International Journal of Difference Equations
ISSN 0973-6069, Volume 8, Number 2, pp. 195–213 (2013)
http://campus.mst.edu/ijde

On the Boundedness of Positive Solutions of a Reciprocal Max-Type Difference Equation with Periodic Parameters

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Abstract

We investigate the boundedness of positive solutions of a reciprocal max-type difference equation with positive periodic parameters and arbitrary delays. We give sufficient conditions on parameters and their periods for every solution to be unbounded. We also introduce the idea of *extended periodicity* of unbounded solutions, and then give sufficient conditions on the delays such that particular patterns of the extended periodicity of unbounded solutions are obtained.

AMS Subject Classifications: 2010 MSC: 39A20.
Keywords: Reciprocal max-type difference equations, unbounded solutions, eventually periodic solutions, extended periodic solutions.

1 Introduction

We examine the boundedness nature of positive solutions of the reciprocal max-type difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-k}}, \frac{B_n}{x_{n-l}}\right\}, \quad n = 0, 1, \ldots, \tag{1.1}$$

Received May 19, 2013; Accepted June 1, 2013
Communicated by Martin Bohner

where

(i) the delays k and l are arbitrary with $k, l \in \{0, 1, \ldots\}$ and $k < l$;

(ii) the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are periodic sequences of positive real numbers with periods p and q, respectively;

(iii) the initial conditions $x_{-l}, x_{-l+1}, \ldots, x_{-1}, x_0$ are positive.

Since the early 1990's to the present, difference equations with the maximum (or minimum) function and reciprocal arguments have rapidly evolved into a diverse family of equations. Relevant to our investigation, in 1998, Al-Amleh, Hoag, and Ladas [2] investigated one of the earliest autonomous reciprocal max-type equations,

$$x_{n+1} = \max\left\{\frac{a}{x_n}, \frac{A}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where $a, A \in \mathbf{R} - \{0\}$. One of the major results that they obtained was that when $a = 1$, $A \in (0, \infty)$, and initial conditions are positive, every solution is periodic with

1. period two if $A \in (0, 1)$;

2. period three if $A = 1$;

3. period four if $A \in (1, \infty)$.

However, they also showed that every solution is *unbounded* when $a \neq A$, $a, A \in (-\infty, 0)$, and $x_{-1}, x_0 \in \mathbf{R} - \{0\}$.

Replacing the constant coefficient A by the variable coefficient A_n (and replacing a by 1) in (1.2), in 1997, Briden *et al.* [7] studied the resulting nonautonomous equation

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A_n}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive numbers with period two such that

$$A_n = \begin{cases} A_0, & \text{if } n \text{ is even}, \\ A_1, & \text{if } n \text{ is odd}, \end{cases}$$

and where initial conditions are positive. They showed that every positive solution is eventually periodic with

1. period two if $A_0 A_1 \in (0, 1)$;

2. period six if $A_0 A_1 = 1$;

3. period four if $A_0 A_1 \in (1, \infty)$.

As of this point, no unbounded solutions were obtained with nonautonomous difference equations whose solutions are positive.

At the end of the decade, Briden *et al.* [6] and Grove *et al.* [13] then changed the period of $\{A_n\}_{n=0}^{\infty}$ in (1.3) to period three, and unbounded solutions made their first appearance with a positive periodic parameter and positive initial conditions in a reciprocal max-type equation. Specifically, assigning

$$A_n = \begin{cases} A_0, & \text{if } n = 3m, \\ A_1, & \text{if } n = 3m+1, \\ A_2, & \text{if } n = 3m+2, \end{cases}$$

for $m \geq 0$, they proved the following:

1. If $A_n \in (0,1)$ for all $n \geq 0$, then every solution is eventually periodic with period two.

2. If $A_n \in (1,\infty)$ for all $n \geq 0$, then every solution is eventually periodic with period twelve.

3. If $A_{i+1} < 1 < A_i$ for some $i \in \{0,1,2\}$, then every solution is *unbounded*.

4. In all other cases, every solution is eventually periodic with period three.

Upon the discovery that unbounded solutions could indeed occur with nonautonomous reciprocal max-type equations with positive solutions, Kent and Radin [16] in 2003 sought necessary and sufficient conditions for boundedness with the equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots, \tag{1.4}$$

where $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are periodic sequences of positive numbers with minimal periods p and q, respectively, and initial conditions are positive. They found the following:

1. If neither p nor q is a multiple of three, then every positive solution is bounded.

2. If $p = 3k$, $k \in \{1, 2, \ldots\}$, such that for some $i \in \{0,1,2\}$ and for all $j = 0, 1, \ldots, k-1$,
$$A_{1+i+3j} < B_0, B_1, \ldots, B_{q-1} < A_{2+i+3j},$$
then every positive solution is unbounded.

3. If $q = 3k$, $k \in \{1, 2, \ldots\}$, such that for some $i \in \{0,1,2\}$ and for all $j = 0, 1, \ldots, k-1$,
$$B_{1+i+3j} < A_0, A_1, \ldots, A_{p-1} < B_{i+3j},$$
then every positive solution is unbounded.

