International Journal of Difference Equations
ISSN 0973-6069, Volume 6, Number 2, pp. 143–152 (2011)
http://campus.mst.edu/ijde

Discrete Population Models with Asymptotically Constant or Periodic Solutions

Youssef N. Raffoul
University of Dayton
Department of Mathematics
Dayton, OH 45469-2316, USA
yraffoul1@udayton.edu

Abstract

Given an initial function we show by means of fixed point theory that the unique solution of nonlinear difference equations of the form

$$\Delta x(t) = g(x(t)) - g(x(t-L))$$

converges to a pre-determined constant or a periodic solution. Then, we show the solution is stable and that its limit function serves as a global attractor.

AMS Subject Classifications: 39A10,34A97.
Keywords: Nonlinear difference equations, constant solution, periodic solution, contraction mapping, global attractor.

1 Introduction

Let $\mathbb{Z}$ be the set of integers and for $t \in \mathbb{Z}$, we use the contraction mapping principle to study the convergence of solutions of nonlinear difference equations of the form

$$\Delta x(t) = g(x(t)) - g(x(t-L)), \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ and is continuous in x with $\mathbb{R}$ denoting the set of all real numbers. In the continuous case, equation (1.1) has been studied, for the first time by the paper of Cooke and Yorke [4]. In that paper they presented three models that described the growth of a population. They used Lyapunov functionals to arrive at their results. Recently, in the paper [1], Burton utilized the notion of fixed point arguments and relaxed some of the conditions that Cook and Yorke were faced with as the result of using

Received March 31, 2009; Accepted August 25, 2011
Communicated by Allan Peterson

Lyapunov functionals. For more on the studies of stability and periodicity in discrete systems, we refer the reader to [5–8]. This paper is motivated by the papers of [1–3].

In addition to considering (1.1), we analyze the two equations:

$$\Delta x(t) = g(x(t - L_1)) - g(x(t - L_1 - L_2)), \tag{1.2}$$

$$\Delta x(t) = g(t, x(t)) - g(t, x(t - L)), \quad g(t + L, x) = g(t, x) \tag{1.3}$$

where L, L_1 and L_2 are positive integers. We note that every constant is a solution of equations (1.1), (1.2) and (1.3).

Equations (1.1), (1.2) and (1.3) play major roles in populations models. For example, suppose $x(t)$ is the number of individuals in a population at time t. Let the delay L be the life span of each individual. Then the birth rate of the population is some function of $x(t)$, say $g(x(t))$ and the function $g(x(t - L))$ can be thought of as the number of deaths per unit time at time t. Then the difference term $g(x(t)) - g(x(t - L))$ represents the net change in population per unit time. This implies that the growth of the population is governed by (1.1).

In the next section, we will use the contraction mapping principle to determine that constant. First, we state what it means for $x(t)$ to be a solution of (1.1). Note that since (1.1) is autonomous, we loose nothing by starting the solution at 0.

Let $\psi : [-L, 0] \to \mathbb{R}$ be a given bounded initial function. We say $x(t, 0, \psi)$ is a solution of (1.1) if $x(t, 0, \psi) = \psi$ on $[-L, 0]$ and $x(t, 0, \psi)$ satisfies (1.1) for $t \geq 0$.

One may use the method of Lyapunov functionals to show that all solutions are bounded by a constant. However, such a method does not tell us what the constant is. It is of importance to us to know such constants since all of our models have constant solutions.

2 Equation (1.1)

In this section we use the notion of fixed point theory to determine the constant that all solutions converge to. For the next theorem we make the following assumptions. The function g is globally Lipschitz. That is, there exists a constant $q > 0$ such that

$$|g(x) - g(y)| \leq q|x - y|. \tag{2.1}$$

Also we assume that

$$qL \leq \xi \quad \text{for some} \quad 0 < \xi < 1. \tag{2.2}$$

Let $\psi : [-L, 0] \to \mathbb{R}$ be a given initial function. By summing (1.1) from $s = t - L$ to $s = t - 1$ and for any constant c we arrive at the expression

$$x(t) = \sum_{s=t-L}^{t-1} g\big(x(s)\big) + c. \tag{2.3}$$

If $x(t)$ is given by (2.3), then it solves (1.1). Since (2.3) holds for all $t \geq 0$, we have

$$c = \psi(0) - \sum_{s=-L}^{-1} g\big(\psi(s)\big). \tag{2.4}$$

Substituting c of (2.4) in expression (2.3) yields

$$x(t) = \psi(0) - \sum_{s=-L}^{-1} g\big(\psi(s)\big) + \sum_{s=t-L}^{t-1} g\big(x(s)\big). \tag{2.5}$$

