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Positive Periodic Solutions for Neutral Functional Difference Equations

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Abstract

In this paper, we apply fixed point theorem in a cone to obtain sufficient conditions for the existence of single and multiple positive periodic solutions for a class of neutral functional difference equation.

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1 Introduction

In this paper, we investigate the existence of positive periodic solutions of the following neutral functional difference equation

$$\Delta(x(n) - cx(n-m)) = a(n)x(n) + f(n, x(n-m)), \tag{1.1}$$

where $a(n) = a(n+\omega)$, $f(n+\omega, u) = f(n, u)$, constant $c > 0$, $\omega, m \in \mathbb{N}$ where $\mathbb{N}$ denotes the set of positive integers. The operator Δ is defined as $\Delta x(n) = x(n+1) - x(n)$.

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The existence of periodic solutions of functional difference equations has gained the attention of many researchers in recent years, for example, see [1–3,7,10,11,13–16] and the references therein. Jiang [6] had obtained the optimal existence theorem for single and multiple positive periodic solutions to general functional difference equations

$$\Delta x(n) = x(n)[a(n) - g(n, x(n - \tau_1(n)), \cdots, x(n - \tau_k(n)))], \tag{1.2}$$

$$\Delta x(n) = -a(n)x(n) + g(n, x(n - \tau(n))). \tag{1.3}$$

Raffoul [8] had considered the existence of positive periodic solutions for functional difference equations with parameter

$$x(n + 1) = a(n)x(n) \pm \lambda h(n)g(x(n - \tau(n))). \tag{1.4}$$

However, results on periodic solutions of nonlinear neutral functional difference equations are relatively few in the literature. In paper [12], Wu and Liu established existence, multiplicity, and nonexistence of positive periodic solutions for the neutral difference equation

$$\Delta(x(n) - cx(n - \delta)) = a(n)g(x(n))x(n) + \lambda b(n)f(x(n - \tau(n))). \tag{1.5}$$

The key in [12] is that the neutral operator $Ax = x(n) - cx(n)$ has a unique continuous bounded inverse under appropriate conditions. A essential condition about constant c is $|c| < 1$ in [12].

In the continuous case, equations in the form of (1.1) have extensive applications in mathematical ecological models and population models, see biological [4]. In recent paper [9], authors obtained some sufficient conditions to guarantee the existence of periodic solution for (1.1) by using the following well-known Krasnosel′skii's fixed-point theorem:

Theorem 1.1. *Let Ω be a closed convex nonempty subset of a Banach space X. Suppose that Φ and Ψ map Ω into X such that*

(i) Φ is compact and continuous;

(ii) Ψ is a contraction mapping;

(iii) $\forall x, y \in \Omega$, implies $\Phi x + \Psi y \in \Omega$.

Then there exists $z \in \Omega$ with $z = \Phi z + \Psi z$.

To apply the above theorem, authors obtained two available operators from the linear term $a(n)x(n)$ under the restrictive condition $|c| < 1$.

Motivated by [9], we discuss the existence of positive periodic solutions of (1.1). Using a fixed point theorem different from the one used in [9] and [12], we obtain some sufficient conditions for the existence of single and multiple positive periodic solutions for (1.1) when $c > 0$ and $c \neq 1$. Our results are new even for $0 < c < 1$. Moreover, our results admit $c > 1$(see Theorem 3.5 and Theorem 3.7).

2 Preliminaries

Let X be the set of all real ω-periodic sequences. When endowed with the maximum norm $\|x\| = \max_{n\in[0,\omega-1]} |x(n)|$, X is a Banach space.

Let $d \in \mathbb{N}$ and $c \neq 1$, and consider the equations

$$x(n+d) = cx(n) + \gamma(n) \tag{2.1}$$

and

$$x(n+1) = \alpha(n)x(n) + \beta(n), \tag{2.2}$$

where $\gamma \in X, \alpha \in X$ and $\beta \in X$. Set (d,ω) is the greatest common divisor of d and ω, $p = \omega/(d,\omega)$. Assume that $x \in X$ is a solution of (2.1), we obtain that

$$\begin{gathered} c^{-1}x(n+d) - x(n) = c^{-1}\gamma(n), \\ c^{-2}x(n+2d) - c^{-1}x(n+d) = c^{-2}\gamma(n+d), \\ \cdots\cdots\cdots\cdots \\ c^{-p}x(n+pd) - c^{1-p}x(n+(p-1)d) = c^{-p}\gamma(n+(p-1)d). \end{gathered}$$

By summing the above equations and using periodicity of x, we obtain the following result.