On the other hand, it was in part shown in 2008 by Kerbert and Radin [17] that every positive solution of the equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots, \tag{1.5}$$

is unbounded if $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are positive periodic sequences with minimal periods p and q, respectively, and p or q is a multiple of four (together with certain other conditions on A_n and B_n which we will omit for the sake of brevity). Furthermore, note that with the equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-1}}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \ldots, \tag{1.6}$$

where again $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are positive periodic sequences with minimal periods p and q, respectively, it can be shown by the methods used in this paper that every positive solution is unbounded provided that either p or q is a multiple of five (and certain other conditions hold which we will not state here for the sake of brevity).

Around the same time as Kerbert and Radin's investigation, Bidwell and Franke in a seminal paper [5] considered the following equation:

$$x_{n+1} = \max\left\{\frac{A_n^{(0)}}{x_n}, \frac{A_n^{(1)}}{x_{n-1}}, \ldots, \frac{A_n^{(r)}}{x_{n-r}}\right\}, \quad n = 0, 1, \ldots, \tag{1.7}$$

where $r \in \{1, 2, 3, \ldots\}$, $\{A_n^{(i)}\}_{n=0}^{\infty}$, for $i = 0, 1, \ldots, r$, is a periodic sequence of nonnegative numbers with period $p_i \in \{1, 2, \ldots\}$, and initial conditions are positive. They showed that if every solution of (1.7) is *bounded*, then every solution is eventually periodic. The following question then remained: *Under what conditions on the nonnegative periodic parameters is every solution unbounded?*

The works in [16, 17] form partial answers to this question. In the sequel, we generalize the results given in [16, 17] and investigate (1.1).

Specifically, in Section 3, we first determine sufficient conditions on the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$, when they are not necessarily periodic, such that every positive solution is unbounded.

In Section 4, we say more about the particular form that solutions may take in the case when the delays k and l satisfy the relation $l + 2 - k \leq k + 3$, and we give an example that satisfies the relation $l + 2 - k > k + 3$.

In Section 5, we give sufficient conditions on the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$, when they are periodic, such that every positive solution is unbounded.

Section 6 offers an open problem in analyzing an extension of (1.1). In this section we also speculate on some possible biological applications of this extension of (1.1).

In a final note, we mention that from 2005 to the present, there have also been investigations into difference equations with maxima that have been inspired by differential equations with maxima and automatic control (cf. [3]) and that have involved further generalizations of (1.7). All such difference equations have added an order of great complexity. For a sampling of work on these difference equations, see the papers by the following authors: Çinar, Stević, and Yalçinkaya [10]; Iričanin and Elsayad [15], whose results were improved upon in Stević [31, 32]; Liu, Yang, and Stević [20]; Sauer [21, 22]; Stević [24–26, 28, 30]; Sun [33], whose results were extended in Stević [27, 29]; Touafek and Halim [34]; and Yang, Liu, and Lin [35].

2 Preliminaries

First and foremost, we define the following:

Definition 2.1 (Boundedness and Persistence)**.** A positive sequence $\{x_n\}_{n=-r}$ is called *bounded* if there exists a positive constant M such that

$$0 < x_n \leq M \quad \text{for all } n \geq -r;$$

and it *persists* (or is *persistent*) if there exists a positive constant m such that

$$x_n \geq m \quad \text{for all } n \geq -r.$$

As an aside, and based on the results for (1.7) (see Section 1), we note that if every positive solution of our equation, (1.1), is bounded, then every positive solution is eventually periodic.

Definition 2.2 (Eventual Periodicity)**.** A positive sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be *eventually periodic* (or *truncated periodic*) if there exists $N \geq -r$ such that $\{x_n\}_{n=N}^{\infty}$ is a periodic sequence.

On the other hand, if every positive solution, $\{x_n\}_{n=-l}^{\infty}$, of (1.1) is unbounded, in the sequel we will show that they possess what could be described as a form of periodicity *in the limit as n tends to infinity*. Hence, we will need the following definition:

Definition 2.3 (Extended Periodicity)**.** A positive sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be *extended periodic with period* p if there exists positive integers u, v with $u + v = p$ and mutually exclusive sets of positive integers

$$S_1 = \{i_1, i_2, \ldots, i_u\} \subset \{0, 1, \ldots, p-1\}$$

and

$$S_2 = \{j_1, j_2, \ldots, j_v\} \subset \{0, 1, \ldots, p-1\}$$

such that

$$\lim_{n\to\infty} x_{pn+i} = \infty \quad \text{for all } i \in S_1$$

and

$$\lim_{n\to\infty} x_{pn+j} = 0 \quad \text{for all } j \in S_2.$$

The following definitions and remarks pertain to the sufficient conditions that we will need when proving unboundedness for (1.1) and (1.5) in the sequel. For the sake of convenience, we will at certain times set $t = k + l + 2$. In these definitions, note that the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ need not be periodic sequences. The sufficient condition for which every positive solution of (1.1) is unbounded will be based in the sequel on our first definition:

Definition 2.4 (Hypothesis (H))**.** Let $k, l \in \{0, 1, \ldots\}$ with $k < l$. A pair of sequences of positive real numbers, $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$, is said to *satisfy Hypothesis* (H) if there exists $i \in \{0, 1, \ldots\}$ such that for all $n \geq 0$,

$$s_A = \sup\left\{A_{tn+(k+l+3)+i} : n = 0, 1, \ldots\right\} < i_B = \inf\left\{B_{tn+(l+2)+i} : n = 0, 1, \ldots\right\},$$

and

$$s_B = \sup\left\{B_{tn+(k+l+3)+i} : n = 0, 1, \ldots\right\} < i_A = \inf\left\{A_{tn+(k+2)+i} : n = 0, 1, \ldots\right\},$$

with i_A, i_B, s_A, s_B all positive real numbers.

