

The Delta-nabla Calculus of Variations for Composition Functionals on Time Scales

Monika Dryl and Delfim F. M. Torres

Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
3810-193 Aveiro, Portugal
monikadryl@ua.pt, delfim@ua.pt

Abstract

We develop the calculus of variations on time scales for a functional that is the composition of a certain scalar function with the delta and nabla integrals of a vector valued field. Euler–Lagrange equations, transversality conditions, and necessary optimality conditions for isoperimetric problems, on an arbitrary time scale, are proved. Interesting corollaries and examples are presented.

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1 Introduction

We study a general problem of the calculus of variations on an arbitrary time scale \mathbb{T} . More precisely, we consider the problem of extremizing (i.e., minimizing or maximizing) a delta-nabla integral functional

$$\mathcal{L}(x) = H \left(\int_a^b f_1(t, x^\sigma(t), x^\Delta(t)) \Delta t, \dots, \int_a^b f_k(t, x^\sigma(t), x^\Delta(t)) \Delta t, \right. \\ \left. \int_a^b f_{k+1}(t, x^\rho(t), x^\nabla(t)) \nabla t, \dots, \int_a^b f_{k+n}(t, x^\rho(t), x^\nabla(t)) \nabla t \right)$$

possibly subject to boundary conditions and/or isoperimetric constraints. For the interest in studying such type of variational problems in economics, we refer the reader

to [10] and references therein. For a review on general approaches to the calculus of variations on time scales, which allow to obtain both delta and nabla variational calculus as particular cases, see [5, 9, 12]. Throughout the text we assume the reader to be familiar with the basic definitions and results of time scales [3, 4, 7, 8].

The article is organized as follows. In Section 2 we collect some necessary definitions and theorems of the nabla and delta calculus on time scales. The main results are presented in Section 3. We begin by proving general Euler–Lagrange equations (Theorem 3.2). Next we consider the situations when initial or terminal boundary conditions are not specified, obtaining corresponding transversality conditions (Theorems 3.4 and 3.5). The results are applied to quotient variational problems in Corollary 3.6. Finally, we prove necessary optimality conditions for general isoperimetric problems given by the composition of delta-nabla integrals (Theorem 3.9). We end with Section 4, illustrating the new results of the paper with several examples.

2 Preliminaries

In this section we review the main results necessary in the sequel. For basic definitions, notations and results of the theory of time scales, we refer the reader to the books [3, 4].

The following two lemmas are the extension of the Dubois–Reymond fundamental lemma of the calculus of variations [13] to the nabla (Lemma 2.1) and delta (Lemma 2.2) time scale calculus. We remark that all intervals in this paper are time scale intervals.

Lemma 2.1 (See [11]). *Let $f \in C_{ld}([a, b], \mathbb{R})$. If*

$$\int_a^b f(t)\eta^\nabla(t)\nabla t = 0 \text{ for all } \eta \in C_{ld}^1([a, b], \mathbb{R}) \text{ with } \eta(a) = \eta(b) = 0,$$

then $f(t) = c$, for some constant c , for all $t \in [a, b]_\kappa$.

Lemma 2.2 (See [2]). *Let $f \in C_{rd}([a, b], \mathbb{R})$. If*

$$\int_a^b f(t)\eta^\Delta(t)\Delta t = 0 \text{ for all } \eta \in C_{rd}^1([a, b], \mathbb{R}) \text{ with } \eta(a) = \eta(b) = 0,$$

then $f(t) = c$, for some constant c , for all $t \in [a, b]^\kappa$.

Under some assumptions, it is possible to relate the delta and nabla derivatives (Theorem 2.3) as well as the delta and nabla integrals (Theorem 2.4).

Theorem 2.3 (See [1]). *If $f : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable on \mathbb{T}^κ and f^Δ is continuous on \mathbb{T}^κ , then f is nabla differentiable on \mathbb{T}_κ and*

$$f^\nabla(t) = (f^\Delta)^\rho(t) \text{ for all } t \in \mathbb{T}_\kappa. \quad (2.1)$$

If $f : \mathbb{T} \rightarrow \mathbb{R}$ is nabla differentiable on \mathbb{T}_κ and f^∇ is continuous on \mathbb{T}_κ , then f is delta differentiable on \mathbb{T}^κ and

$$f^\Delta(t) = (f^\nabla)^\sigma(t) \text{ for all } t \in \mathbb{T}^\kappa. \quad (2.2)$$

Theorem 2.4 (See [6]). *Let $a, b \in \mathbb{T}$ with $a < b$. If function $f : \mathbb{T} \rightarrow \mathbb{R}$ is continuous, then*

$$\int_a^b f(t) \Delta t = \int_a^b f^\rho(t) \nabla t, \quad (2.3)$$

$$\int_a^b f(t) \nabla t = \int_a^b f^\sigma(t) \Delta t. \quad (2.4)$$

3 Main Results

By $C_{k,n}^1([a, b], \mathbb{R})$ we denote the class of functions $x : [a, b] \rightarrow \mathbb{R}$ such that: if $n = 0$, then x^Δ is continuous on $[a, b]^\kappa$; if $k = 0$, then x^∇ is continuous on $[a, b]_\kappa$; if $k \neq 0$ and $n \neq 0$, then x^Δ is continuous on $[a, b]_{\kappa}^\kappa$ and x^∇ is continuous on $[a, b]_{\kappa}^\kappa$, where $[a, b]_{\kappa}^\kappa := [a, b]^\kappa \cap [a, b]_\kappa$. We consider the following problem of calculus of variations:

$$\mathcal{L}(x) = H \left(\int_a^b f_1(t, x^\sigma(t), x^\Delta(t)) \Delta t, \dots, \int_a^b f_k(t, x^\sigma(t), x^\Delta(t)) \Delta t, \int_a^b f_{k+1}(t, x^\rho(t), x^\nabla(t)) \nabla t, \dots, \int_a^b f_{k+n}(t, x^\rho(t), x^\nabla(t)) \nabla t \right) \longrightarrow \text{extr}, \quad (3.1)$$

$$(x(a) = x_a), \quad (x(b) = x_b), \quad (3.2)$$

where ‘‘extr’’ means ‘‘minimize’’ or ‘‘maximize’’. The parentheses in (3.2), around the end-point conditions, means that those conditions may or may not occur (it is possible that both $x(a)$ and $x(b)$ are free). A function $x \in C_{k,n}^1$ is said to be admissible provided it satisfies the boundary conditions (3.2) (if any is given). For $k = 0$ problem (3.1) reduces to a nabla problem (no delta integral and delta derivative is present); for $n = 0$ problem (3.1) reduces to a delta problem (no nabla integral and nabla derivative is present). We assume that:

1. the function $H : \mathbb{R}^{n+k} \rightarrow \mathbb{R}$ has continuous partial derivatives with respect to its arguments, which we denote by $H'_i, i = 1, \dots, n + k$;
2. functions $(t, y, v) \rightarrow f_i(t, y, v)$ from $[a, b] \times \mathbb{R}^2$ to $\mathbb{R}, i = 1, \dots, n + k$, have partial continuous derivatives with respect to y and v for all $t \in [a, b]$, which we denote by f_{iy} and f_{iv} ;

3. f_i, f_{iy}, f_{iv} are continuous on $[a, b]^\kappa$, $i = 1, \dots, k$, and continuous on $[a, b]_\kappa$, $i = k + 1, \dots, k + n$, for all $x \in C_{k,n}^1$.