Lemma 2.1. *Assume that $c \neq 1$, then* (2.1) *has an unique periodic solution*

$$x(n) = (c^{-p} - 1)^{-1} \sum_{i=1}^{p} c^{-i}\gamma(n + (i-1)d).$$

Similarly, we can obtain the following result for (2.2).

Lemma 2.2. *Assume that $\prod_{i=1}^{\omega} \alpha(i) \neq 1$, then* (2.2) *has an unique periodic solution*

$$x(n) = \sum_{s=n}^{n+\omega-1} \left(\prod_{k=n}^{s} \frac{1}{\alpha(k)}\right) \left(\prod_{k=0}^{\omega-1} \frac{1}{\alpha(k)} - 1\right)^{-1} \beta(s).$$

Let $y(n) = x(n) - cx(n-m)$, then (1.1) can be written as

$$\begin{cases} x(n+m) = cx(n) + y(n+m), \\ y(n+1) = (1 + a(n))y(n) + F(n, x(n-m)), \end{cases} \tag{2.3}$$

where $F(n,u) = ca(n)u + f(n,u)$. Assume that $x \in X$ is a solution of (1.1), then $y \in X$ and

$$x(n) = (c^{-h} - 1)^{-1} \sum_{i=1}^{h} c^{-i}y(n+im), \quad y(n) = \sum_{s=n}^{n+\omega-1} G(n,s)F(s, x(s-m)), \tag{2.4}$$

where $h = \omega/(m,\omega)$,

$$G(n,s) = \left(\prod_{k=n}^{s} \frac{1}{1+a(k)}\right)\left(\prod_{k=0}^{\omega-1} \frac{1}{1+a(k)} - 1\right)^{-1}.$$

We introduce the following conditions.

(H_1) $0 < c < 1$, $-1 < a(n) < 0$ and $F : \mathbb{N} \times [0,\infty) \to [0,\infty)$ is continuous.

(H_2) $c > 1$, $a(n) > 0$ and $F : \mathbb{N} \times [0,\infty) \to [0,\infty)$ is continuous.

If (H_1),(2.4) hold and y is a positive ω-periodic function, then x is also positive ω-periodic function and

$$y(n) = \sum_{s=n}^{n+\omega-1} G(n,s) F\left(s, (c^{-h}-1)^{-1} \sum_{i=1}^{h} c^{-i} y(s+(i-1)m)\right),$$

$$M_1 = \max_{n,s\in[1,\omega]} G(n,s) > 0,\ m_1 = \min_{n,s\in[1,\omega]} G(n,s) > 0.$$

If (H_2),(2.4) hold and y is a negative ω-periodic function, then x is positive ω-periodic function and

$$-y(n) = \sum_{s=n}^{n+\omega-1} (-G(n,s)) F\left(s, (1-c^{-h})^{-1} \sum_{i=1}^{h} c^{-i}(-y(s+(i-1)m))\right),$$

$$M_2 = \max_{n,s\in[1,\omega]} -G(n,s) > 0,\ m_2 = \min_{n,s\in[1,\omega]} -G(n,s) > 0.$$

Define operators $T_i (i = 1,2)$ and cones $K_i (i = 1,2)$ by

$$(T_1 u)(n) = \sum_{s=n}^{n+\omega-1} G(n,s) F\left(s, (c^{-h}-1)^{-1} \sum_{i=1}^{h} c^{-i} u(s+(i-1)m)\right),$$

$$(T_2 u)(n) = \sum_{s=n}^{n+\omega-1} (-G(n,s)) F\left(s, (1-c^{-h})^{-1} \sum_{i=1}^{h} c^{-i} u(s+(i-1)m)\right)$$

and

$$K_1 = \{u \in X : u(n) \geq \delta_1 \|u\|,\ n \in \mathbb{N}\},$$

$$K_2 = \{u \in X : u(n) \geq \delta_2 \|u\|,\ n \in \mathbb{N}\},$$

where $\delta_1 = m_1/M_1, \delta_2 = m_2/M_2$.