In the next theorem we show that, given an initial function $\psi(t) : [-L, 0] \to \mathbb{R}$, the unique solution of (1.1) converges to a unique determined constant.

Theorem 2.1. *Assume* (2.1) *and* (2.2) *hold and let* $\psi : [-L, 0] \to \mathbb{R}$ *be a given initial function. Then, the unique solution* $x(t, 0, \psi)$ *of* (1.1) *satisfies* $x(t, 0, \psi) \to r$, *where* r *is unique and given by*

$$r = \psi(0) + g(r)L - \sum_{s=-L}^{-1} g\big(\psi(s)\big). \tag{2.6}$$

Proof. For $|\cdot|$ denoting the absolute value, the metric space $(\mathbb{R}, |\cdot|)$ is complete. Define a mapping $\mathcal{H} : \mathbb{R} \to \mathbb{R}$, by

$$\mathcal{H}r = \psi(0) + g(r)L - \sum_{s=-L}^{-1} g\big(\psi(s)\big).$$

For $a, b \in \mathbb{R}$, we have

$$\big|\mathcal{H}a - \mathcal{H}b\big| \leq L|g(a) - g(b)| \leq L\, q|a - b| \leq \xi|a - b|.$$

This shows that $\mathbb{H}$ is a contraction on the complete metric space $(\mathbb{R}, |\cdot|)$, and hence $\mathcal{H}$ has a unique fixed point r, which implies that (2.6) has a unique solution. It remains to show that (1.1) has a unique solution and that it converges to the constant r.

Let $||\cdot||$ denote the maximum norm and let $\mathbb{M}$ be the set bounded functions $\phi : [-L, \infty) \to \mathbb{R}$ with $\phi(t) = \psi(t)$ on $[-L, 0], \phi(t) \to r$ as $t \to \infty$. Then $(\mathbb{M}, ||\cdot||)$ defines a complete metric space. For $\phi \in \mathbb{M}$, define $\mathcal{P} : \mathbb{M} \to \mathbb{M}$ by

$$(\mathcal{P}\phi)(t) = \psi(t), \text{ for } -L \leq t \leq 0,$$

and

$$(\mathcal{P}\phi)(t) = \psi(0) - \sum_{s=-L}^{-1} g\big(\psi(s)\big) + \sum_{s=t-L}^{t-1} g\big(\phi(s)\big), \text{ for } t \geq 0. \tag{2.7}$$

For $\phi \in \mathbb{M}$ with $\phi(t) \to r$, we have $\sum_{s=t-L}^{t-1} g\big(\phi(s)\big) \to g(r)L$ as $t \to \infty$. Then, using (2.6) and (2.7), we see that

$$(\mathcal{P}\phi)(t) \to \psi(0) - \sum_{s=-L}^{-1} g\big(\psi(s)\big) + g(r)L = r.$$

Thus, $\mathcal{P} : \mathbb{M} \to \mathbb{M}$. It remains to show that $\mathcal{P}$ is a contraction.

For $a, b \in \mathbb{M}$, we have

$$\begin{aligned} \big|(\mathcal{P}a)(t) - (\mathcal{P}b)(t)\big| &\leq \sum_{s=t-L}^{t-1} |g(a(s)) - g(b(s))| \\ &\leq qL||a-b|| \leq \xi||a-b||. \end{aligned}$$

Thus, $\mathcal{P}$ is a contraction and has a unique fixed point $\phi \in \mathbb{M}$. Based on how the mapping $\mathcal{P}$ was constructed, we conclude the unique fixed point ϕ satisfies (1.1). □

Remark 2.2. For any given initial function, Theorem 2.1 explicitly gives the limit to which the solution converges to. That limit is the unique solution r of (2.6).

