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Asymptotic Behavior of a Higher-Order Recursive Sequence

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Abstract

In this paper, we investigate the boundedness character and the global behavior of positive solutions of the difference equation

$$x_{n+1} = p_n + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \cdots,$$

where $k \in \mathbb{N}$ and $\{p_n\}$ is a sequence of nonnegative real numbers which converges to p and the initial conditions $x_{-k}, \cdots, x_0$ are arbitrary positive real numbers.

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1 Introduction

Our aim in this paper is to study the boundedness character and the global asymptotic behavior of positive solutions of the difference equation

$$x_{n+1} = p_n + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \cdots, \tag{1.1}$$

where $k \in \mathbb{N}$ and $\{p_n\}$ is a sequence of nonnegative real numbers which converges to p and the initial conditions $x_{-k}, \cdots, x_0$ are arbitrary positive real numbers. Eq. (1.1) was studied by many authors with $k = 1$.

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When $k = 1$, in [6] the author studied asymptotic behavior of the positive solutions of the difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \cdots, \tag{1.2}$$

where $\{p_n\}$ is a sequence of nonnegative real numbers which converges to p and the initial conditions x_{-1}, x_0 are arbitrary positive real numbers. It was prove in [6] that if $\lim_{n\to\infty} p_n = p > 1$, then every positive solution of Eq. (1.2) is bounded and converges to $p + 1$, and if $0 \leq p < 1$, then there exist solutions of Eq. (1.2) that are unbounded. In [3–5] the authors studied the Eq. (1.2) where $\{p_n\}$ is a positive two periodic sequence. In [2], when $\{p_n\}$ is a positive bounded sequence, the authors obtained conditions for the boundedness and persistence of solutions and for the existence of unbounded solutions, and they also obtained global attractivity results for Eq. (1.2). For the autonomous case of Eq. (1.1) we can refer to [1].

The paper is organized as follows. In Section 2 we investigate the boundedness character of positive solutions of Eq. (1.1). We prove that if k is odd and $0 \leq p < 1$, then there exist unbounded solutions of Eq. (1.1) and when the case $k \in \mathbb{N}$ and $p \geq 1$, then every positive solution of Eq. (1.1) is bounded. Section 3 is devoted to the global attractivity results of the positive solutions of Eq. (1.1). We show that when the case $k \in \mathbb{N}$ and $p > 1$, then every positive solution of Eq. (1.1) converges to $(p + 1)$.

2 Boundedness Character of Eq. (1.1)

In this section, we investigate the boundedness character of Eq. (1.1). We show that if $p \geq 1$, then every positive solution of Eq. (1.1) is bounded and that when k is odd and $0 \leq p < 1$, then there exist unbounded solutions of Eq. (1.1).

Theorem 2.1. *Suppose that* $\lim_{n\to\infty} p_n = p \geq 1$, *then every positive solution of Eq.* (1.1) *is bounded.*

Proof. First, we assume that $p > 1$. Let $\varepsilon \in (0, p - 1)$, from (1.1) one can see that

$$x_n \geq p - \varepsilon \quad \text{for } n \geq 1.$$

Now, we shall prove that $\{x_n\}$ is bounded. Assume without loss of generality that

$$x_n > p - \varepsilon \quad \text{for } n = -k, -k + 1, \cdots, 0.$$

Then one can find $L \in (p - \varepsilon, p - \varepsilon + 1)$ such that

$$L \leq x_n \leq \frac{L}{L - p + \varepsilon} \quad \text{for } n = -k, -k + 1, \cdots, 0.$$

Set

$$f(u, v) = p - \varepsilon + \frac{v}{u}.$$

Note that since $p > 1$

$$f\left(L, \frac{L}{L - p + \varepsilon}\right) \leq \frac{L}{L - p + \varepsilon} \quad \text{and} \quad f\left(\frac{L}{L - p + \varepsilon}, L\right) = L.$$

Now

$$x_1 = f(x_0, x_{-k}) \leq f\left(L, \frac{L}{L - p + \varepsilon}\right) \leq \frac{L}{L - p + \varepsilon}.$$

In a similar way it is true that $x_1 \geq L$. By induction we obtain that

$$L \leq x_n \leq \frac{L}{L - p + \varepsilon} \quad \text{for } n = -k, -k + 1, \cdots .$$

The proof is complete.

Now, we assume that $p = 1$. Let $\varepsilon \in (0, \delta)$ and $\delta \in (0, 1)$, since $\lim_{n\to\infty} p_n = 1$, from (1.1) one can see that

$$x_n \geq 1 - \varepsilon + \delta \quad \text{for } n \geq 1.$$

Then one can find $L \in (1 - \varepsilon + \delta, 2 - \varepsilon + \delta)$ such that

$$L \leq x_n \leq \frac{L}{L - 1 + \varepsilon + \delta} \quad \text{for } n = -k, -k + 1, \cdots , 0.$$

Therefore, the rest of the proof is similar the above and it is omitted. □

Now we study the boundedness character of Eq. (1.1) for the case k is odd and $0 \leq p < 1$. We prove that in this case, there exist unbounded solutions of Eq. (1.1)