Lemma 2.3. *Assume that (H_i), i=1,2, holds. Then $T_i : K_i \to K_i$ is completely continuous.*

Proof. Here we only prove the case of $i = 1$. By the nonnegativity of F and $G(n,s)$, $(T_1u)(n) \geq 0$ on $[0, \omega-1]$. It is clear that $(T_1u)(n+\omega) = (T_1u)(n)$ and T_1 is completely continuous on bounded subsets of K_1. Noting that

$$\begin{aligned}(T_1u)(n) &= \sum_{s=n}^{n+\omega-1} G(n,s)F\left(s, (c^{-h}-1)^{-1}\sum_{i=1}^{h} c^{-i}u(s+(i-1)m)\right)\\ &\leq M_1 \sum_{s=n}^{n+\omega-1} F\left(s, (c^{-h}-1)^{-1}\sum_{i=1}^{h} c^{-i}u(s+(i-1)m)\right),\\ (T_1u)(n) &\geq m_1 \sum_{s=n}^{n+\omega-1} F\left(s, (c^{-h}-1)^{-1}\sum_{i=1}^{h} c^{-i}u(s+(i-1)m)\right).\end{aligned}$$

we easily obtain that $(T_1u)(n) \geq \delta_1\|T_1u\|$, that is, $T_1(K_1) \subset K_1$. The proof is complete. □

Theorem 2.4. *[5] Let X be a Banach space and K be a cone in X. Suppose Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$ and suppose that*

$$\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

is a completely continuous operator such that

(i) $\|\Phi u\| \leq \|u\|$ for $u \in K\cap\partial\Omega_1$, and there exists $\psi \in K\setminus\{0\}$ such that $u \neq \Phi u+\lambda\psi$ for $u \in K \cap \partial\Omega_2$ and $\lambda > 0$, or

(ii) $\|\Phi u\| \leq \|u\|$ for $u \in K\cap\partial\Omega_2$, and there exists $\psi \in K\setminus\{0\}$ such that $u \neq \Phi u+\lambda\psi$ for $u \in K \cap \partial\Omega_1$ and $\lambda > 0$.

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Positive Periodic Solutions

Put

$$\overline{\varphi}(s) = \max\left\{\frac{ca(n)u+f(n,u)}{-a(n)}, n \in [0,\omega-1], u \in \left[\frac{\delta_1 s}{1-c}, \frac{s}{1-c}\right]\right\},$$

$$\underline{\varphi}(s) = \min\left\{\frac{ca(n)u+f(n,u)}{-a(n)u}, n \in [0,\omega-1], u \in \left[\frac{\delta_1 s}{1-c}, \frac{s}{1-c}\right]\right\},$$

$$\overline{\phi}(s) = \max\left\{\frac{ca(n)u+f(n,u)}{a(n)}, n \in [0,\omega-1], u \in \left[\frac{\delta_2 s}{c-1}, \frac{s}{c-1}\right]\right\},$$

$$\underline{\phi}(s) = \min\left\{\frac{ca(n)u+f(n,u)}{a(n)u}, n \in [0,\omega-1], u \in \left[\frac{\delta_2 s}{c-1}, \frac{s}{c-1}\right]\right\}.$$

Theorem 3.1. *Assume that* (H_1) *holds and there exist two positive constants* a, b *with* $a \neq b$ *such that*

$$\overline{\varphi}(a) \leq a, \quad \underline{\varphi}(b) \geq 1 - c. \tag{3.1}$$

Then the equation (1.1) *has at least one positive solution* $x \in X$ *with*

$$\min\{a, b\} \leq (1 - c)\|x\| \leq \max\{a, b\}.$$

Proof. Without loss of generality, we assume that $a < b$. Let $\Omega_1 = \{x \in X : \|x\| < a\}$ and $\Omega_2 = \{x \in X : \|x\| < b\}$. We claim that

(i) $\|T_1 u\| \leq \|u\|$, $u \in K_1 \cap \partial\Omega_1$.