The "structure" of unbounded solutions of (1.1) under certain conditions will be based in the sequel on the following definition:

Definition 2.5 (Hypothesis (H$'$))**.** Let $k, l \in \{0, 1, \ldots\}$ with $k < l$. A pair of sequences of positive real numbers, $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$, is said to *satisfy Hypothesis* (H') if there exists $i \in \{0, 1, \ldots\}$ such that for all $n \geq 0$,

- $s_A^{(1,1)} = \sup\left\{A_{tn+(k+l+3)+i} : n = 0, 1, \ldots\right\}$
  $< i_B^{(1,1)} = \inf\left\{B_{tn+(l+2)+i} : n = 0, 1, \ldots\right\}$
  and
  $s_B^{(1,2)} = \sup\left\{B_{tn+(k+l+3)+i} : n = 0, 1, \ldots\right\}$
  $< i_A^{(1,2)} = \inf\left\{A_{tn+(k+2)+i} : n = 0, 1, \ldots\right\}$;
- $s_A^{(2,1)} = \sup\left\{A_{tn+(k+l+2)+i} : n = 0, 1, \ldots\right\}$
  $< i_B^{(2,1)} = \inf\left\{B_{tn+(l+1)+i} : n = 0, 1, \ldots\right\}$
  and
  $s_B^{(2,2)} = \sup\left\{B_{tn+(k+l+2)+i} : n = 0, 1, \ldots\right\}$
  $< i_A^{(2,2)} = \inf\left\{A_{tn+(k+1)+i} : n = 0, 1, \ldots\right\}$;

  $\vdots$

- $s_A^{(k+2,1)} = \sup\left\{A_{tn+(l+4)+i} : n = 0, 1, \ldots\right\}$
  $< i_B^{(k+1,1)} = \inf\left\{B_{tn+[l+2-(k-1)]+i} : n = 0, 1, \ldots\right\}$
  and
  $s_B^{(k+1,2)} = \sup\left\{B_{tn+(l+4)+i} : n = 0, 1, \ldots\right\}$
  $< i_A^{(k+1,2)} = \inf\{A_{tn+3+i} : n = 0, 1, \ldots\}$;
- $s_A^{(k+1,1)} = \sup\left\{A_{tn+(l+3)+i} : n = 0, 1, \ldots\right\}$
  $< i_B^{(k+1,1)} = \inf\left\{B_{tn+(l+2-k)+i} : n = 0, 1, \ldots\right\}$
  and
  $s_B^{(k+1,2)} = \sup\left\{B_{tn+(l+3)+i} : n = 0, 1, \ldots\right\}$
  $< i_A^{(k+1,2)} = \inf\{A_{tn+2+i} : n = 0, 1, \ldots\}$;

with the i_A's, i_B's, s_A's, and s_B's all positive real numbers.

Remark 2.6. Observe that if, in Definition 2.5, we have that

(i) $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are each periodic sequences with period $t = k + l + 2$,

(ii) $l + 2 - k \leq k + 3$ (with $k < l$),

then Hypothesis (H$'$) implies that

- $A_1 < B_{l+2}$ and $B_1 < A_{k+2}$;
- $A_0 < B_{l+1}$ and $B_0 < A_{k+1}$;

  $\vdots$
- $A_{l+4} < B_{l+2-(k-1)}$ and $B_{l+4} < A_3$;
- $A_{l+3} < B_{l+2-k}$ and $B_{l+3} < A_2$.

The following definition will be applicable to (1.5) in the sequel:

Definition 2.7 (Hypothesis (H$''$)). Let $k, l \in \{0, 1, \ldots\}$ with $k < l$. A pair of sequences of positive real numbers, $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$, is said to *satisfy Hypothesis* (*H*$''$) if there exists $i \in \{0, 1, \ldots\}$ such that for all $n \geq 0$,

- $s_A^{(1,1)} = \sup\{A_{4n+5+i} : n = 0, 1, \ldots\}$
  $< i_B^{(1,1)} = \inf\{B_{4n+4+i} : n = 0, 1, \ldots\}$
  and
  $s_B^{(1,2)} = \sup\{B_{4n+5+i} : n = 0, 1, \ldots\}$
  $< i_A^{(1,2)} = \inf\{A_{4n+2+i} : n = 0, 1, \ldots\}$;

- $s_A^{(2,1)} = \sup\{A_{4n+3+i} : n = 0, 1, \ldots\}$
  $< i_B^{(2,1)} = \inf\{B_{4n+2+i} : n = 0, 1, \ldots\}$
  and
  $s_B^{(2,2)} = \sup\{B_{4n+3+i} : n = 0, 1, \ldots\}$
  $< i_A^{(2,2)} = \inf\{A_{4n+i} : n = 0, 1, \ldots\}$;

with the i_A's, i_B's, s_A's, and s_B's all positive real numbers.

