A Discretization Method to Solve Fractional Variational Problems with Dependence on Hadamard Derivatives

Ricardo Almeida

Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
3810–193 Aveiro, Portugal
ricardo.almeida@ua.pt

Nuno R. O. Bastos

Department of Mathematics, School of Technology
Polytechnic Institute of Viseu
3504–510 Viseu, Portugal
nbastos@mat.estv.ipv.pt

Delfim F. M. Torres

Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
3810–193 Aveiro, Portugal
delfim@ua.pt

Abstract

We provide a fast and simple method to solve fractional variational problems with dependence on Hadamard fractional derivatives. Using a relation between the Hadamard fractional operator and a sum involving integer-order derivatives, we rewrite the fractional problem into a classical optimal control problem. The latter problem is then solved by application of standard numerical techniques. We illustrate the procedure with an example.

1 Introduction

Fractional calculus deals with noninteger order integrals and derivatives. In this sense, it can be seen as a generalization of ordinary calculus. Similarly to standard (integer-order) calculus, where different notions for integration and differentiation are available, several definitions for fractional integrals and derivatives are also possible. The most common fractional operators are the Riemann–Liouville and Caputo, which are well studied in the literature. Available results include numerical tools to handle them [10]. Here we deal with Hadamard's approach [3, 4, 12], considering problems of the calculus of variations with dependence on this type of differentiation.

One way to find extremizers for fractional variational problems consists to deduce necessary optimality conditions, which in this case turn out to be fractional differential equations [6]. As is frequently noted, in most cases it is impossible to find analytical solutions to such equations [1, 11]. Here, instead of dealing directly with a fractional variational functional, we approximate the fractional derivative by a sum that involves integer-order derivatives only. In this way, we leave the field of fractional variational calculus, reducing the problem to an ordinary optimal control problem. To solve the latter, one can use any well-known technique to find an approximated solution. For this purpose we choose to use a discretization process and the software package Ipopt (Interior Point OPTimizer, `http://projects.coin-or.org/Ipopt`) that is designed to find solutions to large-scale nonlinear mathematical optimization problems [13].

The text is organized as follows. In Section 2 we present a short theoretical exposition of the necessary concepts. Our method is presented in Section 3 and then illustrated, with an example, in Section 4. We end with Section 5 of conclusion.

2 Preliminaries

We recall the necessary concepts and results.

Definition 2.1. Let $x : [a, b] \to \mathbb{R}$, $\alpha > 0$, and $n = [\alpha] + 1$. The (left) Hadamard fractional derivative of order α is defined by

$$_a\mathcal{D}_t^\alpha x(t) = \frac{1}{\Gamma(n - \alpha)} \left(t\frac{d}{dt} \right)^n \int_a^t \left(\ln\frac{t}{\tau} \right)^{n-\alpha-1} \frac{x(\tau)}{\tau} d\tau,$$

where Γ is the Gamma function, that is,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\, dt, \quad z > 0.$$

The Gamma function has the following two nice properties:

$$\Gamma(z + 1) = z\Gamma(z),\ \forall z > 0 \quad \text{and} \quad \Gamma(n) = (n - 1)!,\ \forall n \in \mathbb{N}.$$

If $\alpha = m$ is an integer, then one has (cf. [5])

$$_a\mathcal{D}_t^m x(t) = \left(t\frac{d}{dt}\right)^m x(t).$$

In [9] an expansion formula for the Hadamard fractional derivative with $\alpha \in (0,1)$ is proved. It has the advantage that we only need the first-order derivative of x to approximate $_a\mathcal{D}_t^\alpha x(t)$ by such sum. The result is the following one.