The following norm in $C_{k,n}^1$ is considered:

$$\|x\|_{1,\infty} := \|x^\sigma\|_\infty + \|x^\Delta\|_\infty + \|x^\rho\|_\infty + \|x^\nabla\|_\infty,$$

where $\|x\|_\infty := \sup |x(t)|$.

Definition 3.1. We say that an admissible function \hat{x} is a weak local minimizer (respectively weak local maximizer) to problem (3.1)–(3.2) if there exists $\delta > 0$ such that $\mathcal{L}(\hat{x}) \leq \mathcal{L}(x)$ (respectively $\mathcal{L}(\hat{x}) \geq \mathcal{L}(x)$) for all admissible functions $x \in C_{k,n}^1$ satisfying the inequality $\|x - \hat{x}\|_{1,\infty} < \delta$.

For simplicity, we introduce the operators $[\cdot]^\Delta$ and $[\cdot]^\nabla$ by $[x]^\Delta(t) = (t, x^\sigma(t), x^\Delta(t))$ and $[x]^\nabla(t) = (t, x^\rho(t), x^\nabla(t))$. Along the text, c denotes constants that are generic and may change at each occurrence.

3.1 Euler–Lagrange Equations

Depending on the given boundary conditions, we can distinguish four different problems. The first is problem (P_{ab}) , where the two boundary conditions are specified. To solve this problem we need a type of Euler–Lagrange necessary optimality condition. This is given by Theorem 3.2 below. Next two problems — denoted by (P_a) and (P_b) — occur when $x(a)$ is given and $x(b)$ is free (problem (P_a)) and when $x(a)$ is free and $x(b)$ is specified (problem (P_b)). To solve both of them we need to use an Euler–Lagrange equation and one transversality condition. The last problem — denoted by (P) — occurs when both boundary conditions are not specified. To find a solution for such a problem we need to use an Euler–Lagrange equation and two transversality conditions (one at each time a and b). Transversality conditions are the subject of Section 3.2.

Theorem 3.2 (Euler–Lagrange equations in integral form). *If \hat{x} is a weak local solution to problem (3.1)–(3.2), then the Euler–Lagrange equations¹*

$$\begin{aligned} \sum_{i=1}^k H'_i \cdot \left(f_{iv}[\hat{x}]^\Delta(\rho(t)) - \int_a^{\rho(t)} f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau \right) \\ + \sum_{i=k+1}^{k+n} H'_i \cdot \left(f_{iv}[\hat{x}]^\nabla(t) - \int_a^t f_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau \right) = c, \quad t \in \mathbb{T}_\kappa, \end{aligned} \quad (3.3)$$

¹For brevity, we are omitting the arguments of H'_i , i.e., $H'_i := H'_i(\mathcal{F}_1(\hat{x}), \dots, \mathcal{F}_{k+n}(\hat{x}))$, where $\mathcal{F}_i(\hat{x}) = \int_a^b f_i[\hat{x}]^\Delta(t) \Delta t$, $i = 1, \dots, k$, and $\mathcal{F}_i(\hat{x}) = \int_a^b f_i[\hat{x}]^\nabla(t) \nabla t$, $i = k + 1, \dots, k + n$.

and

$$\begin{aligned} & \sum_{i=1}^k H'_i \cdot \left(f_{iv}[\hat{x}]^\Delta(t) - \int_a^t f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau \right) \\ & + \sum_{i=k+1}^{k+n} H'_i \cdot \left(f_{iv}[\hat{x}]^\nabla(\sigma(t)) - \int_a^{\sigma(t)} f_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau \right) = c, \quad t \in \mathbb{T}^\kappa, \end{aligned} \quad (3.4)$$

hold.

Proof. Suppose that $\mathcal{L}(x)$ has a weak local extremum at \hat{x} . Consider a variation $h \in C_{k,n}^1$ of \hat{x} for which we define the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by $\phi(\varepsilon) = \mathcal{L}(\hat{x} + \varepsilon h)$. A necessary condition for \hat{x} to be an extremizer for $\mathcal{L}(x)$ is given by $\phi'(\varepsilon) = 0$ for $\varepsilon = 0$. Using the chain rule, we obtain that

$$\begin{aligned} 0 = \phi'(0) &= \sum_{i=1}^k H'_i \int_a^b (f_{iy}[\hat{x}]^\Delta(t) h^\sigma(t) + f_{iv}[\hat{x}]^\Delta(t) h^\Delta(t)) \Delta t \\ &+ \sum_{i=k+1}^{k+n} H'_i \int_a^b (f_{iy}[\hat{x}]^\nabla(t) h^\rho(t) + f_{iv}[\hat{x}]^\nabla(t) h^\nabla(t)) \nabla t. \end{aligned}$$

Integration by parts of the first terms of both integrals gives

$$\begin{aligned} \int_a^b f_{iy}[\hat{x}]^\Delta(t) h^\sigma(t) \Delta t &= \int_a^t f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau h(t) \Big|_a^b - \int_a^b \left(\int_a^t f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau \right) h^\Delta(t) \Delta t, \\ \int_a^b f_{iy}[\hat{x}]^\nabla(t) h^\rho(t) \nabla t &= \int_a^t f_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau h(t) \Big|_a^b - \int_a^b \left(\int_a^t f_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau \right) h^\nabla(t) \nabla t. \end{aligned}$$

Thus, the necessary condition $\phi'(0) = 0$ can be written as

$$\begin{aligned} & \sum_{i=1}^k H'_i \left[\int_a^t f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau h(t) \Big|_a^b - \int_a^b \left(\int_a^t f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau \right) h^\Delta(t) \Delta t \right. \\ & \left. + \int_a^b f_{iv}[\hat{x}]^\Delta(t) h^\Delta(t) \Delta t \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=k+1}^{k+n} H'_i \left[\int_a^t f_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau h(t) \Big|_a^b - \int_a^b \left(\int_a^t f_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau \right) h^\nabla(t) \nabla t \right. \\
& \qquad \qquad \qquad \left. + \int_a^b f_{iv}[\hat{x}]^\nabla(t) h^\nabla(t) \nabla t \right] = 0. \quad (3.5)
\end{aligned}$$

In particular, condition (3.5) holds for all variations that are zero at both ends: $h(a) = 0$ and $h(b) = 0$. Then, we obtain:

$$\begin{aligned}
& \int_a^b \sum_{i=1}^k H'_i h^\Delta(t) \left(f_{iv}[\hat{x}]^\Delta(t) - \int_a^t f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau \right) \Delta t \\
& \qquad \qquad \qquad + \int_a^b \sum_{i=k+1}^{k+n} H'_i h^\nabla(t) \left(f_{iv}[\hat{x}]^\nabla(t) - \int_a^t f_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau \right) \nabla t = 0.
\end{aligned}$$

Introducing ξ and χ by

$$\xi(t) := \sum_{i=1}^k H'_i \left(f_{iv}[\hat{x}]^\Delta(t) - \int_a^t f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau \right) \quad (3.6)$$

and

$$\chi(t) := \sum_{i=k+1}^{k+n} H'_i \left(f_{iv}[\hat{x}]^\nabla(t) - \int_a^t f_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau \right), \quad (3.7)$$

we then obtain the following relation:

$$\int_a^b h^\Delta(t) \xi(t) \Delta t + \int_a^b h^\nabla(t) \chi(t) \nabla t = 0. \quad (3.8)$$

We consider two cases. (i) Firstly, we change the first integral of (3.8) and we obtain two nabla-integrals and, subsequently, the equation (3.3). (ii) In the second case, we change the second integral of (3.8) to obtain two delta-integrals, which leads us to (3.4).