Remark 2.8. We note that if, in Definition 2.7, $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are each periodic sequences with period four, then Hypothesis (H$''$) implies that

$$A_1 < B_0, \;\; B_1 < A_2, \;\; A_3 < B_2, \;\; \text{and} \;\; B_3 < A_0.$$

Before leaving this section, we note that the number $k + l + 2$ (which we call "t") is "special" and will play a central role in the results of the sequel. This number was based on an *Ansatz*, which followed from various observations.

First, let us say that it is easy to show that every positive solution of the difference equation

$$x_{n+1} = \frac{1}{x_{n-r}}, \;\; n = 0, 1, \ldots,$$

where $r \in \{0, 1, \ldots\}$ and initial conditions are positive, is periodic with period $2(r+1)$. Secondly, observe the following:

1. We know that with (1.4), the period of either $\{A_n\}_{n=0}^{\infty}$ or $\{B_n\}_{n=0}^{\infty}$ must be a multiple of three in order for every solution to be unbounded. It happens that three is the average of two and four, the respective periods of every positive solution of the equations
$$x_{n+1} = \frac{1}{x_n} \;\; \text{and} \;\; x_{n+1} = \frac{1}{x_{n-1}}.$$
The right-hand sides of these two equations make up the arguments of (1.4).

2. Similarly, we know that with (1.5), the period of either $\{A_n\}_{n=0}^{\infty}$ or $\{B_n\}_{n=0}^{\infty}$ must be a multiple of four in order for every solution to be unbounded. It happens that four is the average of two and six, the respective periods of every positive solution of the equations
$$x_{n+1} = \frac{1}{x_n} \;\; \text{and} \;\; x_{n+1} = \frac{1}{x_{n-2}}.$$
The right-hand sides of these two equations make up the arguments of (1.5).

3. Finally, we know that with (1.6), the period of either $\{A_n\}_{n=0}^{\infty}$ or $\{B_n\}_{n=0}^{\infty}$ must be a multiple of five in order for every solution to be unbounded. It happens that five is the average of four and six, the respective periods of every positive solution of the equations
$$x_{n+1} = \frac{1}{x_{n-1}} \quad \text{and} \quad x_{n+1} = \frac{1}{x_{n-2}}.$$
The right-hand sides of these two equations make up the arguments of (1.6).

Now, we note that the number $t = k + l + 2$ is the average of the respective periods of every positive solution of the equations
$$x_{n+1} = \frac{1}{x_{n-k}} \quad \text{and} \quad x_{n+1} = \frac{1}{x_{n-l}},$$
where
$$\frac{2(k+1) + 2(l+1)}{2} = k + l + 2.$$
Of course, the right-hand sides of these two equations make up the arguments of (1.1).

3 Sufficient Conditions for unbounded Solutions

In this section, we present the main result of this paper. We show that if the (not necessarily periodic) sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ of positive real numbers satisfy Hypothesis (H), then every positive solution is unbounded.

Theorem 3.1 (Unbounded Solutions)**.** *Let $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ be a pair of sequences of positive real numbers which satisfies Hypothesis* (*H*)*. Then every positive solution of* (1.1) *is unbounded.*

Proof. Let $\{x_n\}_{n=-l}^{\infty}$ be a positive solution of (1.1), let $i \in \{0, 1, \ldots, k + l + 1\}$, and suppose that the pair of parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfies Hypothesis (H) with i_A, i_B, s_A, s_B as defined in Definition 2.4. Set $t = k + l + 2$ for the sake of convenience. Then we have the following:

$$\begin{aligned}
x_{tn+(k+l+4)+i} &= \max\left\{\frac{A_{tn+(k+l+3)+i}}{x_{tn+(l+3)+i}}, \frac{B_{tn+(k+l+3)+i}}{x_{tn+(k+3)+i}}\right\} \\
&= \max\left\{\frac{A_{tn+(k+l+3)+i}}{\max\left\{\frac{A_{tn+(l+2)+i}}{x_{tn+(l-k+2)+i}}, \frac{B_{tn+(l+2)+i}}{x_{tn+2+i}}\right\}}, \frac{B_{tn+(k+l+3)+i}}{\max\left\{\frac{A_{tn+(k+2)+i}}{x_{tn+2+i}}, \frac{B_{tn+(k+2)+i}}{x_{tn+(k-l+2)+i}}\right\}}\right\} \\
&= \max\left\{\min\left\{\frac{A_{tn+(k+l+3)+i}x_{tn+(l-k+2)+i}}{A_{tn+(l+2)+i}}, \frac{A_{tn+(k+l+3)+i}x_{tn+2+i}}{B_{tn+(l+2)+i}}\right\},\right.
\end{aligned}$$