Remark 2.3. For an arbitrary initial function, say $\eta : [-L, 0] \to \mathbb{R}$, Theorem 2.1 shows that $x(t, 0, \eta) \to r$. Thus, we may think of r as being a global attractor.

Remark 2.4. We may think of Theorem 2.1 as a stability result. In general, we know that solutions depend continuously on initial functions. That is, solutions which start close remain close on finite intervals. However, under the conditions in Theorem 2.1 such solutions remain close forever, and their asymptotic respective constants remain close too.

The next theorem is a verification of our claim in Remark 2.4.

Theorem 2.5. *Assume the hypotheses of Theorem 2.1. Then every initial function is stable. Moreover, if ψ_1 and ψ_2 are two initial functions with $x(t, 0, \psi_1) \to r_1$, and $x(t, 0, \psi_2) \to r_2$, then $|r_1 - r_2| < \epsilon$ for positive ϵ.*

Proof. Let $||\psi||_{[-L,0]}$ denote the supremum norm of ψ on the interval $[-L, 0]$. Fix an initial function ψ_1 and let ψ_2 be any other initial function. Let $\mathcal{P}_i, i = 1, 2$ be the mapping defined by (2.7). Then by Theorem 2.1 there are unique functions θ_1, θ_2 and unique constants r_1 and r_2 such that

$$\mathcal{P}_1\theta_1 \to \theta_1, \;\; \mathcal{P}_2\theta_2 \to \theta_2, \;\; \theta_1(t) \to r_1, \; \theta_2(t) \to r_2.$$

Let $\epsilon > 0$ be any given positive number and set $\delta = \dfrac{\epsilon(1-qL)}{1+qL}$. Then

$$\begin{aligned}
|\theta_1(t) - \theta_2(t)| &= |(\mathcal{P}_1\theta_1)(t) - (\mathcal{P}_2\theta_2)(t)| \\
&\leq |\psi_1(0) - \psi_2(0)| + \sum_{s=-L}^{-1} |g\big(\psi_1(s)\big) - g\big(\psi_2(s)\big)| \\
&+ \sum_{s=t-L}^{t-1} |g\big(\theta_1(s)\big) - g\big(\theta_2(s)\big)| \\
&\leq |\psi_1(0) - \psi_2(0)| + qL||\psi_1 - \psi_2||_{[-L,0]} + qL||\theta_1 - \theta_2||.
\end{aligned}$$

This yields

$$||\theta_1 - \theta_2|| < \frac{qL+1}{1-qL}||\psi_1 - \psi_2||_{[-L,0]} < \epsilon,$$

provided that

$$||\psi_1 - \psi_2||_{[-L,0]} < \frac{\epsilon(1-qL)}{1+qL} := \delta.$$

This shows that

$$|x(t,0,\psi_1) - x(t,0,\psi_2)| < \epsilon, \text{ whenever } ||\psi_1 - \psi_2||_{[-L,0]} < \delta.$$

For the rest of the proof we note that $|\theta_i(t) - k_i| \to 0$, as $t \to \infty$ implies that

$$\begin{aligned}
|r_1 - r_2| &= |r_1 - \theta_1(t) + \theta_1(t) - \theta_2(t) + \theta_2(t) - r_2| \\
&\leq |r_1 - \theta_1(t)| + ||\theta_1 - \theta_2|| + |\theta_2(t) - r_2| \to ||\theta_1 - \theta_2||, \text{ (as } t \to \infty) \\
&< \epsilon.
\end{aligned}$$

This completes the proof. □

Next we study the periodicity of solutions of (1.3)

$$\Delta x(t) = g(t, x(t)) - g(t, x(t-L)),$$

where,

$$g(t+L, x) = g(t, x) \tag{2.8}$$

for all x. Condition (2.8) allows us to put (1.3) into the form

$$\begin{aligned}
\Delta x(t) &= g(t, x(t)) - g(t-L, x(t-L)) \\
&= \Delta \sum_{s=t-L}^{t-1} g(s, x(s)).
\end{aligned} \tag{2.9}$$

Again, we assume that he function $g(t,x)$ is globally Lipschitz. That is, there exists a constant $q > 0$ such that

$$|g(t,x) - g(t,y)| \leq q|x-y|,$$

where (2.2) holds.

