

In the next example we present an equation for which every constant $\lambda \in \mathbb{R}$ is the limit of certain solution but if λ is the limit of a full solution, then λ is nonnegative.

Example 5.7. Assume a is nonnegative, the series $\sum_{n=1}^{\infty} a_n$ is convergent and there exists an index $p > 1$ such that $a_p = 0$, $a_{p+1} = 1$. Consider the equation

$$\Delta x_n = a_n |x_{n-1}|.$$

By Corollary 3.2, every number $\lambda \in \mathbb{R}$ is the limit of certain solution. Assume x is a full convergent solution. Then x is nondecreasing and

$$x_{p+1} - x_p = a_p |x_{p-1}| = 0, \quad x_{p+2} = x_{p+1} + a_{p+1}|x_p| = x_{p+1} + |x_{p+1}| \geq 0.$$

Hence $\lim x_n \geq 0$.

In the next example we present an equation for which every constant $\lambda \in \mathbb{R}$ is the limit of certain solution but if λ is the limit of a full solution, then $\lambda < 2$.

Example 5.8. Assume a is positive, the series $\sum_{n=1}^{\infty} n a_n$ is convergent and $a_p = 1$ for certain p. Consider the equation

$$\Delta^2 x_n = a_n x_n^2.$$

By Corollary 3.2, every number $\lambda \in \mathbb{R}$ is the limit of certain solution. Assume x is a full convergent solution. Let $\lim x_n = \lambda \geq 2$. For $n \in \mathbb{N}$ let $z_n = \Delta x_n$. Then $z_n = \mathrm{o}(1)$ and

$$\Delta z_n = \Delta^2 x_n = a_n x_n^2 \geq 0.$$

Hence the sequence (z_n) is nondecreasing and nonpositive. If $z_q = 0$ for certain q then $z_n = 0$ for $n \geq q$. Thus $a_n x_n^2 = \Delta^2 x_n = \Delta z_n = 0$ for large n and $\lim x_n = \lambda \geq 2$. It is impossible. Hence $z_n < 0$ for all n. This means that (x_n) is decreasing. Hence $x_n > \lambda \geq 2$ for any n. Since $a_p = 1$, we have

$$x_{p+2} - 2x_{p+1} + x_p = \Delta^2 x_p = a_p x_p^2 = x_p^2.$$

Since $t^2 - t > t$ for $t > 2$ we obtain

$$x_{p+2} = 2x_{p+1} - x_p + x_p^2 > x_p^2 - x_p > x_p.$$

This contradicts the fact that (x_n) is decreasing. Hence $\lim x_n < 2$ for any full convergent solution x.

In the last example we can see the importance of the set $\Gamma(\varphi)$ in the case $I(\sigma) \neq \emptyset$. We present a family of equations for which, under assumption $a_n \in \Gamma(\varphi)$ for all n, any real λ is the limit of certain full solution. On the other hand, if $a_p \notin \Gamma(\varphi)$ for certain p, then for every p-solution x we have $x_n = -1$ for any $n > p$.

Example 5.9. Let $\varphi(t) = t+1$. Then $\Gamma(\varphi) = (-\infty, 1) \cup (1, \infty)$. Consider the equation

$$\Delta x_n = a_n \varphi(x_n).$$

If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then by Corollary 3.2, every $\lambda \in \mathbb{R}$ is the limit of certain solution. Moreover, if $a_n \neq -1$ for all $n \in \mathbb{N}$, then by Corollary 5.5 for every $\lambda \in \mathbb{R}$ there exists a full solution convergent to λ. Assume $a_p = -1$ for certain p. Then writing the equation in the form $x_{n+1} = (1 + a_n)x_n + a_n$ we obtain

$$x_{p+1} = (1 + (-1))x_p + (-1) = -1,$$
$$x_{p+2} = (1 + a_{p+1})(-1) + a_{p+1} = -1$$

and so on. Hence $x_n = -1$ for any $n > p$.

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