$$
\begin{aligned}
&\min\left\{\frac{B_{tn+(k+l+3)+i}x_{tn+2+i}}{A_{tn+(k+2)+i}}, \frac{B_{tn+(k+l+3)+i}x_{tn+(k-l+2)+i}}{B_{tn+(k+2)+i}}\right\}\Bigg\} \\
\leq\ & \max\left\{\frac{A_{tn+(k+l+3)+i}x_{tn+2+i}}{B_{tn+(l+2)+i}}, \frac{B_{tn+(k+l+3)+i}x_{tn+2+i}}{A_{tn+(k+2)+i}}\right\} \\
=\ & \max\left\{\frac{A_{tn+(k+l+3)+i}}{B_{tn+(l+2)+i}}, \frac{B_{tn+(k+l+3)+i}}{A_{tn+(k+2)+i}}\right\}\cdot x_{tn+2+i} \\
\leq\ & \max\left\{\frac{s_A}{i_B}, \frac{s_B}{i_A}\right\}\cdot x_{tn+2+i}.
\end{aligned}
$$

Let

$$\alpha = \max\left\{\frac{s_A}{i_B}, \frac{s_B}{i_A}\right\}.$$

Then clearly $\alpha < 1$ by Hypothesis (H). Since we have just shown that for all $n \geq 0$,

$$x_{tn+(k+l+4)+i} = x_{tn+(k+l+2)+2+i} = x_{t(n+1)+2+i} \leq \alpha \cdot x_{tn+2+i}$$

and hence

$$x_{tn+2+i} \leq \alpha^n \cdot x_{2+i},$$

it follows that

$$x_{tn+2+i} \downarrow 0.$$

As a consequence, we have

$$
\begin{aligned}
x_{tn+l+3+i} &= \max\left\{\frac{A_{tn+(l+2)+i}}{x_{tn+(l-k+2)+i}}, \frac{B_{tn+(l+2)+i}}{x_{tn+2+i}}\right\} \geq \frac{B_{tn+(l+2)+i}}{x_{tn+2+i}} \\
&\geq \frac{i_B}{x_{tn+2+i}} \longrightarrow \infty \quad \text{as } n \to \infty.
\end{aligned}
$$

Therefore, $\{x_n\}_{n=-l}^{\infty}$ is unbounded, and we are done. □

4 Extended Periodicity of Positive Solutions of (1.1)

In this section, we first assume that in (1.1) the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfy Hypothesis (H′) and the delays k and l satisfy the condition $l+2-k \leq k+3$, with $k, l \in \{0, 1, \ldots\}$ and $k < l$. We then show that every solution is extended periodic with period $k+l+2$.

We next take a specific look at (1.5), whose parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfy Hypothesis (H″) and whose delays $k = 0$ and $l = 2$ satisfy the condition $l+2-k > k+3$. We then show that every positive solution of (1.5) is extended periodic with period 4.

Note that we refer to the condition $l+2-k \leq k+3$ as "Case 1" and the condition $l+2-k > k+3$ as "Case 2".

4.1 Case 1

Set $t = k + l + 3$ for convenience. Suppose that in (1.1) the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfy Hypothesis (H′) and the delays k and l satisfy the condition $l+2-k \le k+3$, with $k, l \in \{0, 1, \ldots\}$ and $k < l$. Let $\{x_n\}_{n=-l}^{\infty}$ be a positive solution of (1.1). Then, repeatedly applying the same technique as that used in the proof of Theorem 3.1, we obtain the $k+1$ limits

$$\lim_{n\to\infty} x_{tn+j+i} = 0,$$

for $j = -(k-2), -(k-3), \ldots, -2, -1, 0, 1, 2$ and for some $i \in \{0, 1, \ldots, k+l+1\}$.

We next compute the limits of x_{tn+j+i} as $n \to \infty$ for $j = 3, 4, \ldots, k+3, k+4, \ldots, l+2, l+3$ (note that, because $k < l$, we have $k+3 \le l+2$):

$$\begin{aligned} x_{tn+3+i} &= \max\left\{\frac{A_{tn+2+i}}{x_{tn-(k-2)+i}}, \frac{B_{tn+2+i}}{x_{tn-(l-2)+i}}\right\} \ge \frac{A_{tn+2+i}}{x_{tn-(k-2)+i}} \\ &\ge \frac{i_A^{(k+1,2)}}{x_{tn-(k-2)+i}} \longrightarrow \infty \quad \text{as } n \to \infty; \end{aligned}$$

$$\begin{aligned} x_{tn+4+i} &= \max\left\{\frac{A_{tn+3+i}}{x_{tn-(k-3)+i}}, \frac{B_{tn+3+i}}{x_{tn-(l-3)+i}}\right\} \ge \frac{A_{tn+3+i}}{x_{tn-(k-3)+i}} \\ &\ge \frac{i_A^{(k,2)}}{x_{tn-(k-3)+i}} \longrightarrow \infty \quad \text{as } n \to \infty; \end{aligned}$$