Theorem 2.6. *Assume conditions* (2.8) *and* (2.2)*. If* (1.3) *has an* L*-periodic solution, then that solution is constant.*

Proof. First, conditions (2.8) and (2.2) give the existence of a unique solution of (1.3). By summing (2.9) from $u = -1$ to $u = t-1$ we arrive at the expression

$$x(t) = x(-1) + \sum_{s=t-L}^{t-1} g(s, x(s)) - \sum_{s=-L}^{-1} g(s, x(s)). \tag{2.10}$$

Set

$$G(t) := g(t, x(t)).$$

Since $x(t+L) = x(t)$, we have by (2.8) that $G(t)$ satisfies $G(t+L) = G(t)$. This implies that the sum of G over any interval of length L is constant. In other words,

$$\sum_{s=t-L}^{t-1} g(s, x(s)) = \sum_{s=t-L}^{t-1} G(s) = \sum_{s=-L}^{-1} G(s) = \sum_{s=-L}^{-1} g(s, x(s)).$$

It then follows from (2.10) that $x(t) = x(-1)$ for all $t \geq -L$. Hence x is constant. □

3 Equation (1.2)

In this section we extend the results of the previous section to Equation (1.2). The term L_1 of Equation (1.2) can be seen as a lag time between conception and birth. Then the term $g(x(t-L_1))$ can be considered as the number of births at time t. Again, we will make use of the contraction mapping principle to show the behavior of the solutions of Equation (1.2) is the same as of Equation (1.1) and that there are no periodic solutions, except constants.

As before, we rewrite Equation (1.2) in the summed form

$$x(t) = \sum_{s=t-L_1-L_2}^{t-L_1-1} g\big(x(s)\big) + c, \tag{3.1}$$

where c is to be specified shortly. Note that if $c = 0$, then the summation term in (3.1) represents the number of individuals born in the past generation $[t-L_2, t-1]$ (conceived in $[t-L_1-L_2-1, t-L_1]$).

To fully describe the solutions of (1.2) we need to ask for an initial function $\psi : [-L_1-L_2, 0] \to \mathbb{R}$. By assuming condition (2.1) and the contraction condition

$$qL_2 \leq \xi, \ \ \xi < 1, \tag{3.2}$$

Equation (1.2) has the unique solution $x(t, 0, \psi)$ satisfying

$$x(t) = \psi(0) - \sum_{s=-L_1-L_2}^{-L_1-1} g\big(\psi(s)\big) + \sum_{s=t-L_1-L_2}^{t-L_1-1} g\big(x(s)\big). \tag{3.3}$$

We have the following result.

Theorem 3.1. *Assume* (2.1) *and* (3.2) *hold and let* $\psi : [-L_1 - L_2, 0] \to \mathbb{R}$ *be a given initial function. Then, the unique solution* $x(t, 0, \psi)$ *of* (1.2) *satisfies* $x(t, 0, \psi) \to r$, *where* r *is unique and satisfies*

$$r = \psi(0) + g(r)L_2 - \sum_{s=-L_1-L_2}^{-L_1-1} g\big(\psi(s)\big). \tag{3.4}$$

Proof. For $|\cdot|$ denoting the absolute value, the metric space $(\mathbb{R}, |\cdot|)$ is complete. Define a mapping $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{H}r = \psi(0) + g(r)L - \sum_{s=-L_1-L_2}^{-L_1-1} g\big(\psi(s)\big).$$

For $a, b \in \mathbb{R}$, we have

$$\big|\mathcal{H}a - \mathcal{H}b\big| \le L_2|g(a) - g(b)| \le L_2\, q|a - b| \le \xi|a - b|.$$

This shows that $\mathcal{H}$ is a contraction on the complete metric space $(\mathbb{R}, |\cdot|)$ and hence $\mathcal{H}$ has a unique fixed point r. Consequently, (3.4) has a unique solution. It remains to show that (1.2) has a unique solution and that it converges to the constant r.