(ii) $u \neq T_1 u + \lambda$, $\forall u \in K_1 \cap \partial\Omega_2$ and $\lambda > 0$.

From (3.1), we have that

$$F(n, u) \leq -aa(n), \ \forall 0 \leq n \leq \omega - 1, \ \forall \ \frac{a\delta_1}{1 - c} \leq u \leq \frac{a}{1 - c}, \tag{3.2}$$

$$F(n, u) \geq -a(n)u(1 - c), \ \forall 0 \leq n \leq \omega - 1, \ \forall \ \frac{b\delta_1}{1 - c} \leq u \leq \frac{b}{1 - c}. \tag{3.3}$$

To justify (i), let $u \in K_1 \cap \partial\Omega_1$, then $\|u\| = a$ and $\delta_1 a \leq u(n) \leq a$ for $0 \leq n \leq \omega - 1$. It follows that

$$\begin{aligned} (T_1 u)(n) &= \sum_{s=n}^{n+\omega-1} G(n, s) F\left(s, (c^{-h} - 1)^{-1} \sum_{i=1}^{h} c^{-i} u(s + (i - 1)m)\right) \\ &\leq -a \sum_{s=n}^{n+\omega-1} G(n, s) a(s) = a = \|u\| \end{aligned}$$

since

$$\frac{a\delta_1}{1 - c} \leq (c^{-h} - 1)^{-1} \sum_{i=1}^{h} c^{-i} u(s + (i - 1)m) \leq \frac{a}{1 - c}.$$

This means that

$$\|T_1 u\| \leq \|u\|, \ \forall u \in K_1 \cap \partial\Omega_1.$$

Next, we prove (ii). If not, there exist $u_0 \in K_1 \cap \partial\Omega_2$ and $\lambda_0 > 0$ such that

$$u_0 = T_1 u_0 + \lambda_0.$$

Since $u_0 \in K_1 \cap \partial\Omega_2$, then $\|u_0\| = b$ and $\delta b \le u_0(n) \le b$. Put $\chi = \min\{u_0(n), 0 \le n \le \omega - 1\}$, then we have $\chi = u_0(n)$ for some $n \in [0, \omega - 1]$. Thus it follows that

$$\begin{aligned} u_0(n) &= (T_1u_0)(n) + \lambda_0 \\ &= \sum_{s=n}^{n+\omega-1} G(n,s)F\left(s, (c^{-h}-1)^{-1}\sum_{i=1}^{h} c^{-i}u_0(s+(i-1)m)\right) + \lambda_0 \\ &\ge \sum_{s=n}^{n+\omega-1} G(n,s)a(s)(c-1)(c^{-h}-1)^{-1}\sum_{i=1}^{h} c^{-i}u_0(s+(i-1)m) + \lambda_0 \\ &\ge \chi \sum_{s=n}^{n+\omega-1} G(n,s)(-a(s)) + \lambda_0 = \chi + \lambda_0 \end{aligned}$$

and this implies $\chi > \chi$, a contradiction. Therefore, it follows by Theorem 2.4 that T_1 has a fixed point $u \in K_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Furthermore, $a \le \|u\| \le b$ and $u(n) \ge \delta_1 a$, which means that

$$x = (c^{-h}-1)^{-1}\sum_{i=1}^{h} c^{-i}u(n+im)$$

is one positive ω-periodic solution of (1.1) and $\min\{a,b\} \le (1-c)\|x\| \le \max\{a,b\}$. The proof is complete. □

Corollary 3.2. *Assume that* (H_1) *holds and one of the following conditions holds:*

(i) $\overline{\varphi}_0 < 1$ *and* $\underline{\varphi}_\infty > 1-c$,

(ii) $\overline{\varphi}_\infty < 1$ *and* $\underline{\varphi}_0 > 1-c$,

where

$$\overline{\varphi}_0 = \lim_{s\to 0^+}\frac{\overline{\varphi}(s)}{s},\ \ \overline{\varphi}_\infty = \lim_{s\to+\infty}\frac{\overline{\varphi}(s)}{s},$$
$$\underline{\varphi}_0 = \lim_{s\to 0^+}\underline{\varphi}(s),\ \ \underline{\varphi}_\infty = \lim_{s\to+\infty}\underline{\varphi}(s).$$

Then the equation (1.1) *has at least one positive solution* $x \in X$.