$$\vdots$$

$$\begin{aligned} x_{tn+(k+3)+i} &= \max\left\{\frac{A_{tn+(k+2)+i}}{x_{tn+2+i}}, \frac{B_{tn+(k+2)+i}}{x_{tn-(k-l+2)+i}}\right\} \ge \frac{A_{tn+(k+2)+i}}{x_{tn+2+i}} \\ &\ge \frac{i_A^{(1,2)}}{x_{tn+2+i}} \longrightarrow \infty \quad \text{as } n \to \infty; \end{aligned}$$

$$\begin{aligned} x_{tn+(k+4)+i} &= \max\left\{\frac{A_{tn+(k+3)+i}}{x_{tn+3+i}}, \frac{B_{tn+(k+3)+i}}{x_{tn-(k-l+3)+i}}\right\} \ge \frac{B_{tn+(k+3)+i}}{x_{tn+(k-l+3)+i}} \\ &\ge \frac{i_B^{((l-k-1)+1,1)}}{x_{tn+(k-l+3)+i}} \longrightarrow \infty \quad \text{as } n \to \infty, \end{aligned}$$

[note that

(i) $1 \le l-k-1 \le k$ since $k+1 \le l$ and $l-k+2 \le k+3$ (cf. Definition 2.5)

(ii) $-(k-2) \le k-l+3 \le 2$ since $l-k+2 \le k+3 \le l+2$];

$$\vdots$$

$$\begin{aligned} x_{tn+(l+2)+i} &= \max\left\{\frac{A_{tn+(l+1)+i}}{x_{tn+(l-k+1)+i}}, \frac{B_{tn+(l+1)+i}}{x_{tn+1+i}}\right\} \geq \frac{B_{tn+(l+1)+i}}{x_{tn+1+i}} \\ &\geq \frac{i_B^{(2,1)}}{x_{tn+1+i}} \longrightarrow \infty \text{ as } n \to \infty; \end{aligned}$$

$$\begin{aligned} x_{tn+(l+3)+i} &= \max\left\{\frac{A_{tn+(l+2)+i}}{x_{tn+(l-k+2)+i}}, \frac{B_{tn+(l+2)+i}}{x_{tn+2+i}}\right\} \geq \frac{B_{tn+(l+2)+i}}{x_{tn+2+i}} \\ &\geq \frac{i_B^{(1,1)}}{x_{tn+2+i}} \longrightarrow \infty \text{ as } n \to \infty. \end{aligned}$$

Therefore, we have extended periodicity with period $t = k + l + 2$ where, for some $i \in \{0, 1, \dots, k + l + 1\}$,

$$\lim_{n\to\infty} x_{tn+j+i} = 0,$$

for $j = -(k - 2), -(k - 3), \dots, 1, 2$, and

$$\lim_{n\to\infty} x_{tn+j+i} = \infty,$$

for $j = 3, 4, \dots, k + 3, k + 4, \dots, l + 2, l + 3$.

4.2 Case 2: An Example

We consider (1.5). Then its delays $k = 0$ and $l = 2$ satisfy the condition $l+2-k > k+3$ with $k < l$. Suppose that its parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfy Hypothesis (H$''$) with $i_A^{(1,2)}, i_A^{(2,2)}, i_B^{(1,1)}, i_B^{(2,1)}$ and $s_A^{(1,1)}, S_A^{(2,1)} S_B^{(1,2)}, S_B^{(2,2)}$ as defined in Definition 2.4. We obtain an extended solution whose form differs from the case when $l + 2 - k \leq k + 3$.

We first let $\{x_n\}_{n=-l}^{\infty}$ be a positive solution of (1.5) and let $i \in \{0, 1, 2, 3\}$. Then we have the following for all $n \geq 0$:

$$\begin{aligned} x_{4n+6+i} &= \max\left\{\frac{A_{4n+5+i}}{x_{4n+5+i}}, \frac{B_{4n+5+i}}{x_{4n+3+i}}\right\} \\ &= \max\left\{\frac{A_{4n+5+i}}{\max\left\{\frac{A_{4n+4+i}}{x_{tn4+i}}, \frac{B_{4n+4+i}}{x_{4n+2+i}}\right\}}, \frac{B_{4n+5+i}}{\max\left\{\frac{A_{4n+2+i}}{x_{4n+2+i}}, \frac{B_{4n+2+i}}{x_{4n+i}}\right\}}\right\} \\ &= \max\left\{\min\left\{\frac{A_{4n+5+i}x_{4n+4+i}}{A_{4n+4+i}}, \frac{A_{4n+5+i}x_{4n+2+i}}{B_{4n+4+i}}\right\}\right., \end{aligned}$$

$$\begin{aligned}
&\min\left\{\frac{B_{4n+5+i}x_{4n+2+i}}{A_{4n+2+i}}, \frac{B_{4n+5+i}x_{4n+i}}{B_{4n+2+i}}\right\}\Bigg\} \\
\leq\ &\max\left\{\frac{A_{4n+5+i}x_{4n+2+i}}{B_{4n+4+i}}, \frac{B_{4n+5+i}x_{4n+2+i}}{A_{4n+2+i}}\right\} \\
=\ &\max\left\{\frac{A_{4n+5+i}}{B_{4n+4+i}}, \frac{B_{4n+5+i}}{A_{4n+2+i}}\right\}\cdot x_{4n+2+i} \\
\leq\ &\max\left\{\frac{s_A^{(1,1)}}{i_B^{(1,1)}}, \frac{s_B^{(1,2)}}{i_A^{(1,2)}}\right\}\cdot x_{4n+2+i}.
\end{aligned}$$