Theorem 2.2 (See [9]). *Let $x \in C^2[a,b]$ and $N \geq 2$. Then,*

$$_a\mathcal{D}_t^\alpha x(t) = {}_a\widetilde{\mathcal{D}}_t^\alpha x(t) + E(t),$$

where

$$_a\widetilde{\mathcal{D}}_t^\alpha x(t) = A\left(\ln\frac{t}{a}\right)^{-\alpha} x(t) + B\left(\ln\frac{t}{a}\right)^{1-\alpha} tx^{(1)}(t) + \sum_{p=2}^N C_p\left(\ln\frac{t}{a}\right)^{1-\alpha-p} V_p(t)$$

with

$$A = \frac{1}{\Gamma(1-\alpha)}\left[1 + \sum_{p=2}^N \frac{\Gamma(p+\alpha-1)}{\Gamma(\alpha)(p-1)!}\right],$$

$$B = \frac{1}{\Gamma(2-\alpha)}\left[1 + \sum_{p=1}^N \frac{\Gamma(p+\alpha-1)}{\Gamma(\alpha-1)p!}\right],$$

$$C_p = \frac{\Gamma(p+\alpha-1)}{\Gamma(-\alpha)\Gamma(1+\alpha)(p-1)!}, \quad p \in \{2,\ldots,N\},$$

$$V_p(t) = \int_a^t (p-1)\left(\ln\frac{\tau}{a}\right)^{p-2} \frac{x(\tau)}{\tau}d\tau, \quad p \in \{2,\ldots,N\},$$

and the error $|E(t)| = \left|_a\mathcal{D}_t^\alpha x(t) - {}_a\widetilde{\mathcal{D}}_t^\alpha x(t)\right|$ is bounded by

$$\tilde{E}(x,t) := \max_{\tau \in [a,t]}\left|x^{(1)}(t) + tx^{(2)}(t)\right|\frac{\exp((1-\alpha)^2 + 1 - \alpha)}{\Gamma(2-\alpha)(1-\alpha)N^{1-\alpha}}\left(\ln\frac{t}{a}\right)^{1-\alpha}(t-a). \quad (2.1)$$

In [9] the authors obtain a more general form of expansion, with sums up to the nth derivative. Expansion formulas for the Hadamard fractional integrals are also given.

3 Fractional Variational Problems and Discretization

Necessary optimality conditions for the Hadamard fractional calculus of variations are trivially obtained as corollaries from the results of [7, 8]. However, in general, there

are no analytical methods to solve such fractional differential equations. Therefore, to establish simple numerical methods is an important issue. Here we provide a direct method to address such variational problems. Our idea is simple but effective and easy to implement.

The problem of the fractional calculus of variations under consideration is stated in the following way: minimize the functional

$$J(x) = \int_a^b L\left(t, x(t), {_a}\mathcal{D}_t^\alpha x(t)\right) dt \tag{3.1}$$

subject to the boundary conditions

$$x(a) = x_a \quad \text{and} \quad x(b) = x_b,$$

where x_a and x_b are two given reals and $0 < \alpha < 1$. For other types of fractional variational problems we refer the reader to [6]. Note that functions x should be absolutely continuous in order to guarantee the existence of the Hadamard fractional derivatives [5, Lemma 2.34]. To apply our method, more smoothness is, however, required: admissible functions are assumed to be C^2, in agreement with Theorem 2.2. To solve the problem, we first replace the fractional operator by the sum given in Theorem 2.2. By doing so, we get a classical optimal control problem: minimize the functional

$$
\begin{aligned}
J_{approx}(x, u, V_2, \ldots, V_n) = \int_a^b L\Bigg(&t, x(t), A\left(\ln\frac{t}{a}\right)^{-\alpha} x(t) \\
&+ B\left(\ln\frac{t}{a}\right)^{1-\alpha} tu(t) + \sum_{p=2}^N C_p \left(\ln\frac{t}{a}\right)^{1-\alpha-p} V_p(t)\Bigg) dt
\end{aligned} \tag{3.2}
$$

subject to the dynamic constraints

$$
\begin{cases}
x'(t) = u(t), \\
V_p'(t) = (p-1)\left(\ln\dfrac{t}{a}\right)^{p-2}\dfrac{x(t)}{t}, \quad p \in \{2, \ldots, N\},
\end{cases}
$$

and the boundary conditions

$$
\begin{cases}
x(a) = x_a, \\
x(b) = x_b, \\
V_p(a) = 0, \quad p \in \{2, \ldots, N\}.
\end{cases}
$$

This is a classical optimal control problem with state functions x and $V_p, p \in \{2, \ldots, N\}$, and control u. We solve it applying any standard discretization procedure [2].

