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Oscillation of Some Fourth-Order Difference Equations

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Abstract

We shall establish some new criteria for the oscillation of solutions of the fourth-order difference equation

$$\Delta^2\left(a(k)\left(\Delta^2 x(k)\right)^\alpha\right)+q(k)f\left(x\left(g(k)\right)\right)=0$$

with the property that $x(k)/k^2 \to 0$ as $k \to \infty$.

Keywords: Difference equation, fourth order, nonlinear, oscillation.
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1 Introduction

Consider the fourth-order nonlinear difference equation

$$\Delta^2\left(a(k)\left(\Delta^2 x(k)\right)^\alpha\right)+q(k)f\left(x\left(g(k)\right)\right)=0, \tag{1.1}$$

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where Δ is the forward difference operator defined by $\Delta x(k) = x(k+1) - x(k)$ and α is the ratio of positive odd integers. We assume that $a, q : \mathbb{N}_K \to (0, \infty)$ for some $K \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, where $\mathbb{N}_K = \{K, K+1, \dots\}$, $g : \mathbb{N}_K \to \mathbb{N}_0$ is nondecreasing such that $g(k) \le k$ for all $k \in \mathbb{N}_K$ and $\lim_{k\to\infty} g(k) = \infty$, and $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing satisfying $xf(x) > 0$ for $x \ne 0$.

By a solution of equation (1.1) we mean a nontrivial sequence $\{x(k)\}$ satisfying equation (1.1) for all $k \in \mathbb{N}_K$, where $K \in \mathbb{N}_0$. A solution $\{x(k)\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative, and it is called nonoscillatory otherwise. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The problem of determining the oscillation and nonoscillation of solutions of difference equations has been a very active area in the last three decades, and many references and summaries of known results can be found in the monographs by Agarwal et. al. [1,4,5]. The results of this paper complement those recently established in [2,3,6–9].

2 Main Results

We assume

$$\left.\begin{aligned} &\sum_{j=n_0\in\mathbb{N}_0}^{\infty} \left(\frac{1}{a(j)}\right)^{\frac{1}{\alpha}} = \infty, \\ \lim_{k\to\infty} \frac{1}{k^2} \sum_{s=n_0}^{k-1} \sum_{j=n_0}^{s-1} &\left(\frac{1}{a(j)}\right)^{\frac{1}{\alpha}} > 0, \quad \lim_{k\to\infty} \frac{1}{k^2} \sum_{s=n_0}^{k-1} \sum_{j=n_0}^{s-1} \left(\frac{j}{a(j)}\right)^{\frac{1}{\alpha}} > 0. \end{aligned}\right\} \tag{2.1}$$

Now we establish the following result.

Theorem 2.1. *Let condition* (2.1) *hold. If x is a nontrivial solution of equation* (1.1) *such that $x(k)/k^2 \to 0$ as $k \to \infty$, then*

$$x(k) > 0, \quad \Delta x(k) > 0, \quad \Delta^2 x(k) < 0, \quad \Delta\left(a(k)\left(\Delta^2 x(k)\right)^{\alpha}\right) > 0 \tag{2.2}$$

for $k \ge n_0 \in \mathbb{N}_0$ and

$$a(k)\left(\Delta^2 x(k)\right)^{\alpha} \to 0 \quad \textit{and} \quad \Delta\left(a(k)\left(\Delta^2 x(k)\right)^{\alpha}\right) \to 0 \quad \textit{as} \quad k \to \infty.$$

Proof. Let x be a nonoscillatory solution of equation (1.1), say, $x(k) > 0$ for $k \ge n_0$. Summing equation (1.1) from n_0 to $k-1 \ge n_0$, we obtain

$$\Delta\left(a(k)\left(\Delta^2 x(k)\right)^{\alpha}\right) = \Delta\left(a(n_0)\left(\Delta^2 x(n_0)\right)^{\alpha}\right) - \sum_{j=n_0}^{k-1} q(j) f\left(x\left(g(j)\right)\right).$$