Let

$$\alpha = \max\left\{\frac{s_A^{(1,1)}}{i_B^{(1,1)}}, \frac{s_B^{(1,2)}}{i_A^{(1,2)}}\right\}.$$

Then clearly $\alpha < 1$ by Hypothesis (H$''$). Since we have just shown that for all $n \geq 0$,

$$x_{4n+6+i} = x_{4n+4+2+i} = x_{4(n+1)+2+i} \leq \alpha \cdot x_{4n+2+i}$$

and hence

$$x_{4n+2+i} \leq \alpha^n \cdot x_{2+i},$$

it follows that

$$x_{4n+2+i} \downarrow 0.$$

Similarly, it can be shown that

$$x_{4n+i} \downarrow 0.$$

We next compute the limits, as $n \to \infty$, of the terms x_{4n+1+i} and x_{4n+3+i}:

$$\begin{aligned}
x_{4n+1+i} &= \max\left\{\frac{A_{4n+i}}{x_{4n+i}}, \frac{B_{4n+i}}{x_{4n-2+i}}\right\} \geq \frac{A_{4n+i}}{x_{4n+i}} \\
&\geq \frac{i_A^{(2,2)}}{x_{4n+i}} \longrightarrow \infty \quad \text{as } n \to \infty;
\end{aligned}$$

$$\begin{aligned}
x_{4n+3+i} &= \max\left\{\frac{A_{4n+2+i}}{x_{4n+2+i}}, \frac{B_{4n+2+i}}{x_{4n+i}}\right\} \geq \frac{A_{4n+2+i}}{x_{4n+2+i}} \\
&\geq \frac{i_A^{(1,2)}}{x_{4n+2+i}} \longrightarrow \infty \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, we have extended periodicity with period four, where, for some $i \in \{0, 1, 2, 3\}$,

$$\lim_{n\to\infty} x_{4n+i} = 0,$$

$$\lim_{n\to\infty} x_{4n+1+i} = \infty,$$

$$\lim_{n\to\infty} x_{4n+2+i} = 0,$$

$$\lim_{n\to\infty} x_{4n+3+i} = \infty.$$

5 Unboundedness in the Case when the Parameters of (1.1) are Periodic

We are now ready to consider the case when the parameters $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ of (1.1) are positive *periodic* sequences with minimal periods p and q, respectively. As in Section 4, we first look at the case when the delays k and l satisfy the condition $l+2-k \leq k+3$, with $k, l \in \{0, 1, \ldots\}$ and $k < l$, and give sufficient conditions for every positive solution to be unbounded. We then give specific sufficient conditions for every positive solution of (1.5), whose delays $k=0$ and $l=2$ satisfy the condition $l+2-k > k+3$, to be unbounded.

Note that we refer to the condition $l+2-k \leq k+3$ as "Case 1" and the condition $l+2-k > k+3$ as "Case 2."

5.1 Case 1

We again set $t = k+l+2$ for the sake of convenience.

Corollary 5.1 ($\{B_n\}_{n=0}^{\infty}$ Periodic with Period a Multiple of $k+l+2$)**.** *Consider* (1.1). *Let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with period $p \in \{1, 2, \ldots\}$, and let $\{B_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with period $(k+l+2)r$, for some $r \in \{1, 2, \ldots\}$. Suppose that for some $i \in \{0, 1, \ldots, k+l+1\}$, we have*

$$\begin{aligned} &\max\left\{B_{tj+(l+3)+i}, B_{tj+(l+4)+i}, \ldots, B_{tj+(k+l+3)+i} \ : \ j = 0, 1, \ldots, r-1\right\} \\ &\quad < \ A_0, A_1, \ldots, A_{p-1} \\ &\quad < \ \min\left\{B_{tj+(l+2-k)+i}, B_{tj+(l+3-k)+i}, \ldots, B_{tj+(l+2)+i} \ : \ j = 0, 1, \ldots, r-1\right\}. \end{aligned}$$

Then every positive solution of (1.1) *is unbounded. Furthermore, every positive solution is extended periodic with period $k+l+2$.*

Proof. The pair of sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfies Hypothesis (H′) (Definition 2.5). □

