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A Multiplicity Result for Nonautonomous Discrete Dirichlet Problems via Variational Methods

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Abstract

The aim of this paper is to present several new and general results concerning the existence and multiplicity of solutions for autonomous discrete Dirichlet problems. The approach is based on variational methods and critical points.

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1 Introduction

Let $\mathbb{N}$ be positive integers. Denote $\mathbb{N}(a) = \{a, a+1, a+2, \cdots\}$ and $\mathbb{N}(a, b) = \{a, a+1, \cdots, b\}$ with $a < b$ for any $a, b \in \mathbb{N}$. The purpose of this study is to obtain triple solutions of nonautonomous discrete Dirichlet boundary value problems

$$\begin{cases} \Delta^2 u(k-1) + \lambda f(k, u(k), \Delta u(k-1)) = 0, \quad k \in \mathbb{N}(a, b), \\ u(a-1) = u(b+1) = 0, \end{cases} \tag{P_λ}$$

where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator.

It is well known that variational methods are a powerful tool in dealing with many nonlinear Dirichlet boundary value problems for ordinary differential equation, see [4–7, 10, 17]. This technique is closely connected to the existence of critical points for a specific functional. Further, using the variational methods and critical points, one has a better, more complete, relevant, and accurate solution to the given equation, which is from the corresponding Dirichlet boundary value problem, such as [1, 3]. Recently, the theory of multiple critical points for a functional has been widely investigated for Dirichlet boundary value problems for difference equations. There is an increasing

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interest in the existence of solutions to the discrete Dirichlet problem which arise in a variety of different areas of applied mathematics and physics. Results on this topic are usually achieved by using various variational methods and theorems [2, 14] and the references therein. Further details can be found in, for example, [11, 12, 18]. A few authors have gradually paid more attention to applying critical point theory to deal with problems for nonlinear second-order discrete Dirichlet problem. Among the literature, very little work has been to show the existence of solutions for nonautonomous discrete Dirichlet boundary value problems. In [1], Agarwal et al have employed critical point theory to establish the existence of multiple solutions of some regular as well as singular nonautonomous discrete boundary value problem

$$\begin{cases} \Delta^2 y(k-1) + f(k, y(k)) = 0, \quad k \in [1, T], \\ y(0) = 0 = y(T+1). \end{cases}$$

Inspired by [8, 9, 13, 15, 16], we consider a nonautonomous discrete Dirichlet problem (P_λ) with assumptions on the nonlinearity. The goal of this paper is to establish a new and general result on the existence of multiple positive solutions to the problem (P_λ). Our results lead to new existence principles and generalize and improve several of the above cited results, for instance, in [1].

Theorem 1.1. *[2] Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Set*

$$\varphi_1(r) := \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\Psi(u) - \inf_{u \in \overline{\Phi^{-1}((-\infty, r))}^w} \Psi(u)}{r - \Phi(u)},$$

$$\varphi_2(r) := \inf_{u \in \Phi^{-1}((-\infty, r))} \sup_{v \in \Phi^{-1}([r, \infty))} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)},$$

for each $r > \inf_X \Phi$, where $\overline{\Phi^{-1}((-\infty, r))}^w$ is the closure of $\Phi^{-1}((-\infty, r))$ in the weak topology. Assume that

(i) $\lim_{\|u\| \to \infty} (\Phi(u) + \lambda \Psi(u)) = \infty$ *for all* $\lambda \in [0, \infty)$,

(ii) *there is $r \in \mathbb{R}$, such that $r > \inf_X \Phi$ and $\varphi_1(r) < \varphi_2(r)$.*

Then for each $\lambda \in \left(\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right)$, the equation

$$\Phi'(u) + \lambda \Psi'(u) = 0$$

has at least three critical points in X.

2 Main Results

Let $H = \{u : \mathbb{N}(a-1, b+1) \to \mathbb{R} : u(a-1) = 0 = u(b+1)\}$ be a $(b-a)$-dimensional Hilbert space with inner product $(u, v) = \sum_{k=a}^{b} u(k)v(k)$ for $u, v \in H$ and norm given by

$$\|u\| = \Big(\sum_{k=a}^{b+1} |\Delta u(k-1)|^2\Big)^{\frac{1}{2}}.$$

We define functionals $\Phi, \Psi : H \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\sum_{k=a}^{b+1} |\Delta u(k-1)|^2, \quad \Psi(u) = -\sum_{k=a}^{b} F(k, u(k)), \tag{2.1}$$