Theorem 3.3. *Assume that* (H_1) *holds. There exist* $N+1$ *positive constants* $p_1 < p_2 < \cdots < p_N < p_{N+1}$ *such that one of the following conditions is satisfied:*

(i) $\overline{\varphi}(p_{2k-1}) < p_{2k-1},\ k = 1,2,\ldots,[(N+2)/2],\ \underline{\varphi}(p_{2k}) > 1-c,\ \ k = 1,2,\ldots,[(N+1)/2]$,

(ii) $\underline{\varphi}(p_{2k-1}) > 1-c,\ k = 1,2,\ldots,[(N+2)/2],\ \overline{\varphi}(p_{2k}) < p_{2k},\ \ k = 1,2,\ldots,[(N+1)/2]$,

where $[d]$ *denotes the integer part of* d*. Then the equation* (1.1) *has at least* N *positive solutions* $x_k \in X, k = 1,2,\ldots,N$ *with* $p_k < (1-c)\|x_k\| < p_{k+1}$.

Proof. It is enough to prove case (i). Since $\overline{\varphi}, \underline{\varphi} : (0, \infty) \to [0, \infty)$ are continuous, there exist $p_k < a_k < b_k < p_{k+1},\ k = 1, 2, \dots, N$ such that

$$\overline{\varphi}(a_{2k-1}) \leq a_{2k-1},\ \underline{\varphi}(b_{2k-1}) \geq (1-c),\ k = 1, 2, \dots, [(N+1)/2],$$

$$\underline{\varphi}(a_{2k}) \geq (1-c),\ \overline{\varphi}(b_{2k}) \leq b_{2k},\ k = 1, 2, \dots, [N/2].$$

It follows by Theorem 3.1 that equation (1.1) has at least one positive periodic solution $x_k \in X$ for every pair of numbers $\{a_k, b_k\}$ with $p_k < a_k \leq (1-c)\|x_k\| \leq b_k < p_{k+1}$. The proof is complete. □

Corollary 3.4. *Assume that* (H_1) *holds and one of the following conditions holds:*

(i) $\overline{\varphi}_0 < 1$, $\overline{\varphi}_\infty < 1$ *and there exists a positive constant* b *such that* $\underline{\varphi}(b) > 1 - c$*;*

(ii) $\underline{\varphi}_0 > 1 - c$, $\underline{\varphi}_\infty > 1 - c$ *and there exists a positive constant* b *such that* $\overline{\varphi}(b) < b$.

Then the equation (1.1) *has at least two positive solutions* $x_1, x_2 \in X$ *with*

$$0 < (1-c)\|x_1\| < b < (1-c)\|x_2\| < \infty.$$

Theorem 3.5. *Assume that* (H_2) *holds and there exist two positive constants* a, b *with* $a \neq b$ *such that*

$$\overline{\phi}(a) \leq a, \quad \underline{\phi}(b) \geq c - 1. \tag{3.4}$$

Then the equation (1.1) *has at least one positive solution* $x \in X$ *with*

$$\min\{a, b\} \leq (c-1)\|x\| \leq \max\{a, b\}.$$

The proof of Theorem 3.5 is similar to that of Theorem 3.1 and we omit it.