We claim that $\Delta\left(a(n_0)\left(\Delta^2 x(n_0)\right)^{\alpha}\right) > 0$. To prove it, assume the contrary:

$$\Delta\left(a(n_0)\left(\Delta^2 x(n_0)\right)^{\alpha}\right) \le 0.$$

Then $\Delta\left(a(k)\left(\Delta^2 x(k)\right)^\alpha\right)$ is nonpositive and nonincreasing for $k \geq n_0$, and for some $n_1 > n_0 + 1$ we have

$$\Delta\left(a(n_1)\left(\Delta^2 x(n_1)\right)^\alpha\right) = \Delta\left(a(n_0)\left(\Delta^2 x(n_0)\right)^\alpha\right) - \sum_{j=n_0}^{n_1-1} q(j) f\left(x\left(g(j)\right)\right),$$

that is,

$$\Delta\left(a(k)\left(\Delta^2 x(k)\right)^\alpha\right) \leq \Delta\left(a(n_0)\left(\Delta^2 x(n_0)\right)^\alpha\right) < 0 \quad \text{for} \quad k \geq n_1.$$

Consequently,

$$a(k)\left(\Delta^2 x(k)\right)^\alpha \to -\infty \quad \text{as} \quad k \to \infty,$$

irrespective of $a(n_0)\left(\Delta^2 x(n_0)\right)^\alpha$. This in turn implies $\Delta x(k) \to -\infty$ as $k \to \infty$, and hence $x(k) \to -\infty$ as $k \to \infty$, contrary to the hypothesis that $x(k) > 0$ for $k \geq n_0$. This contradiction proves

$$\Delta\left(a(n_0)\left(\Delta^2 x(n_0)\right)^\alpha\right) > 0.$$

Since n_0 is arbitrary, we conclude that

$$\Delta\left(a(k)\left(\Delta^2 x(k)\right)^\alpha\right) > 0 \quad \text{for} \quad k \geq n_0.$$

Now it is easy to see that $\Delta\left(a(k)\left(\Delta^2 x(k)\right)^\alpha\right) \to 0$ as $k \to \infty$. If this is not true, then there exists a constant $C_1 > 0$ such that

$$\Delta\left(a(k)\left(\Delta^2 x(k)\right)^\alpha\right) > C_1 \quad \text{for} \quad k \geq n_2 \quad \text{for some} \quad n_2 \geq n_0.$$

However, this implies

$$x(k) \geq C \sum_{s=n_2}^{k-1} \sum_{i=n_0}^{s-1} \left(\frac{i}{a(i)}\right)^{1/\alpha}$$

for some constant $C > 0$ and $n_3 \geq n_2$, which contradicts the asymptotic behavior $\lim_{k\to\infty} x(k)/k^2 = 0$.

Next we shall prove that $\Delta^2 x(k) < 0$ for some $k \geq n_0$. If $a(n_0)\left(\Delta^2 x(n_0)\right)^\alpha > 0$, then $a(k)\left(\Delta^2 x(k)\right)^\alpha > 0$ for $k \geq n_0$, and there would exist constants $b_1 > 0$ and $\bar{n}_1 > n_0$ such that

$$a(k)\left(\Delta^2 x(k)\right)^\alpha > b_1 \quad \text{for} \quad k \geq \bar{n}_1.$$

However this again leads to the contradiction that

$$x(k) \geq b \sum_{s=n_0}^{k-1} \sum_{i=n_0}^{s-1} a^{1/\alpha}(i)$$

for some constant $b > 0$ and $\bar{n}_2 > \bar{n}_1$.

Moreover, we must have $a(k)\left(\Delta^2 x(k)\right)^\alpha \to 0$ as $k \to \infty$, for otherwise we would again be led to the contradiction that $x(k) \to -\infty$ as $k \to \infty$. Continuing this process, we deduce that $\Delta x(k) > 0$ for $k \geq n_0$. This completes the proof. □

In order to characterize the behavior of solutions, we reformulate Theorem 2.1 as follows.

Corollary 2.2. *Let condition* (2.1) *hold and let x be a nontrivial solution of equation* (1.1) *such that* $\lim_{k\to\infty} x(k)/k^2 = 0$*. Then either*

(I) *x is oscillatory on $[n_0, \infty)$, or else,*

(II) *$\Delta x(k) > 0$ ($\Delta x(k) < 0$) for $k \geq n_1$ for some $n_1 \geq n_0$ and $x(k)$ ($-x(k)$) satisfies the inequalities* (2.2) *of Theorem 2.1. In particular, x ($-x$) increases (decreases) monotonically for $k \geq n_0$.*

If x is a nontrivial solution of equation (1.1) such that $x(k) \to 0$ as $k \to \infty$, it cannot satisfy the inequalities in (2.2) of Theorem 2.1. Thus we conclude by Corollary 2.2 that such an x is oscillatory.