(i) Using relation (2.3) of Theorem 2.4, we obtain:

$$\int_a^b (h^\Delta(t))^\rho \xi^\rho(t) \nabla t + \int_a^b h^\nabla(t) \chi(t) \nabla t = 0.$$

Using (2.1) of Theorem 2.3 we have

$$\int_a^b h^\nabla(t) (\xi^\rho(t) + \chi(t)) \nabla t = 0.$$

By the Dubois–Reymond Lemma 2.1

$$\xi^\rho(t) + \chi(t) = \text{const} \quad (3.9)$$

and we obtain (3.3).

(ii) From (3.8), and using relation (2.4) of Theorem 2.4,

$$\int_a^b h^\Delta(t)\xi(t)\Delta t + \int_a^b (h^\nabla(t))^\sigma \chi^\sigma(t)\Delta t = 0.$$

Using (2.2) of Theorem 2.3, we get:

$$\int_a^b h^\Delta(t)(\xi(t) + \chi^\sigma(t))\Delta t = 0.$$

From the Dubois–Reymond Lemma 2.2, it follows that $\xi(t) + \chi^\sigma(t) = \text{const}$. Hence, we obtain the Euler–Lagrange equation (3.4). \square

A time scale \mathbb{T} is said to be regular if the following two conditions are satisfied simultaneously for all $t \in \mathbb{T}$: $\sigma(\rho(t)) = t$ and $\rho(\sigma(t)) = t$. For regular time scales, the Euler–Lagrange equations (3.3) and (3.4) coincide; on a general time scale, they are different. Such a difference is illustrated in Example 3.3.

Example 3.3. Let us consider the irregular time scale $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty} [2k, 2k + 1]$. We show that for this time scale there is a difference between the Euler–Lagrange equations (3.3) and (3.4). The forward and backward jump operators are given by

$$\sigma(t) = \begin{cases} t, & t \in \bigcup_{k=0}^{\infty} [2k, 2k + 1), \\ t + 1, & t \in \bigcup_{k=0}^{\infty} \{2k + 1\}, \end{cases} \quad \rho(t) = \begin{cases} t, & t \in \bigcup_{k=0}^{\infty} (2k, 2k + 1], \\ t - 1, & t \in \bigcup_{k=1}^{\infty} \{2k\}, \\ 0, & t = 0. \end{cases}$$

For $t = 0$ and $t \in \bigcup_{k=0}^{\infty} (2k, 2k + 1)$, equations (3.3) and (3.4) coincide. We can distinguish between them for $t \in \bigcup_{k=0}^{\infty} \{2k + 1\}$ and $t \in \bigcup_{k=1}^{\infty} \{2k\}$. In what follows we use the notations (3.6) and (3.7). If $t \in \bigcup_{k=0}^{\infty} \{2k + 1\}$, then we obtain from (3.3) and (3.4) the

Euler–Lagrange equations $\xi(t) + \chi(t) = c$ and $\xi(t) + \chi(t+1) = c$, respectively. If $t \in \bigcup_{k=1}^{\infty} \{2k\}$, then the Euler–Lagrange equation (3.3) has the form $\xi(t-1) + \chi(t) = c$ while (3.4) takes the form $\xi(t) + \chi(t) = c$.

3.2 Natural Boundary Conditions

In this section we consider the situation when we want to minimize or maximize the variational functional (3.1), but boundary conditions $x(a)$ and/or $x(b)$ are free.

Theorem 3.4 (Transversality condition at the initial time $t = a$). *Let \mathbb{T} be a time scale for which $\rho(\sigma(a)) = a$. If \hat{x} is a weak local solution to (3.1) with $x(a)$ not specified, then*

$$\sum_{i=1}^k H'_i \cdot f_{iv}[\hat{x}]^\Delta(a) + \sum_{i=k+1}^{k+n} H'_i \cdot \left(f_{iv}[\hat{x}]^\nabla(\sigma(a)) - \int_a^{\sigma(a)} f_{iy}[\hat{x}]^\nabla(t) \nabla t \right) = 0 \quad (3.10)$$

holds together with the Euler–Lagrange equations (3.3) and (3.4).

Proof. From (3.5) and (3.9), we have

$$\sum_{i=1}^k H'_i \int_a^t f_{iy}[\hat{x}]^\Delta(\tau) \Delta \tau h(t) \Big|_a^b + \sum_{i=k+1}^{k+n} H'_i \int_a^t f_{iy}[\hat{x}]^\nabla(\tau) \nabla \tau h(t) \Big|_a^b + \int_a^b h^\nabla(t) \cdot c \nabla t = 0.$$

Therefore,

$$\sum_{i=1}^k H'_i \int_a^t f_{iy}[\hat{x}]^\Delta(\tau) \Delta \tau h(t) \Big|_a^b + \sum_{i=k+1}^{k+n} H'_i \int_a^t f_{iy}[\hat{x}]^\nabla(\tau) \nabla \tau h(t) \Big|_a^b + h(t) \cdot c|_a^b = 0.$$

Next, we deduce that

$$\begin{aligned} h(b) & \left[\sum_{i=1}^k H'_i \int_a^b f_{iy}[\hat{x}]^\Delta(\tau) \Delta \tau + \sum_{i=k+1}^{k+n} H'_i \int_a^b f_{iy}[\hat{x}]^\nabla(\tau) \nabla \tau + c \right] \\ & - h(a) \left[\sum_{i=1}^k H'_i \int_a^a f_{iy}[\hat{x}]^\Delta(\tau) \Delta \tau + \sum_{i=k+1}^{k+n} H'_i \int_a^a f_{iy}[\hat{x}]^\nabla(\tau) \nabla \tau + c \right] = 0, \end{aligned} \quad (3.11)$$

where

$$c = \xi(\rho(t)) + \chi(t). \quad (3.12)$$

The Euler–Lagrange equation (3.3) of Theorem 3.2 (or (3.12)) is given at $t = \sigma(a)$ as

$$\begin{aligned} \sum_{i=1}^k H'_i \left(f_{iv}[\hat{x}]^\Delta(\rho(\sigma(a))) - \int_a^{\rho(\sigma(a))} f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau \right) \\ + \sum_{i=k+1}^{k+n} H'_i \left(f_{iv}[\hat{x}]^\nabla(\sigma(a)) - \int_a^{\sigma(a)} f_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau \right) = c. \end{aligned}$$