The error of approximating an Hadamard derivative is given by Theorem 2.2. Using the bound for the error (2.1), it is possible to get a bound for the error of approximating the variational functional (3.1) by the approximated value (3.2).

Theorem 3.1. *Let $x \in C^2[a, b]$ and $N \geq 2$. The error of approximating the variational functional* (3.1) *by* (3.2) *is bounded as follows:*

$$|J(x) - J_{approx}(x, u, V_2, \ldots, V_n)| \leq M \int_a^b \tilde{E}(x, t)dt,$$

where

$$M := \max_{t \in [a,b]} |\partial_3 L(t, x(t), {}_a\mathcal{D}_t^\alpha x(t))|$$

and $\tilde{E}(x, t)$ is given by (2.1).

Proof. It follows from Taylor's theorem and Theorem 2.2. $\qquad\square$

4 An Example

Let us minimize the functional

$$J(x) = \int_1^2 \left({}_1\mathcal{D}_t^{0.5}x(t) - \frac{\sqrt{\ln t}}{\Gamma(1.5)} \right)^2 dt \qquad (4.1)$$

subject to the boundary conditions

$$x(1) = 0 \quad \text{and} \quad x(2) = \ln 2. \qquad (4.2)$$

It is easy to check that the (global) solution is $\overline{x}(t) = \ln t$. Indeed,

$$_1\mathcal{D}_t^{0.5}\overline{x}(t) = \frac{\sqrt{\ln t}}{\Gamma(1.5)}$$

and the functional (4.1) is nonnegative with $J(\overline{x}) = 0$. Using the approach explained in Section 3, we approximate this problem by the following one: minimize the functional

$$J_{approx}(x, u, V_2, \ldots, V_N) = \int_1^2 \left(A(\ln t)^{-0.5}x(t) + B(\ln t)^{0.5}tu(t) \right.$$

$$\left. + \sum_{p=2}^N C_p(\ln t)^{0.5-p}V_p(t) - \frac{\sqrt{\ln t}}{\Gamma(1.5)} \right)^2 dt \quad (4.3)$$

subject to the dynamic constraints

$$\begin{cases} x'(t) = u(t), \\ V_p'(t) = (p-1)(\ln t)^{p-2}\dfrac{x(t)}{t}, \quad p \in \{2, \ldots, N\}, \end{cases} \qquad (4.4)$$

and the boundary conditions

$$\begin{cases} x(1) = 0, \\ x(2) = \ln 2, \\ V_p(1) = 0, \quad p \in \{2, \dots, N\}. \end{cases} \tag{4.5}$$

Fix $N = 3$ and $k \in \mathbb{N}$ and consider the maximum error given by

$$E_k(x, \tilde{x}) = \max_{i \in \{1, \dots, k\}} |x(t_i) - \tilde{x}(t_i)|.$$

We apply a direct discretization process with k points to problem (4.3)–(4.5), and then we use Ipopt [13] to find a solution for it. In Figure 4.1 we show the results for the state function x with $k = 100$ and $k = 250$. From these results, we obtain the errors $E_{100}(x, \tilde{x}) = 0.029558802$ and $E_{250}(x, \tilde{x}) = 0.011928571$, and we can see that the error decreases as the number of points k increases. As is shown in Figure 4.1, when $k = 250$
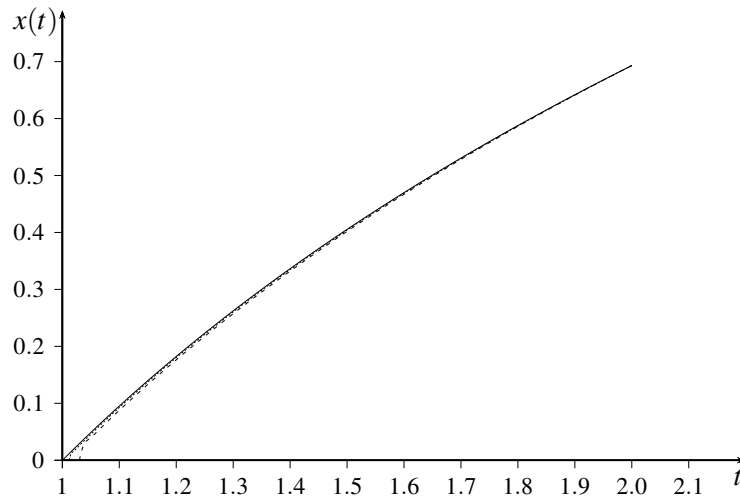
Figure 4.1: Exact solution to (4.1)–(4.2) (continuous line) versus numerical approximations to (4.3)–(4.5) with $k = 250$ (dotted line) and $k = 100$ (dashed line).