Let $||\cdot||$ denote the maximum norm and let $\mathbb{M}$ be the set of bounded functions $\phi : [-L_1 - L_2, \infty) \to \mathbb{R}$ with $\phi(t) = \psi(t)$ on $[-L_1 - L_2, 0],\ \phi(t) \to r$ as $t \to \infty$. Then $(\mathbb{M}, ||\cdot||)$ defines a complete metric space. For $\phi \in \mathbb{M}$, define $\mathcal{P} : \mathbb{M} \to \mathbb{M}$ by

$$(\mathcal{P}\phi)(t) = \psi(t), \text{ for } -L_1 - L_2 \le t \le 0,$$

and

$$(\mathcal{P}\phi)(t) = \psi(0) - \sum_{s=-L_1-L_2}^{-L_1-1} g\big(\psi(s)\big) + \sum_{s=t-L_1-L_2}^{t-L_1-1} g\big(\phi(s)\big), \text{ for } t \ge 0. \tag{3.5}$$

For $\phi \in \mathbb{M}$ with $\phi(t) \to r$, we have $\sum_{s=t-L_1-L_2}^{t-L_1-1} g\big(\phi(s)\big) \to g(r)L_2$ as $t \to \infty$. Then, using (3.4) and (3.5), we see that

$$(\mathcal{P}\phi)(t) \to \psi(0) - \sum_{s=-L_1-L_2}^{-L_1-1} g\big(\psi(s)\big) + g(r)L_2 = r.$$

Thus, $\mathcal{P} : \mathbb{M} \to \mathbb{M}$. It remains to show that $\mathcal{P}$ is a contraction.
For $a, b \in \mathbb{M}$, we have

$$\begin{aligned} \big|(\mathcal{P}a)(t) - (\mathcal{P}b)(t)\big| &\le \sum_{s=t-L_1-L_2}^{t-L_1 1} |g(a(s)) - g(b(s))| \\ &\le qL_2||a - b|| \le \xi||a - b||. \end{aligned}$$

Thus, $\mathcal{P}$ is a contraction and has a unique fixed point $\phi \in \mathbb{M}$. Based on how the mapping $\mathcal{P}$ was constructed we deduce the unique fixed point ϕ satisfies (1.2). □

It is worth mentioning that a parallel theorem to Theorem 2.6 can be easily stated and proved for Equation (1.2).

We end this paper by adding an L-periodic function to the right hand side of Equation (1.3). Particularly, we consider

$$\Delta x(t) = -g(t, x(t)) + g(t, x(t-L)) + h(t),\ t \geq 0, \tag{3.6}$$

where

$$g(t+L, x) = g(t, x) \text{ and } h(t+L) = h(t), \tag{3.7}$$

$$\sum_{s=-L}^{-1} h(s) = 0, \tag{3.8}$$

and

$$|g(t, x) - g(t, y)| \leq q|x - y|, \tag{3.9}$$

for some positive constant q. We remark that Equation (3.6) has no constant solution, unless $h(t)$ is identically zero for all $t \in \mathbb{Z}$.

Let $\psi : [-L, 0] \to \mathbb{R}$ be a given initial function and put (3.6) in the form

$$x(t) = \psi(0) + \sum_{s=-L}^{-1} g\big(s, \psi(s)\big) - \sum_{s=t-L}^{t-1} g\big(s, x(s)\big) + \sum_{s=0}^{t-1} h(s). \tag{3.10}$$

We define the constant c by

$$c = \psi(0) + \sum_{s=-L}^{-1} g\big(s, \psi(s)\big). \tag{3.11}$$

We have the following result.