We conclude that

$$\sum_{i=1}^k H'_i \cdot f_{iv}[\hat{x}]^\Delta(a) + \sum_{i=k+1}^{k+n} H'_i \cdot \left(f_{iv}[\hat{x}]^\nabla(\sigma(a)) - \int_a^{\sigma(a)} f_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau \right) = c.$$

Restricting the variations h to those such that $h(b) = 0$, it follows from (3.11) that $h(a) \cdot c = 0$. From the arbitrariness of h , we conclude that $c = 0$. Hence, we obtain (3.10). \square

Theorem 3.5 (Transversality condition at the terminal time $t = b$). *Let \mathbb{T} be a time scale for which $\sigma(\rho(b)) = b$. If \hat{x} is a weak local solution to (3.1) with $x(b)$ not specified, then*

$$\sum_{i=1}^k H'_i \left(f_{iv}[\hat{x}]^\Delta(\rho(b)) + \int_{\rho(b)}^b f_{iy}[\hat{x}]^\Delta(t) \Delta t \right) + \sum_{i=k+1}^{k+n} H'_i \cdot f_{iv}[\hat{x}]^\nabla(b) = 0 \quad (3.13)$$

holds together with the Euler–Lagrange equations (3.3) and (3.4).

Proof. The calculations in the proof of Theorem 3.4 give us (3.11). When $h(a) = 0$, the Euler–Lagrange equation (3.4) of Theorem 3.2 has the following form at $t = \rho(b)$:

$$\begin{aligned} \sum_{i=1}^k H'_i \left(f_{iv}[\hat{x}]^\Delta(\rho(b)) - \int_a^{\rho(b)} f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau \right) \\ + \sum_{i=k+1}^{k+n} H'_i \left(f_{iv}[\hat{x}]^\nabla(\sigma(\rho(b))) - \int_a^{\sigma(\rho(b))} f_{iy}[\hat{x}]^\nabla(t) \nabla\tau \right) = c. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{i=1}^k H'_i \left(f_{iv}[\hat{x}]^\Delta(\rho(b)) - \int_a^{\rho(b)} f_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau \right) \\ + \sum_{i=k+1}^{k+n} H'_i \left(f_{iv}[\hat{x}]^\nabla(b) - \int_a^b f_{iy}[\hat{x}]^\nabla(t) \nabla\tau \right) = c. \quad (3.14) \end{aligned}$$

We obtain (3.13) from (3.11) and (3.14). \square

Several new interesting results can be immediately obtained from Theorems 3.2, 3.4 and 3.5. An example of such results is given by Corollary 3.6.

Corollary 3.6. *If \hat{x} is a solution to the problem*

$$\mathcal{L}(x) = \frac{\int_a^b f_1(t, x^\sigma(t), x^\Delta(t)) \Delta t}{\int_a^b f_2(t, x^\rho(t), x^\nabla(t)) \nabla t} \rightarrow \text{extr},$$

$$(x(a) = x_a), \quad (x(b) = x_b),$$

then the Euler–Lagrange equations

$$\frac{1}{\mathcal{F}_2} \left(f_{1v}[\hat{x}]^\Delta(\rho(t)) - \int_a^{\rho(t)} f_{1y}[\hat{x}]^\Delta(\tau) \Delta \tau \right) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2} \left(f_{2v}[\hat{x}]^\nabla(t) - \int_a^t f_{2y}[\hat{x}]^\nabla(\tau) \nabla \tau \right) = c$$

and

$$\frac{1}{\mathcal{F}_2} \left(f_{1v}[\hat{x}]^\Delta(t) - \int_a^t f_{1y}[\hat{x}]^\Delta(\tau) \Delta \tau \right) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2} \left(f_{2v}[\hat{x}]^\nabla(\sigma(t)) - \int_a^{\sigma(t)} f_{2y}[\hat{x}]^\nabla(\tau) \nabla \tau \right) = c$$

hold for all $t \in [a, b]_\kappa^\kappa$, where

$$\mathcal{F}_1 := \int_a^b f_1(t, \hat{x}^\sigma(t), \hat{x}^\Delta(t)) \Delta t \quad \text{and} \quad \mathcal{F}_2 := \int_a^b f_2(t, \hat{x}^\rho(t), \hat{x}^\nabla(t)) \nabla t.$$

Moreover, if $x(a)$ is free and $\rho(\sigma(a)) = a$, then

$$\frac{1}{\mathcal{F}_2} f_{1v}[\hat{x}]^\Delta(a) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2} \left(f_{2v}[\hat{x}]^\nabla(\sigma(a)) - \int_a^{\sigma(a)} f_{2y}[\hat{x}]^\nabla(t) \nabla t \right) = 0;$$

if $x(b)$ is free and $\sigma(\rho(b)) = b$, then

$$\frac{1}{\mathcal{F}_2} \left(f_{1v}[\hat{x}]^\Delta(\rho(b)) + \int_{\rho(b)}^b f_{1y}[\hat{x}]^\Delta(t) \Delta t \right) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2} f_{2v}[\hat{x}]^\nabla(b) = 0.$$

3.3 Isoperimetric Problems

Let us consider the general composition isoperimetric problem on time scales subject to given boundary conditions. The problem consists of minimizing or maximizing

$$\mathcal{L}(x) = H \left(\int_a^b f_1(t, x^\sigma(t), x^\Delta(t)) \Delta t, \dots, \int_a^b f_k(t, x^\sigma(t), x^\Delta(t)) \Delta t, \right. \\ \left. \int_a^b f_{k+1}(t, x^\rho(t), x^\nabla(t)) \nabla t, \dots, \int_a^b f_{k+n}(t, x^\rho(t), x^\nabla(t)) \nabla t \right) \quad (3.15)$$

in the class of functions $x \in C_{k,n}^1$ satisfying the boundary conditions

$$x(a) = x_a, \quad x(b) = x_b, \quad (3.16)$$

and the generalized isoperimetric constraint

$$\mathcal{K}(x) = P \left(\int_a^b g_1(t, x^\sigma(t), x^\Delta(t)) \Delta t, \dots, \int_a^b g_m(t, x^\sigma(t), x^\Delta(t)) \Delta t, \right. \\ \left. \int_a^b g_{m+1}(t, x^\rho(t), x^\nabla(t)) \nabla t, \dots, \int_a^b g_{m+p}(t, x^\rho(t), x^\nabla(t)) \nabla t \right) = d, \quad (3.17)$$

where $x_a, x_b, d \in \mathbb{R}$. We assume that:

1. the functions $H : \mathbb{R}^{n+k} \rightarrow \mathbb{R}$ and $P : \mathbb{R}^{m+p} \rightarrow \mathbb{R}$ have continuous partial derivatives with respect to all their arguments, which we denote by $H'_i, i = 1, \dots, n+k$, and $P'_i, i = 1, \dots, m+p$;
2. functions $(t, y, v) \rightarrow f_i(t, y, v), i = 1, \dots, n+k$, and $(t, y, v) \rightarrow g_j(t, y, v), j = 1, \dots, m+p$, from $[a, b] \times \mathbb{R}^2$ to \mathbb{R} , have partial continuous derivatives with respect to y and v for all $t \in [a, b]$, which we denote by f_{iy}, f_{iv} , and g_{jy}, g_{jv} ;
3. for all $x \in C_{k+m, n+p}^1$, f_i, f_{iy}, f_{iv} and g_j, g_{jy}, g_{jv} are continuous in $t \in [a, b]^\kappa, i = 1, \dots, k, j = 1, \dots, m$, and continuous in $t \in [a, b]_\kappa, i = k+1, \dots, k+n, j = m+1, \dots, m+p$.