For $k \geq n_0 \in \mathbb{N}_0$, we let

$$Q(k) = \left(\frac{1}{a(k)} \sum_{s=k+1}^{\infty} \sum_{j=s+1}^{\infty} q(j) \right)^{1/\alpha}.$$

Now we shall present the following comparison result.

Theorem 2.3. *Let condition* (2.1) *hold. If the second-order difference equation*

$$\Delta^2 y(k) + Q(k) f^{1/\alpha}\left(y\left(g(k)\right)\right) = 0 \tag{2.3}$$

is oscillatory, then every solution x of equation (1.1) *such that $x(k)/k^2 \to 0$ as $k \to \infty$ is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1), say, $x(k) > 0$ for $k \geq n_0 \in \mathbb{N}_0$. By Theorem 2.1, x satisfies the inequalities (2.2). Summing equation (1.1) twice from $k+1 > n_0$ to u and letting $u \to \infty$, we get

$$-\Delta^2 x(k) \geq \left(\frac{1}{a(k)} \sum_{s=k+1}^{\infty} \sum_{j=s+1}^{\infty} q(j) f\left(x\left(g(j)\right)\right) \right)^{1/\alpha} \geq Q(k) f^{1/\alpha}\left(x\left(g(k)\right)\right) \tag{2.4}$$

for $k \geq n_0$. Summing both sides of (2.4) from $k+1 \geq n_0$ to $u \geq k+1$ and letting $u \to \infty$, we find

$$\Delta x(k) \geq \sum_{j=k+1}^{\infty} Q(j) f^{1/\alpha}\left(x\left(g(j)\right)\right). \tag{2.5}$$

Summing both sides of (2.5) from n_0 to $k-1 > n_0$, we have

$$x(k) \geq x(n_0) + \sum_{s=n_0}^{k-1} \sum_{j=s+1}^{\infty} Q(j) f^{1/\alpha}\left(x\left(g(j)\right)\right).$$

Now we define the sequence $\{y_m(k)\}$ by

$$\begin{aligned} y_0(k) &= x(k) \\ y_{m+1}(k) &= x(n_0) + \sum_{s=n_0}^{k-1} \sum_{j=s+1}^{\infty} Q(j) f^{1/\alpha}\left(y_m\left(g(j)\right)\right), \quad m \in \mathbb{N}_0,\ k \geq n_0. \end{aligned}$$

It is easy to check that the sequence $\{y_m(k)\}$ is well defined as an increasing sequence and satisfies

$$x(n_0) \leq y_m(k) \leq x(k) \quad \text{for} \quad k \geq n_0 \quad \text{and} \quad m \in \mathbb{N}_0.$$

Hence there exists a sequence $\{y(k)\}$ for $k \geq n_0$ such that

$$\lim_{m\to\infty} y_m(k) = y(k)$$

and

$$x(n_0) \leq y(k) \leq x(k) \quad \text{for} \quad k \geq n_0.$$

From Lebesgue's dominated convergence theorem, it follows that

$$x(k) = x(n_0) + \sum_{s=n_0}^{k-1} \sum_{j=s+1}^{\infty} Q(j) f^{1/\alpha}\left(x\left(g(j)\right)\right) \quad \text{for} \quad k \geq n_0.$$

Taking the difference twice, we conclude that x is nonoscillatory, which contradicts the hypotheses. This completes the proof. □

The following result is immediate.

Theorem 2.4. *Let condition* (2.1) *hold. Then every solution x of equation* (1.1) *such that $\lim_{k\to\infty} x(k)/k^2 = 0$ is oscillatory if one of the following conditions holds:*

$$\int^{\pm\infty} f^{-1/\alpha}(u)\mathrm{d}u < \infty \quad \textit{and} \quad \sum_{s=n_0}^{\infty} \Delta g(s) \sum_{j=s+1}^{\infty} Q(j) = \infty;$$

$$\limsup_{k\to\infty} \frac{1}{k^2} \sum_{s=n_0}^{k-1} \sum_{j=s+1}^{\infty} Q(j) > 1;$$

$$\int_{\pm 0} f^{-1/\alpha}(u)\mathrm{d}u < \infty \quad \textit{and} \quad \sum_{j=n_0}^{\infty} Q(j) f^{1/\alpha}\left(g(j)\right) = \infty.$$

Next, we shall establish the following result.