Corollary 3.6. *Assume that* (H_2) *holds and one of the following conditions holds:*

(i) $\overline{\phi}_0 < 1$ *and* $\underline{\phi}_\infty > c - 1$,

(ii) $\overline{\phi}_\infty < 1$ *and* $\underline{\phi}_0 > c - 1$,

where

$$\overline{\phi}_0 = \lim_{s \to 0^+} \frac{\overline{\phi}(s)}{s},\ \ \overline{\phi}_\infty = \lim_{s \to +\infty} \frac{\overline{\phi}(s)}{s},$$
$$\underline{\phi}_0 = \lim_{s \to 0^+} \underline{\phi}(s),\ \ \underline{\phi}_\infty = \lim_{s \to +\infty} \underline{\phi}(s).$$

Then the equation (1.1) *has at least one positive solution* $x \in X$.

Theorem 3.7. *Assume that* (H_2) *holds. There exist* $N + 1$ *positive constants* $p_1 < p_2 < \cdots < p_N < p_{N+1}$ *such that one of the following conditions is satisfied:*

(i) $\overline{\phi}(p_{2k-1}) < p_{2k-1},\ k = 1, 2, \dots, [(N+2)/2]$, $\underline{\phi}(p_{2k}) > c-1,\ \ k = 1, 2, \dots, [(N+1)/2]$,

(ii) $\underline{\phi}(p_{2k-1}) > c-1,\ k = 1, 2, \ldots, [(N+2)/2], \overline{\phi}(p_{2k}) < p_{2k},\ \ k = 1, 2, \ldots, [(N+1)/2],$

where $[d]$ *denotes the integer part of d. Then the equation* (1.1) *has at least* N *positive solutions* $x_k \in X, k = 1, 2, \ldots, N$ *with* $p_k < (c-1)\|x_k\| < p_{k+1}$.

Corollary 3.8. *Assume that* (H_2) *holds and one of the following conditions holds:*

(i) $\overline{\phi}_0 < 1,\ \overline{\phi}_\infty < 1$ *and there exists a positive constant* b *such that* $\underline{\phi}(b) > c-1$*;*

(ii) $\underline{\phi}_0 > c-1,\ \underline{\phi}_\infty > c-1$ *and there exists a positive constant* b *such that* $\overline{\phi}(b) < b$.

Then the equation (1.1) *has at least two positive solutions* $x_1, x_2 \in X$ *with*

$$0 < (c-1)\|x_1\| < b < (c-1)\|x_2\| < \infty.$$

Example 3.9. Consider the difference equation

$$\Delta(x(n) - 0.5x(n+30)) = -\frac{1}{3}x(n) + \frac{x(n+30)}{5+5x(n+30)} + b(n)x^3(n+30), \tag{3.5}$$

where b is an ω-periodic function with $b(n) > 0$ for all $n \in [1, \omega]$.

Obviously, $f(n,x) = x/5(x+1) + b(n)x^3$ and $\overline{\varphi}_0 = 0.1,\ \underline{\varphi}_\infty = +\infty$. By Corollary 3.2,

$$\Delta(x(n)-0.5x(n+30-p\omega)) = -\frac{1}{3}x(n)+\frac{x(n+30-p\omega)}{5+5x(n+30-p\omega)}+b(n)x^3(n+30-p\omega) \tag{3.6}$$

has at least one positive ω-periodic solution, where $p \in \mathbb{N}$ and $p > 30$. Hence, (3.5) has at least one positive ω-periodic solution.

Example 3.10. Consider the difference equation

$$\Delta(x(n)-2x(n-m)) = x(n)-\frac{5}{3}x(n-m)+\lambda\left(x^2(n-m)+\frac{1}{2+x(n-m)}\right), \tag{3.7}$$

where λ is a positive real parameter.

In (3.7), $f(n,x) = -5x/3 + \lambda(x^2 + 1/(2+x))$. There exist $0 < p_1 < 1$ and $p_3 > 1$ such that

$$\underline{\phi}(p_1) > c-1,\ \underline{\phi}(p_3) > c-1$$

since $\underline{\phi}_0 = \underline{\phi}_\infty = +\infty$. If $\lambda > 0$ is sufficiently small, $\overline{\phi}(1) < 1$. By Theorem 3.7, there exists a constant $\lambda^* > 0$ such that (3.7) has at least two positive ω-periodic solutions for $\lambda \in (0, \lambda^*)$.

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