Definition 3.7. We say that an admissible function \hat{x} is a weak local minimizer (respectively a weak local maximizer) to the isoperimetric problem (3.15)–(3.17), if there exists a $\delta > 0$ such that $\mathcal{L}(\hat{x}) \leq \mathcal{L}(x)$ (respectively $\mathcal{L}(\hat{x}) \geq \mathcal{L}(x)$) for all admissible functions $x \in C_{k+m, n+p}^1$ satisfying the boundary conditions (3.16), the isoperimetric constraint (3.17), and inequality $\|x - \hat{x}\|_{1, \infty} < \delta$.

Let us define u and w by

$$u(t) := \sum_{i=1}^m P'_i \left(g_{iv}[\hat{x}]^\Delta(t) - \int_a^t g_{iy}[\hat{x}]^\Delta(\tau) \Delta\tau \right) \quad (3.18)$$

and

$$w(t) := \sum_{i=m+1}^{m+p} P'_i \left(g_{iv}[\hat{x}]^\nabla(t) - \int_a^t g_{iy}[\hat{x}]^\nabla(\tau) \nabla\tau \right), \quad (3.19)$$

where we omit, for brevity, the argument of P'_i : $P'_i := P'_i(\mathcal{G}_1(\hat{x}), \dots, \mathcal{G}_{m+p}(\hat{x}))$ with $\mathcal{G}_i(\hat{x}) = \int_a^b g_i(t, \hat{x}^\sigma(t), \hat{x}^\Delta(t)) \Delta t$, $i = 1, \dots, m$, and $\mathcal{G}_i(\hat{x}) = \int_a^b g_i(t, \hat{x}^\rho(t), \hat{x}^\nabla(t)) \nabla t$, $i = m+1, \dots, m+p$.

Definition 3.8. An admissible function \hat{x} is said to be an extremal for \mathcal{K} if $u(t) + w(\sigma(t)) = \text{const}$ and $u(\rho(t)) + w(t) = \text{const}$ for all $t \in [a, b]_\kappa^\kappa$. An extremizer (i.e., a weak local minimizer or a weak local maximizer) to problem (3.15)–(3.17) that is not an extremal for \mathcal{K} is said to be a normal extremizer; otherwise (i.e., if it is an extremal for \mathcal{K}), the extremizer is said to be abnormal.

Theorem 3.9 (Optimality condition to the isoperimetric problem (3.15)–(3.17)). *Let ξ and χ be given as in (3.6) and (3.7), and u and w be given as in (3.18) and (3.19). If \hat{x} is a normal extremizer to the isoperimetric problem (3.15)–(3.17), then there exists a real number λ such that*

1. $\xi^\rho(t) + \chi(t) - \lambda(u^\rho(t) + w(t)) = \text{const}$;
2. $\xi(t) + \chi^\sigma(t) - \lambda(u^\rho(t) + w(t)) = \text{const}$;
3. $\xi^\rho(t) + \chi(t) - \lambda(u(t) + w^\sigma(t)) = \text{const}$;
4. $\xi(t) + \chi^\sigma(t) - \lambda(u(t) + w^\sigma(t)) = \text{const}$;

for all $t \in [a, b]_\kappa^\kappa$.

Proof. We prove the first item of Theorem 3.9. The other items are proved in a similar way. Consider a variation of \hat{x} such that $\bar{x} = \hat{x} + \varepsilon_1 h_1 + \varepsilon_2 h_2$, where $h_i \in C_{k+m, n+p}^1$ and $h_i(a) = h_i(b) = 0$, $i = 1, 2$, and parameters ε_1 and ε_2 are such that $\|\bar{x} - \hat{x}\|_{1, \infty} < \delta$ for some $\delta > 0$. Function h_1 is arbitrary and h_2 will be chosen later. Define

$$\begin{aligned} \bar{\mathcal{K}}(\varepsilon_1, \varepsilon_2) = \mathcal{K}(\bar{x}) = P \left(\int_a^b g_1(t, \bar{x}^\sigma(t), \bar{x}^\Delta(t)) \Delta t, \dots, \int_a^b g_m(t, \bar{x}^\sigma(t), \bar{x}^\Delta(t)) \Delta t, \right. \\ \left. \int_a^b g_{m+1}(t, \bar{x}^\rho(t), \bar{x}^\nabla(t)) \nabla t, \dots, \int_a^b g_{m+p}(t, \bar{x}^\rho(t), \bar{x}^\nabla(t)) \nabla t \right) - d. \end{aligned}$$

A direct calculation gives

$$\begin{aligned} \left. \frac{\partial \bar{\mathcal{K}}}{\partial \varepsilon_2} \right|_{(0,0)} &= \sum_{i=1}^m P'_i \int_a^b (g_{iy}[\hat{x}]^\Delta(t) h_2^\sigma(t) + g_{iv}[\hat{x}]^\Delta(t) h_2^\Delta(t)) \Delta t \\ &\quad + \sum_{i=m+1}^{m+p} P'_i \int_a^b (g_{iy}[\hat{x}]^\nabla(t) h_2^\rho(t) + g_{iv}[\hat{x}]^\nabla(t) h_2^\nabla(t)) \nabla t. \end{aligned}$$

Integration by parts of the first terms of both integrals gives:

$$\begin{aligned} \sum_{i=1}^m P'_i &\left[\int_a^t g_{iy}[\hat{x}]^\Delta(\tau) \Delta \tau h_2(t) \Big|_a^b - \int_a^b \left(\int_a^t g_{iy}[\hat{x}]^\Delta(\tau) \Delta \tau \right) h_2^\Delta(t) \Delta t \right. \\ &\quad \left. + \int_a^b g_{iv}[\hat{x}]^\Delta(t) h_2^\Delta(t) \Delta t \right] \\ &+ \sum_{i=m+1}^{m+p} P'_i \left[\int_a^t g_{iy}[\hat{x}]^\nabla(\tau) \nabla \tau h_2(t) \Big|_a^b - \int_a^b \left(\int_a^t g_{iy}[\hat{x}]^\nabla(\tau) \nabla \tau \right) h_2^\nabla(t) \nabla t \right. \\ &\quad \left. + \int_a^b g_{iv}[\hat{x}]^\nabla(t) h_2^\nabla(t) \nabla t \right]. \end{aligned}$$