Theorem 2.2. *Consider Eq.* (1.1) *when the case k is odd. If* $0 \leq p < 1$*, then there exist solutions of Eq.* (1.1) *that are unbounded.*

Proof. It is clear that since $\lim_{n\to\infty} p_n = p$, we may assume $p - \varepsilon < p_n < p + \varepsilon$, where ε is a sufficiently small arbitrary positive number. Now, let $0 < p < 1$, $\delta \in (0, 1 - p)$, $\varepsilon \in (0, \delta)$. Choose the initial conditions such that

$$p < x_{-k+1}, x_{-k+3}, \cdots , x_0 < p+\delta \ \text{ and } \ x_{-k}, x_{-k+2}, \cdots , x_{-1} > \frac{1}{1 - p - \delta} > 1+p+\delta.$$

Then,

$$x_1 = p_0 + \frac{x_{-k}}{x_0} > p - \varepsilon + \frac{x_{-k}}{p + \delta} > p - \varepsilon + x_{-k}$$

and

$$x_2 = p_1 + \frac{x_{-k+1}}{x_1} < p + \varepsilon + \frac{p + \delta}{x_1} < p + \varepsilon + \frac{1}{x_1} < p + \varepsilon < 1.$$

Further we have

$$x_3 = p_2 + \frac{x_{-k+2}}{x_2} > p - \varepsilon + \frac{x_{-k+2}}{p+\delta} > p - \varepsilon + x_{-k+2}$$

and

$$x_4 = p_3 + \frac{x_{-k+3}}{x_3} < p + \varepsilon + \frac{p+\delta}{x_3} < p + \varepsilon + \frac{1}{x_3} < p + \varepsilon < 1.$$

Therefore, we obtain $p - \varepsilon < x_{k+1} < 1$ and $x_{k+2} > 2\left(p - \varepsilon\right) + x_{-k}$.

By induction, for $i = 1, 2, \cdots$, we have $p - \varepsilon < x_{(k+1)i} < 1$ and $x_{(k+1)i+1} > (i+1)\left(p - \varepsilon\right) + x_{-k}$. Thus,

$$\lim_{i\to\infty} x_{(k+1)i+1} = \infty \quad \text{and} \quad \lim_{i\to\infty} x_{(k+1)i} = p.$$

Now, let $p = 0$, $\delta \in (0,1)$, $\varepsilon \in (0,\delta)$ and choose the initial conditions such that

$$x_{-k}, x_{-k+2}, \cdots, x_{-1} > \frac{1}{(1-\delta-\varepsilon)}$$

and

$$0 < x_{-k+1}, x_{-k+3}, \cdots, x_0 < 1.$$

So, we have

$$x_1 > \frac{x_{-k}}{x_0} > \frac{1}{(1-\delta-\varepsilon)}$$

and

$$x_2 < \varepsilon + \frac{x_{-k+1}}{x_1} < \varepsilon + \frac{1}{x_1} < \varepsilon + 1 - \delta - \varepsilon = 1 - \delta.$$

Further we have

$$x_3 > \frac{x_{-k+2}}{x_2} > \frac{1}{(1-\delta)(1-\delta-\varepsilon)} > \frac{1}{(1-\delta-\varepsilon)}$$

and

$$x_4 < \varepsilon + \frac{x_{-k+3}}{x_3} < \varepsilon + \frac{1}{x_3} < \varepsilon + 1 - \delta - \varepsilon = 1 - \delta$$

Therefore, we obtain

$$0 < x_{2n} < 1 - \delta \quad \text{and} \quad x_{2n+1} > \frac{1}{\left(1-\delta\right)^n\left(1-\delta-\varepsilon\right)}.$$

Thus, we obtain

$$\lim_{n\to\infty} x_{2n+1} = \infty \quad \text{and} \quad \lim_{n\to\infty} x_{2n} = 0.$$

The proof is complete. □

3 Global Attractivity of Eq. (1.1)

In this section, when $k \in \mathbb{N}$, we show that if $p > 1$, then every positive solution of Eq. (1.1) converges to $(p+1)$.

Theorem 3.1. *Consider Eq.* (1.1) *when the case* $k \in \mathbb{N}$. *Assume that* $p > 1$. *Then every positive solution of Eq.* (1.1) *converges to* $(p+1)$.

Proof. Since $p > 1$, by Theorem 2.1 every positive solution of Eq. (1.1) is bounded, then we have the following

$$s = \liminf_{n\to\infty} x_n \quad \text{and} \quad S = \limsup_{n\to\infty} x_n.$$

Then it is easy to see from Eq. (1.1) that

$$s \geq p + \frac{s}{S} \quad \text{and} \quad S \leq p + \frac{S}{s}.$$

Thus, we have

$$sS \geq pS + s \quad \text{and} \quad Ss \leq ps + S.$$

This implies that

$$pS + s \leq Ss \leq ps + S.$$

Then, we get

$$p(S-s) \leq (S-s).$$

Since $p > 1$, if $s < S$, then we arrive a contradiction. Thus, the proof is complete. □

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