Theorem 2.5. *Let condition* (2.1) *hold and assume that there exists a nondecreasing sequence ξ such that $g(k) < \xi(k) < k - 1$ for $k \geq n_0$. Moreover, assume that*

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \textit{for} \quad xy > 0. \tag{2.6}$$

If there exist a constant $C \in (0, 1)$ and an $\bar{n}_0 > n_0$ such that all bounded solutions of the delay second-order difference equation

$$\Delta^2 y(k) - Cq(k) f\left((\xi(k) - g(k))\, a^{-1/\alpha}(\xi(k))\right) f(g(k))\, f\left(y^{1/\alpha}(\xi(k))\right) = 0$$

are oscillatory, then every solution x such that $\lim_{k\to\infty} x(k)/k^2 = 0$ is oscillatory.

Proof. Let x be a nonoscillatory solution of equation (1.1), say, $x(k) > 0$ for $k \geq n_0 \in \mathbb{N}_0$. By Theorem 2.1, we see that x satisfies (2.2). Thus there exist a constant $b > 0$ and an $n_1 \geq n_0$ such that

$$x(g(k)) \geq b g(k) \Delta x(g(k)) \quad \text{for} \quad k \geq n_1. \tag{2.7}$$

Using (2.6) and (2.7) in equation (1.1), we get

$$\Delta^2 \left(a(k) (\Delta y(k))^\alpha\right) + \bar{b} f(g(k))\, f(y(g(k))) \leq 0 \quad \text{for} \quad k \geq n_1, \tag{2.8}$$

where $y(k) = \Delta x(k)$ for $k \geq n_1$ and $\bar{b} = f(b)$. Clearly, we see that $y(k) > 0$, $\Delta y(k) < 0$ and $\Delta\left(a(k)(\Delta y(k))^\alpha\right) > 0$ for $k \geq n_1$. Now for $t \geq s \geq n_1$, we obtain

$$y(s) \geq (t - s)(-\Delta y(t)).$$

Replacing s and t by $g(k)$ and $\xi(k)$ respectively, we find

$$\begin{aligned} y(g(k)) &\leq (\xi(k) - g(k))(-\Delta y(\xi(k))) \\ &= \frac{\xi(k) - g(k)}{a^{1/\alpha}(\xi(k))} \left(-a(\xi(k))(\Delta y(\xi(k)))^\alpha\right)^{1/\alpha} \quad \text{for} \quad k \geq n_2 \geq n_1. \end{aligned} \tag{2.9}$$

Using (2.6) and (2.9) in (2.8), we have

$$\Delta^2 z(k) \geq \bar{b} q(k) f(g(k))\, f\left(\frac{\xi(k) - g(k)}{a^{1/\alpha}(\xi(k))}\right) f\left(z^{1/\alpha}(\xi(k))\right) \quad \text{for} \quad k \geq n_2,$$

where $z(k) = -a(k)(\Delta y(k))^\alpha$ for $k \geq n_2$. Using an argument similar to that in the proof of Theorem 2.3, we arrive at the desired contradiction. This completes the proof. □

The following result is immediate.

Theorem 2.6. *Let condition* (2.1) *hold and assume that there exists a nondecreasing sequence $\{\xi(k)\}$ such that $g(k) < \xi(k) < k$ for $k \geq n_0$. Then every solution x of equation* (1.1) *such that $\lim_{k\to\infty} x(k)/k^2 = 0$ is oscillatory if one of the following conditions holds:*

(i) $f(x) = x^\alpha$ *and either*

$$\limsup_{k\to\infty} \sum_{j=\xi(k)}^{k-1} q(j)g^\alpha(j) \left(\frac{(\xi(j) - g(j))^\alpha}{a(\xi(j))} \right) (\xi(k) - \xi(j)) > 1$$

or

$$\limsup_{k\to\infty} \sum_{\sigma=\xi(k)}^{k-1} \sum_{j=\sigma}^{k-1} q(j)g^\alpha(j) \frac{(\xi(j) - g(j))^\alpha}{a(\xi(j))} > 1;$$

(ii) $f(x) = x^\beta$, $\beta \in (0, \alpha)$ *is the ratio of positive odd integers, and either*

$$\limsup_{k\to\infty} \sum_{j=\xi(k)}^{k-1} q(j)g^\beta(j) \left(\frac{\xi(j) - g(j)}{a^{1/\alpha}(\xi(j))} \right)^\beta (\xi(k) - \xi(j)) > 0$$

or

$$\limsup_{k\to\infty} \sum_{\sigma=\xi(k)}^{k-1} \sum_{j=\sigma}^{k-1} q(j)g^\beta(j) \left(\frac{\xi(j) - g(j)}{a^{1/\alpha}(\xi(j))} \right)^\beta > 0.$$

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