Since $h_2(a) = h_2(b) = 0$, we have

$$\begin{aligned} \int_a^b \sum_{i=1}^m P'_i h_2^\Delta(t) &\left(g_{iv}[\hat{x}]^\Delta(t) - \int_a^t g_{iy}[\hat{x}]^\Delta(\tau) \Delta \tau \right) \Delta t \\ &+ \int_a^b \sum_{i=m+1}^{m+p} P'_i h_2^\nabla(t) \left(g_{iv}[\hat{x}]^\nabla(t) - \int_a^t g_{iy}[\hat{x}]^\nabla(\tau) \nabla \tau \right) \nabla t. \end{aligned}$$

Therefore,

$$\left. \frac{\partial \bar{\mathcal{K}}}{\partial \varepsilon_2} \right|_{(0,0)} = \int_a^b h_2^\Delta(t) u(t) \Delta t + \int_a^b h_2^\nabla(t) w(t) \nabla t.$$

Using relation (2.1) of Theorem 2.3, we obtain that

$$\int_a^b (h_2^\Delta)^\rho(t) u^\rho(t) \nabla t + \int_a^b h_2^\nabla(t) w(t) \nabla t = \int_a^b h_2^\nabla(t) (u^\rho(t) + w(t)) \nabla t.$$

By the Dubois–Reymond Lemma 2.1, there exists a function h_2 such that $\left. \frac{\partial \bar{\mathcal{K}}}{\partial \varepsilon_2} \right|_{(0,0)} \neq 0$.

Since $\bar{\mathcal{K}}(0,0) = 0$, there exists a function ε_2 , defined in the neighborhood of zero, such that $\bar{\mathcal{K}}(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = 0$, i.e., we may choose a subset of variations \hat{x} satisfying the isoperimetric constraint. Let us consider the real function

$$\bar{\mathcal{L}}(\varepsilon_1, \varepsilon_2) = \mathcal{L}(\bar{x}) = H \left(\int_a^b f_1(t, \bar{x}^\sigma(t), \bar{x}^\Delta(t)) \Delta t, \dots, \int_a^b f_k(t, \bar{x}^\sigma(t), \bar{x}^\Delta(t)) \Delta t, \right. \\ \left. \int_a^b f_{k+1}(t, \bar{x}^\rho(t), \bar{x}^\nabla(t)) \nabla t, \dots, \int_a^b f_{k+n}(t, \bar{x}^\rho(t), \bar{x}^\nabla(t)) \nabla t \right).$$

The point $(0,0)$ is an extremal of $\bar{\mathcal{L}}$ subject to the constraint $\bar{\mathcal{K}} = 0$ and $\nabla \bar{\mathcal{K}}(0,0) \neq 0$. By the Lagrange multiplier rule, there exists $\lambda \in \mathbb{R}$ such that $\nabla (\bar{\mathcal{L}}(0,0) - \lambda \bar{\mathcal{K}}(0,0)) = 0$. Because $h_1(a) = h_2(b) = 0$, we have

$$\left. \frac{\partial \bar{\mathcal{L}}}{\partial \varepsilon_1} \right|_{(0,0)} = \sum_{i=1}^k H_i' \int_a^b (f_{iy}[\hat{x}]^\Delta(t) h_1^\sigma(t) + f_{iv}[\hat{x}]^\Delta(t) h_1^\Delta(t)) \Delta t \\ + \sum_{i=k+1}^{k+n} H_i' \int_a^b (f_{iy}[\hat{x}]^\nabla(t) h_1^\rho(t) + f_{iv}[\hat{x}]^\nabla(t) h_1^\nabla(t)) \nabla t.$$

Integrating by parts, and using $h_1(a) = h_1(b) = 0$, gives

$$\left. \frac{\partial \bar{\mathcal{L}}}{\partial \varepsilon_1} \right|_{(0,0)} = \int_a^b h_1^\Delta(t) \xi(t) \Delta t + \int_a^b h_1^\nabla(t) \chi(t) \nabla t.$$

Using (2.3) of Theorem 2.4 and (2.1) of Theorem 2.3, we obtain that

$$\left. \frac{\partial \bar{\mathcal{L}}}{\partial \varepsilon_1} \right|_{(0,0)} = \int_a^b (h_1^\Delta)^\rho(t) \xi^\rho(t) \nabla t + \int_a^b h_1^\nabla(t) \chi(t) \nabla t = \int_a^b h_1^\nabla(t) (\xi^\rho(t) + \chi(t)) \nabla t$$

and

$$\left. \frac{\partial \bar{\mathcal{K}}}{\partial \varepsilon_1} \right|_{(0,0)} = \sum_{i=1}^m P_i' \int_a^b (g_{iy}[\hat{x}]^\Delta(t) h_1^\sigma(t) + g_{iv}[\hat{x}]^\Delta(t) h_1^\Delta(t)) \Delta t \\ + \sum_{i=m+1}^{m+p} P_i' \int_a^b (g_{iy}[\hat{x}]^\nabla(t) h_1^\rho(t) + g_{iv}[\hat{x}]^\nabla(t) h_1^\nabla(t)) \nabla t.$$

Integrating by parts, and recalling that $h_1(a) = h_1(b) = 0$,

$$\left. \frac{\partial \bar{\mathcal{K}}}{\partial \varepsilon_1} \right|_{(0,0)} = \int_a^b h_1^\Delta(t) u(t) \Delta t + \int_a^b h_1^\nabla(t) w(t) \nabla t.$$

Using relation (2.3) of Theorem 2.4 and relation (2.1) of Theorem 2.3, we obtain that

$$\left. \frac{\partial \bar{\mathcal{K}}}{\partial \varepsilon_1} \right|_{(0,0)} = \int_a^b (h_1^\Delta)^\rho(t) u^\rho(t) \nabla t + \int_a^b h_1^\nabla(t) w(t) \nabla t = \int_a^b h_1^\nabla(t) (u^\rho(t) + w(t)) \nabla t.$$

Since $\left. \frac{\partial \bar{\mathcal{L}}}{\partial \varepsilon_1} \right|_{(0,0)} - \lambda \left. \frac{\partial \bar{\mathcal{K}}}{\partial \varepsilon_1} \right|_{(0,0)} = 0$, we have

$$\int_a^b h_1^\nabla(t) [\xi^\rho(t) + \chi(t) - \lambda (u^\rho(t) + w(t))] \nabla t = 0$$

for any $h_1 \in C_{k+m, n+p}$. Therefore, by the Dubois–Reymond Lemma 2.1, one has $\xi^\rho(t) + \chi(t) - \lambda (u^\rho(t) + w(t)) = c$, where $c \in \mathbb{R}$. \square

Remark 3.10. One can easily cover both normal and abnormal extremizers with Theorem 3.9, if in the proof we use the abnormal Lagrange multiplier rule [13].

4 Illustrative Examples

We begin with a nonautonomous problem.