Corollary 5.2 ($\{A_n\}_{n=0}^{\infty}$ Periodic with Period a Multiple of $k+l+2$)**.** *Consider* (1.1). *Let $\{B_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with period $q \in \{1, 2, \ldots\}$, and let $\{A_n\}_{n=0}^{\infty}$ be a periodic sequence of positive real numbers with period $(k+l+2)s$, for some $s \in \{1, 2, \ldots\}$. Suppose that for some $i \in \{0, 1, \ldots, k+l+1\}$, we have*

$$\begin{aligned} &\max\left\{A_{tj+(l+3)+i}, A_{tj+(l+4)+i}, \ldots, A_{tj+(k+l+3)+i} \ : \ j = 0, 1, \ldots, s-1\right\} \\ &\quad < \ B_0, B_1, \ldots, B_{q-1} \\ &\quad < \ \min\left\{A_{tj+2+i}, A_{tj+3+i}, \ldots, A_{tj+(k+2)+i} \ : \ j = 0, 1, \ldots, s-1\right\}. \end{aligned}$$

Then every positive solution of (1.1) *is unbounded. Furthermore, every positive solution is extended periodic with period $k+l+2$.*

Proof. The pair of sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfies Hypothesis (H′) (Definition 2.5). □

5.2 Case 2: An Example

Corollary 5.3 ($\{B_n\}_{n=0}^{\infty}$ Periodic with Period a Multiple of Four)**.** *Consider* (1.1). *Let* $\{A_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $p \in \{1, 2, \ldots\}$, *and let* $\{B_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $4r$, *for some* $r \in \{1, 2, \ldots\}$. *Suppose that for some* $i \in \{0, 1, 2, 3\}$, *we have*

$$\max\{B_{tj+3+i}, B_{tj+5+i} \ : \ j = 0, 1, \ldots, r-1\}$$

$$< A_0, A_1, \ldots, A_{p-1}$$

$$< \min\{B_{4j+i}, B_{tj+2+i} \ : \ j = 0, 1, \ldots, r-1\}.$$

Then every positive solution of (1.5) *is unbounded. Furthermore, every positive solution is extended periodic with period four.*

Proof. The pair of sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfies Hypothesis (H$''$) (Definition 2.7). □

Corollary 5.4 ($\{A_n\}_{n=0}^{\infty}$ Periodic with Period a Multiple of Four)**.** *Consider* (1.1). *Let* $\{B_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $q \in \{1, 2, \ldots\}$, *and let* $\{A_n\}_{n=0}^{\infty}$ *be a periodic sequence of positive real numbers with period* $4s$, *for some* $s \in \{1, 2, \ldots\}$. *Suppose that for some* $i \in \{0, 1, 2, 3\}$, *we have*

$$\max\{A_{4j+3+i}, A_{4j+5+i} \ : \ j = 0, 1, \ldots, s-1\}$$

$$< B_0, B_1, \ldots, B_{q-1}$$

$$< \min\{A_{4j+i}, A_{4j+2+i} \ : \ j = 0, 1, \ldots, s-1\}.$$

Then every positive solution of (1.5) *is unbounded. Furthermore, every positive solution is extended periodic with period four.*

Proof. The pair of sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ satisfies Hypothesis (H$''$) (Definition 2.7). □

6 Future Goals

We conclude with an open problem and with suggestions for two potential biological applications that could be considered novel for max-type difference equations. First, we state our open problem.

Open Problem 6.1. Consider Bidewll and Franke's equation (1.7), where $\{A_n^{(i)}\}_{n=0}^{\infty}$ are *nonnegative*. Find necessary and sufficient conditions on the periods and periodic parameters such that every solution is bounded.

Secondly, we remark that max-type difference equations in essence belong to a larger group of difference equations, namely, piecewise-defined difference equations (see, e.g., [1, 4, 12, 14, 18]). Some attractive features of piecewise-defined difference equations which make them especially suited to serve as models of biological processes and systems include their "decision-making" properties with the incorporation of thresholds and their sometimes eventually periodic or unbounded behavior. Max-type difference equations such as the one in Open Problem 6.1 have the additional desirable feature of consisting of an arbitrary number of variable parameters.

Piecewise-defined difference equations, as well as differential equations with maxima (cf. [3]), the counterparts to max-type difference equations, have been used as models for *neural networks* (see, e.g., [8, 9]).

Less frequently, piecewise-defined difference equations have been applied to the area of *morphogenesis* (see, e.g., [11, 23]), an area that investigates the embryological development and differentiation of organisms or the origins of their growth and shape from the embryonic to the final adult stage. Morphogenesis involves decision-making when it comes to the proliferation of cells, the death of cells, and the differentiation of cells. The study of morphogenesis includes such things as analyzing the occurrence of repetitive patterns of development, for example, the stripes of a zebra or the appearance of five appendages and then five fingers and five toes in the human. There is also abnormal morphogenesis, as seen in the development of cancer in which there is an excessive, almost unbounded proliferation of cells.

We therefore propose that in the future one might consider max-type equations such as the one in Open Problem 6.1 or modifications thereof as candidates for the modeling of the following:

(1) Huge networks of neurons both during normal functioning when transmission of electrochemical signals is rhythmic and oscillatory, and during abnormal functioning as with, for example, epileptic seizures when transmission of electrochemical signals can be characterized as hypersynchronous and hyperexcited;

(2) Morphogenetic processes both during normal growth in which there is, at least externally, symmetry and periodicity, and during abnormal growth as with, for example, cancer in which there is the excessive, almost unbounded proliferation of cells.

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