for any $u \in H$, where $F(k, \xi) = \int_a^{\xi} f(k, s, s')ds$ for $\xi \in \mathbb{R}$. Clearly, $\Phi, \Psi \in C^1(H, \mathbb{R})$. Making use of the summation by parts formula for each $u, h \in H$, we have

$$\begin{aligned}
\Phi'(u)(h) &= \lim_{t\to 0} \frac{\Phi(u+th) - \Phi(u)}{t} \\
&= \sum_{k=a}^{b+1} \Delta u(k-1)\Delta h(k-1) \\
&= \sum_{k=a}^{b} \Delta u(k-1)\Delta h(k-1) - u(b)\Delta h(b) \\
&= \Delta u(k-1)h(k-1)|_a^{b+1} - \sum_{k=a}^{b} \Delta^2 u(k-1)h(k) - u(b)\Delta h(b) \\
&= -\sum_{k=a}^{b} \Delta^2 u(k-1)h(k).
\end{aligned}$$

Furthermore, for each $u, h \in H$, we get

$$\Psi'(u)(h) = \lim_{t\to 0} \frac{\Psi(u+th) - \Psi(u)}{t} = -\sum_{k=a}^{b} f(k, u(k))h(k).$$

Thus, for each $k \in \mathbb{N}(a, b)$ and $u, h \in H$, the relation

$$(\Phi + \lambda\Psi)'(u)(h) = -\sum_{k=a}^{b} \big(\Delta^2 u(k-1) + \lambda f(k, u(k), \Delta u(k-1))\big)h(k) = 0$$

is equivalent to

$$\Delta^2 u(k-1) + \lambda f\big(k, u(k), \Delta u(k-1)\big) = 0.$$

Thus, the critical points in H of the equation $\Phi'(u) + \lambda\Psi'(u) = 0$ are precisely the solutions of the problem (P_λ). Therewith we will apply Theorem 1.1 to the problem (P_λ).

Lemma 2.1. *If* $u \in H$, *then*

$$\max_{k\in\mathbb{N}(a,b)} |u(k)| \leq \frac{(b+1)^{\frac{1}{2}}}{2}\|u\|. \tag{2.2}$$

Proof. Let $k_0 \in \mathbb{N}(a,b)$ such that $|u(k_0)| = \max_{k\in\mathbb{N}(a,b)} |u(k)|$. Since $u(a-1) = u(b+1) = 0$ for each $u \in H$, by Hölder's inequality, we have

$$\begin{aligned} |u(k_0)| &= \Big|\sum_{k=a}^{k_0} \Delta u(k-1)\Big| \leq \sum_{k=a}^{k_0} |\Delta u(k-1)| \\ &\leq (k_0 - a + 1)^{\frac{1}{2}}\Big(\sum_{k=a}^{k_0} |\Delta u(k-1)|^2\Big)^{\frac{1}{2}}, \end{aligned} \tag{2.3}$$

$$|u(k_0)| \leq \sum_{k=k_0+1}^{b+1} |\Delta u(k-1)| \leq (b - k_0 + 1)^{\frac{1}{2}}\Big(\sum_{k=k_0+1}^{b+1} |\Delta u(k-1)|^2\Big)^{\frac{1}{2}}. \tag{2.4}$$

If $\sum_{k=a}^{k_0} |\Delta u(k-1)|^2 \leq \frac{b+1}{4(k_0-a+1)}\|u\|^2$, combining (2.3), we obtain (2.2). If $\sum_{k=a}^{k_0} |\Delta u(k-1)|^2 > \frac{b+1}{4(k_0-a+1)}\|u\|^2$, then $\sum_{k=k_0+1}^{b+1} |\Delta u(k-1)|^2 = \|u\|^2 - \sum_{k=a}^{k_0} |\Delta u(k-1)|^2 < \big(1 - \frac{b+1}{4(k_0-a+1)}\big)\|u\|^2$, combining (2.4), we get

$$|u(k_0)| < (b - k_0 + 1)^{\frac{1}{2}}\Big(1 - \frac{b+1}{4(k_0-a+1)}\Big)^{\frac{1}{2}}\|u\|.$$