Example 4.1. Consider the problem

$$\mathcal{L}(x) = \frac{\int_0^1 tx^\Delta(t) \Delta t}{\int_0^1 (x^\nabla(t))^2 \nabla t} \longrightarrow \min, \quad (4.1)$$

$$x(0) = 0, \quad x(1) = 1.$$

If x is a local minimizer to problem (4.1), then the Euler–Lagrange equations of Corollary 3.6 must hold, i.e.,

$$\frac{1}{\mathcal{F}_2} \rho(t) - 2 \frac{\mathcal{F}_1}{\mathcal{F}_2^2} x^\nabla(t) = c \quad \text{and} \quad \frac{1}{\mathcal{F}_2} t - 2 \frac{\mathcal{F}_1}{\mathcal{F}_2^2} x^\nabla(\sigma(t)) = c,$$

where $\mathcal{F}_1 := \mathcal{F}_1(x) = \int_0^1 tx^\Delta(t)\Delta t$ and $\mathcal{F}_2 := \mathcal{F}_2(x) = \int_0^1 (x^\nabla(t))^2\nabla t$. Let us consider the second equation. Using (2.2) of Theorem 2.3, it can be written as

$$\frac{1}{\mathcal{F}_2}t - 2\frac{\mathcal{F}_1}{\mathcal{F}_2^2}x^\Delta(t) = c. \quad (4.2)$$

Solving equation (4.2) and using the boundary conditions $x(0) = 0$ and $x(1) = 1$,

$$x(t) = \frac{1}{2Q} \int_0^t \tau \Delta \tau - t \left(\frac{1}{2Q} \int_0^1 \tau \Delta \tau - 1 \right), \quad (4.3)$$

where $Q := \frac{\mathcal{F}_1}{\mathcal{F}_2}$. Therefore, the solution depends on the time scale. Let us consider two examples: $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$. With $\mathbb{T} = \mathbb{R}$, from (4.3) we obtain

$$x(t) = \frac{1}{4Q}t^2 + \frac{4Q-1}{4Q}t, \quad x^\Delta(t) = x^\nabla(t) = x'(t) = \frac{1}{2Q}t + \frac{4Q-1}{4Q}. \quad (4.4)$$

Substituting (4.4) into \mathcal{F}_1 and \mathcal{F}_2 gives $\mathcal{F}_1 = \frac{12Q+1}{24Q}$ and $\mathcal{F}_2 = \frac{48Q^2+1}{48Q^2}$, that is,

$$Q = \frac{2Q(12Q+1)}{48Q^2+1}. \quad (4.5)$$

Solving equation (4.5) we get $Q \in \left\{\frac{3-2\sqrt{3}}{12}, \frac{3+2\sqrt{3}}{12}\right\}$. Because (4.1) is a minimizing problem, we select $Q = \frac{3-2\sqrt{3}}{12}$ and we get the extremal

$$x(t) = -(3+2\sqrt{3})t^2 + (4+2\sqrt{3})t. \quad (4.6)$$

If $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$, then from (4.3) we obtain $x(t) = \frac{1}{8Q} \sum_{k=0}^{2t-1} k + \frac{8Q-1}{8Q}t$, that is,

$$x(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{8Q-1}{16Q}, & \text{if } t = \frac{1}{2}, \\ 1, & \text{if } t = 1. \end{cases}$$

Direct calculations show that

$$\begin{aligned} x^\Delta(0) &= \frac{x(\frac{1}{2}) - x(0)}{\frac{1}{2}} = \frac{8Q - 1}{8Q}, & x^\Delta\left(\frac{1}{2}\right) &= \frac{x(1) - x(\frac{1}{2})}{\frac{1}{2}} = \frac{8Q + 1}{8Q}, \\ x^\nabla\left(\frac{1}{2}\right) &= \frac{x(\frac{1}{2}) - x(0)}{\frac{1}{2}} = \frac{8Q - 1}{8Q}, & x^\nabla(1) &= \frac{x(1) - x(\frac{1}{2})}{\frac{1}{2}} = \frac{8Q + 1}{8Q}. \end{aligned} \quad (4.7)$$

Substituting (4.7) into the integrals \mathcal{F}_1 and \mathcal{F}_2 gives

$$\mathcal{F}_1 = \frac{8Q + 1}{32Q}, \quad \mathcal{F}_2 = \frac{64Q^2 + 1}{64Q^2}, \quad Q = \frac{\mathcal{F}_1}{\mathcal{F}_2} = \frac{2Q(8Q + 1)}{64Q^2 + 1}.$$

Thus, we obtain the equation $64Q^2 - 16Q - 1 = 0$. The solutions to this equation are:

$$Q \in \left\{ \frac{1 - \sqrt{2}}{8}, \frac{1 + \sqrt{2}}{8} \right\}. \text{ We are interested in the minimum value } Q, \text{ so we select}$$

$$Q = \frac{1 - \sqrt{2}}{8} \text{ to get the extremal}$$

$$x(t) = \begin{cases} 0, & \text{if } t = 0, \\ 1 + \frac{\sqrt{2}}{2}, & \text{if } t = \frac{1}{2}, \\ 1, & \text{if } t = 1. \end{cases} \quad (4.8)$$

Note that the extremals (4.6) and (4.8) are different: for (4.6) one has $x(1/2) = \frac{5}{4} + \frac{\sqrt{3}}{2}$.

We now present a problem where, in contrast with Example 4.1, the extremal does not depend on the time scale \mathbb{T} .

Example 4.2. Consider the autonomous problem

$$\mathcal{L}(x) = \frac{\int_0^2 (x^\Delta(t))^2 \Delta t}{\int_0^2 [x^\nabla(t) + (x^\nabla(t))^2] \nabla t} \longrightarrow \min, \quad (4.9)$$

$$x(0) = 0, \quad x(2) = 4.$$

If x is a local minimizer to (4.9), then the Euler–Lagrange equations must hold, i.e.,

$$\frac{2}{\mathcal{F}_2} x^\nabla(t) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2} (2x^\nabla(t) + 1) = c \quad \text{and} \quad \frac{2}{\mathcal{F}_2} x^\Delta(t) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2} (2x^\Delta(t) + 1) = c, \quad (4.10)$$

where $\mathcal{F}_1 := \mathcal{F}_1(x) = \int_0^2 (x^\Delta(t))^2 \Delta t$ and $\mathcal{F}_2 := \mathcal{F}_2(x) = \int_0^2 [x^\nabla(t) + (x^\nabla(t))^2] \nabla t$.

Choosing one of the equations of (4.10), for example the first one, we get

$$x^\nabla(t) = \left(c + \frac{\mathcal{F}_1}{\mathcal{F}_2^2} \right) \frac{\mathcal{F}_2^2}{2\mathcal{F}_2 - 2\mathcal{F}_1}. \quad (4.11)$$

Using (4.11) with boundary conditions $x(0) = 0$ and $x(2) = 4$, we obtain, for any given time scale \mathbb{T} , the extremal $x(t) = 2t$.

In the previous two examples, the variational functional is given by the ratio of a delta and a nabla integral. We now discuss a variational problem where the composition is expressed by the product of three time-scale integrals.