the approximation is almost indistinguishable from the exact solution.

5   Conclusion

The Hadamard fractional operator was introduced by J. Hadamard in 1892 [3]. This notion is different from the better known Riemann–Liouville and Caputo derivatives, in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains a logarithmic function of arbitrary exponent. Recently, there has been an increasing interest in studying variational problems with generalized fractional operators, that is, with fractional operators depending on a general kernel — see, e.g., [7, 8]. Such generalized

theory includes, as a particular case, the Hadamard variational calculus. However, available results are, excluding the particular cases of Riemann–Liouville and Caputo, only analytical. Here we investigated nonstandard fractional variational problems depending on Hadamard fractional derivatives. Relation between the Hadamard fractional operator and a sum involving integer-order derivatives is used to rewrite the fractional problem into a classical optimal control problem. The latter problem is then solved by application of standard numerical techniques. The effectiveness of the numerical approach is illustrated with an example.

Acknowledgments

References

[1] R. Almeida and D. F. M. Torres, Leitmann's direct method for fractional optimization problems, Appl. Math. Comput. **217** (2010), no. 3, 956–962.

[2] J. T. Betts, *Practical methods for optimal control using nonlinear programming*, Advances in Design and Control, 3, SIAM, Philadelphia, PA, 2001.

[3] J. Hadamard, Essai sur l'etude des fonctions donnees par leur developpment de Taylor, J. Pure Appl. Math. **4** (1892), no. 8, 101–186.

[4] A. A. Kilbas, Hadamard-type fractional calculus, J. Korean Math. Soc. **38** (2001), no. 6, 1191–1204.

[5] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.

[6] A. B. Malinowska and D. F. M. Torres, *Introduction to the fractional calculus of variations*, Imp. Coll. Press, London, 2012.

[7] T. Odzijewicz, A. B. Malinowska and D. F. M. Torres, Generalized fractional calculus with applications to the calculus of variations, Comput. Math. Appl. **64** (2012), no. 10, 3351–3366.

[8] T. Odzijewicz, A. B. Malinowska and D. F. M. Torres, A generalized fractional calculus of variations, Control Cybernet. **42** (2013), no. 2, 443–458.

[9] S. Pooseh, R. Almeida and D. F. M. Torres, Expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative, Numer. Funct. Anal. Optim. **33** (2012), no. 3, 301–319.

[10] S. Pooseh, R. Almeida and D. F. M. Torres, Discrete direct methods in the fractional calculus of variations, Comput. Math. Appl. **66** (2013), no. 5, 668–676.

[11] S. Pooseh, R. Almeida and D. F. M. Torres, Numerical approximations to fractional problems of the calculus of variations and optimal control. In: Fractional Calculus in Analysis, Dynamics and Optimal Control (Ed: J. Cresson), Chap. V, Mathematics Research Developments, Nova Science Publishers, New York, 2014.

[12] D. Qian, Z. Gong and C. Li, A generalized Gronwall inequality and its application to fractional differential equations with Hadamard derivatives, 3rd Conference on Nonlinear Science and Complexity (NSC10), Cankaya University, Ankara, Turkey, 28–31 July, 2010.

[13] A. Wächter and L. T. Biegler, On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming, Math. Program. **106** (2006), no. 1, Ser. A, 25–57.