Theorem 3.2. *Assume* (2.2) *and* (3.7)–(3.9) *and let* $\psi : [-L, 0] \to \mathbb{R}$ *be a given initial function. Then, the unique solution* $x(t, 0, \psi)$ *of* (3.6) *satisfies* $x(t, 0, \psi) \to r(t)$, *where* $r(t)$ *is the unique* L*-periodic function satisfying*

$$r(t) = c - \sum_{s=t-L}^{t-1} g\big(s, r(s)\big) + \sum_{s=0}^{t-1} h(s). \tag{3.12}$$

Proof. Let $||\cdot||$ denote the maximum norm and let $\mathbb{M}$ be the set of L-periodic sequences $\phi : \mathbb{Z} \to \mathbb{Z}$. Then $(\mathbb{M}, ||\cdot||)$ defines a Banach space of L-periodic sequences. For $\phi \in \mathbb{M}$, define $\mathcal{P} : \mathbb{M} \to \mathbb{M}$ by

$$(\mathcal{P}\phi)(t) = c - \sum_{s=t-L}^{t-1} g\big(s, \phi(s)\big) + \sum_{s=0}^{t-1} h(s).$$

It is clear from conditions (3.7) and (3.8) that $\mathcal{P}$ maps $\mathbb{M}$ into $\mathbb{M}$. Also, due to conditions (2.2) and (3.9), the mapping $\mathcal{P}$ is a contraction. Hence, Equation (3.12) has a unique fixed point r in $\mathbb{M}$, which solves (3.6). It remains to show that $(\mathcal{P}\phi)(t) \to r(t)$.

Let $\mathbb{C}$ be the complete metric space of sequences $\phi : [-L, \infty) \to \mathbb{R}$ satisfying $\phi(t) = \psi(t)$ on $[-L, 0]$, $\phi(t) \to r(t)$ as $t \to \infty$. Define a mapping $\mathcal{P} : \mathbb{C} \to \mathbb{C}$ by

$$(\mathcal{P}\phi)(t) = \psi(t) \text{ on } [-L, 0],$$

and

$$(\mathcal{P}\phi)(t) = c - \sum_{s=t-L}^{t-1} g\big(s, \phi(s)\big) + \sum_{s=0}^{t-1} h(s), \quad t \geq 0.$$

It is clear from the definition of $\mathcal{P}$ that $(\mathcal{P}\phi)(0) = \psi(0)$ and hence $\mathcal{P} : \mathbb{C} \to \mathbb{C}$. Also, by (2.2) and (3.9) $\mathcal{P}$ is a contraction. Since $r(t)$ is a periodic sequence of period L, there exists a positive constant K such that $-K \leq r(t) \leq K$. Since g is periodic in t and is uniformly continuous in x and since $\phi(t) \to r(t)$, it follows that

$$\Big| \sum_{s=t-L}^{t-1} g\big(s, \phi(s)\big) - \sum_{s=t-L}^{t-1} g\big(s, r(s)\big) \Big| \to 0, \text{ as } t \to \infty.$$

This implies that

$$\big|(\mathcal{P}\phi)(t) - r(t)\big| = \Big| \sum_{s=t-L}^{t-1} g\big(s, \phi(s)\big) - \sum_{s=t-L}^{t-1} g\big(s, r(s)\big) \Big| \to 0,$$

as $t \to \infty$. □

References

[1] T. A. Burton, Fixed points and differential equations with asymptotically constant or periodic solutions, *Electron. J. Qual. Theory Differ. Equ.* 2004 (2004), no.11, 1–31.

[2] L. K. Cooke, Functional-differential equations; some models and perturbation problems, *Differential and Dynamical Systems,* Academic Press, New York (1967), 167–183.

[3] L. K. Cooke, An epidemic equation with immigration, *Math. Biosciences,* 29 (1976), 135–158.

[4] L. K. Cooke, and J. A. Yorke, Some equations modeling growth processs and gonorrhea epidemics, *Math. Biosciences,* 16 (1973), 75–101.

[5] S. Elaydi, Periodicity and stability of linear Volterra difference systems, *J. Math. Anal. Appl.* 181(1994), 483–492.

[6] Y. N. Raffoul, Positive periodic solutions of nonlinear functional difference equations, *Electron. J. Differential Equations,* 2002 (2002), no. 2, 339–351.

[7] Y. N. Raffoul, Stability and periodicity in completely delayed equations, *J. Math. Anal. Appl.,* 324 (2006), no. 2, 1356–1362.

[8] Y. N. Raffoul, and T. C. Tisdell, Positive periodic solutions of functional discrete systems and population models, *Adv. Difference Equ.,* 2005 (2005), 3, 369–380.