Example 4.3. Consider the problem

$$\mathcal{L}(x) = \left(\int_0^1 tx^\Delta(t)\Delta t \right) \left(\int_0^1 x^\Delta(t)(1+t)\Delta t \right) \left(\int_0^1 (x^\nabla(t))^2 \nabla t \right) \longrightarrow \min, \\ x(0) = 0, \quad x(1) = 1. \quad (4.12)$$

If x is a local minimizer to problem (4.12), then the Euler–Lagrange equations must hold, and we can write that

$$(\mathcal{F}_1\mathcal{F}_3 + \mathcal{F}_2\mathcal{F}_3)t + \mathcal{F}_1\mathcal{F}_3 + 2\mathcal{F}_1\mathcal{F}_2x^\nabla(\sigma(t)) = c, \quad (4.13)$$

where c is a constant, $\mathcal{F}_1 := \mathcal{F}_1(x) = \int_0^1 tx^\Delta(t)\Delta t$, $\mathcal{F}_2 := \mathcal{F}_2(x) = \int_0^1 x^\Delta(t)(1+t)\Delta t$,

and $\mathcal{F}_3 := \mathcal{F}_3(x) = \int_0^1 (x^\nabla(t))^2 \nabla t$. Using relation (2.2), we can write (4.13) as

$$(\mathcal{F}_1\mathcal{F}_3 + \mathcal{F}_2\mathcal{F}_3)t + \mathcal{F}_1\mathcal{F}_3 + 2\mathcal{F}_1\mathcal{F}_2x^\Delta(t) = c. \quad (4.14)$$

Using the boundary conditions $x(0) = 0$ and $x(1) = 1$, we get from (4.14) that

$$x(t) = \left(1 + Q \int_0^1 \tau \Delta \tau \right) t - Q \int_0^t \tau \Delta \tau, \quad (4.15)$$

where $Q = \frac{\mathcal{F}_1\mathcal{F}_3 + \mathcal{F}_2\mathcal{F}_3}{2\mathcal{F}_1\mathcal{F}_2}$. Therefore, the solution depends on the time scale. Let us

consider $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \left\{ 0, \frac{1}{2}, 1 \right\}$. With $\mathbb{T} = \mathbb{R}$, expression (4.15) gives

$$x(t) = \left(\frac{2+Q}{2} \right) t - \frac{Q}{2}t^2, \quad x^\Delta(t) = x^\nabla(t) = x'(t) = \frac{2+Q}{2} - Qt. \quad (4.16)$$

Substituting (4.16) into \mathcal{F}_1 , \mathcal{F}_2 and \mathcal{F}_3 gives:

$$\mathcal{F}_1 = \frac{6-Q}{12}, \quad \mathcal{F}_2 = \frac{18-Q}{12}, \quad \mathcal{F}_3 = \frac{Q^2+12}{12}.$$

One can proceed by solving the equation $Q^3 - 18Q^2 + 60Q - 72 = 0$, to find the extremal

$$x(t) = \left(\frac{2+Q}{2}\right)t - \frac{Q}{2}t^2 \text{ with } Q = 2\sqrt[3]{9 + \sqrt{17}} + \frac{9 - \sqrt{17}}{8}\sqrt[3]{(9 + \sqrt{17})^2 + 6}.$$

Let us consider now the time scale $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$. From (4.15), we obtain

$$x(t) = \left(\frac{4+Q}{4}\right)t - \frac{Q}{4}\sum_{k=0}^{2t-1} k = \begin{cases} 0, & \text{if } t = 0, \\ \frac{4+Q}{8}, & \text{if } t = \frac{1}{2}, \\ 1, & \text{if } t = 1. \end{cases} \quad (4.17)$$

Substituting (4.17) into $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_3 , we obtain

$$\mathcal{F}_1 = \frac{4-Q}{16}, \quad \mathcal{F}_2 = \frac{20-Q}{16}, \quad \mathcal{F}_3 = \frac{Q^2+16}{16}$$

and the equation $Q^3 - 18Q^2 + 48Q - 96 = 0$. Solving this equation, we find the extremal

$$x(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{5 + \sqrt[3]{5} + \sqrt[3]{25}}{4}, & \text{if } t = \frac{1}{2}, \\ 1, & \text{if } t = 1. \end{cases}$$

Finally, we apply the results of Section 3.3 to an isoperimetric variational problem.

Example 4.4. Let us consider the problem of extremizing

$$\mathcal{L}(x) = \frac{\int_0^1 (x^\Delta(t))^2 \Delta t}{\int_0^1 tx^\nabla(t) \nabla t}$$

subject to the boundary conditions $x(0) = 0$ and $x(1) = 1$, and the constraint

$$\mathcal{K}(t) = \int_0^1 tx^\nabla(t) \nabla t = 1.$$

Applying Theorem 3.9, we get the nabla differential equation

$$\frac{2}{\mathcal{F}_2}x^\nabla(t) - \left(\lambda + \frac{\mathcal{F}_1}{(\mathcal{F}_2)^2}\right)t = c. \quad (4.18)$$

Solving this equation, we obtain

$$x(t) = \left(1 - Q \int_0^1 \tau \nabla \tau\right)t + Q \int_0^t \tau \nabla \tau, \quad (4.19)$$

where $Q = \frac{\mathcal{F}_2}{2} \left(\frac{\mathcal{F}_1}{(\mathcal{F}_2)^2} + \lambda \right)$. Therefore, the solution of equation (4.18) depends on

the time scale. As before, let us consider $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \left\{ 0, \frac{1}{2}, 1 \right\}$.

For $\mathbb{T} = \mathbb{R}$, we obtain from (4.19) that $x(t) = \frac{2-Q}{2}t + \frac{Q}{2}t^2$. Substituting this expression for x into the integrals \mathcal{F}_1 and \mathcal{F}_2 , gives $\mathcal{F}_1 = \frac{Q^2 + 12}{12}$ and $\mathcal{F}_2 = \frac{Q + 6}{12}$.

Using the given isoperimetric constraint, we obtain $Q = 6$, $\lambda = 8$, and $x(t) = 3t^2 - 2t$.

Let us consider now the time scale $\mathbb{T} = \left\{ 0, \frac{1}{2}, 1 \right\}$. From (4.19), we have

$$x(t) = \frac{4-3Q}{4}t + Q \sum_{k=1}^{2t} \frac{k}{4} = \begin{cases} 0, & \text{if } t = 0, \\ \frac{4-Q}{8}, & \text{if } t = \frac{1}{2}, \\ 1, & \text{if } t = 1. \end{cases}$$

Simple calculations show that

$$\mathcal{F}_1 = \sum_{k=0}^1 \frac{1}{2} \left(x^\Delta \left(\frac{k}{2} \right) \right)^2 = \frac{1}{2} (x^\Delta(0))^2 + \frac{1}{2} \left(x^\Delta \left(\frac{1}{2} \right) \right)^2 = \frac{Q^2 + 16}{16},$$

$$\mathcal{F}_2 = \sum_{k=1}^2 \frac{1}{4} k x^\nabla \left(\frac{k}{2} \right) = \frac{1}{4} x^\nabla \left(\frac{1}{2} \right) + \frac{1}{2} x^\nabla(1) = \frac{Q + 12}{16}$$

and $\mathcal{K}(t) = \frac{Q + 12}{16} = 1$. Therefore, $Q = 4$, $\lambda = 6$, and we have the extremal

$$x(t) = \begin{cases} 0, & \text{if } t \in \left\{ 0, \frac{1}{2} \right\}, \\ 1, & \text{if } t = 1. \end{cases}$$

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