Now we claim that the inequality

$$(b - k_0 + 1)\Big(1 - \frac{b+1}{4(k_0-a+1)}\Big) \leq \frac{b+1}{4} \tag{2.5}$$

holds. In fact, we define a continuous function $w : (a-1, b+1) \to \mathbb{R}$ by $w(t) = \frac{1}{b-t+1} + \frac{1}{t}$. This function $w(t)$ can attain its minimum $\frac{4}{b+1}$ at $t = \frac{b+1}{2}$. Since

$k_0 \in \mathbb{N}(a,b)$, we have $w(k_0) \geq \dfrac{4}{b+1}$. That is, $\dfrac{4}{b+1} \leq \dfrac{1}{b-k_0+1} + \dfrac{1}{k_0}$. Further, it is not difficult to check that $\dfrac{k_0}{b+1} \geq 1 - \dfrac{b+1}{4(k_0-a+1)}$. Therefore, the inequality (2.5) holds. □

Theorem 2.2. *Assume that $f(k,u,v) \in C\big(\mathbb{N}(a,b)\times\mathbb{R}^2,\mathbb{R}\big)$, $f(t,u,v) \not\equiv 0$ on $\mathbb{N}(a,b)\times\mathbb{R}^2$, and there exist four positive constants α, β, γ, δ with $\alpha < (2(b+1))^{\frac{1}{2}}\beta$ and $\delta < 2$ such that*

(A$_1$) $\max_{|\xi|\leq\alpha} F(k,\xi) < \dfrac{\alpha^2}{\alpha^2+2(b+1)\beta^2}\displaystyle\sum_{k=a}^{b} F(k,2\beta), \quad (k,\xi) \in \mathbb{N}(a,b)\times[-\alpha,\alpha],$

(A$_2$) $F(k,\xi) \leq \gamma(1+|\xi|^\delta), \quad (k,\xi) \in \mathbb{N}(a,b)\times\mathbb{R}.$

Then for each $\lambda \in \Big(\dfrac{4\beta^2}{\sum_{k=a}^{b} F(k,2\beta) - \max_{|\xi|\leq\alpha} F(k,\xi)}, \dfrac{2\alpha^2}{(b+1)\max_{|\xi|\leq\alpha} F(k,\xi)}\Big)$, the problem ($P_\lambda$) has at least three nontrivial solutions.

Proof. Let H be the Hilbert space defined in Section 2 and Φ, Ψ be as defined by (2.1), $\Phi : H \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on H^*, $\Psi : H \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Now, set $u_0(k) = 0$ for each $k \in \mathbb{N}(a-1,b+1)$, it is easy to check that $u_0 \in H$ and $\Phi(u_0) = \Psi(u_0) = 0$. It follows from $\delta < 2$, assumption (A_2) and Lemma 2.1 that for each $u \in H$,

$$\begin{aligned}\Phi(u)+\lambda\Psi(u) &= \frac{1}{2}\sum_{k=a}^{b+1}|\Delta u(k-1)|^2 - \lambda\sum_{k=a}^{b}F(k,u(k))\\ &\geq \frac{1}{2}\|u\|^2 - \lambda\gamma\sum_{k=a}^{b}(1+|u(k)|^\delta).\end{aligned}$$

Thus, we obtain that $\lim_{\|u\|\to\infty}(\Phi(u)+\lambda\Psi(u)) = \infty$ for all $\lambda \in [0,\infty)$. Hence, condition (i) of Theorem 1.1 holds. In order to prove (ii) of Theorem 1.1, we claim that

$$\varphi_1(r) \leq \frac{\max_{|\xi|\leq\sqrt{\frac{r(b+1)}{2}}} F(k,\xi)}{r}, \tag{2.6}$$

$$\varphi_2(r) \geq 2\frac{\sum_{k=a}^{b} F(k,v(k)) - \max_{|\xi|\leq\sqrt{\frac{r(b+1)}{2}}} F(k,\xi)}{\|v\|^2}, \tag{2.7}$$

for each $r > 0$ and for any $v \in H$. In fact, taking into account that $\overline{\Phi^{-1}((-\infty,r))^w} = \Phi^{-1}((-\infty,r))$ and by the definition of r, it follows that

$$\Phi^{-1}((-\infty,r]) \subseteq \{u \in H : |u(k) \leq \alpha, \forall k \in \mathbb{N}(a,b)\}.$$

For $u \in H$, $r > 0$ and $v \in H$ such that $\frac{\|v\|^2}{2} \geq r$, by (2.2), we have

$$\begin{aligned}
\varphi_1(r) &= \inf_{u \in \Phi^{-1}((-\infty,r))} \frac{\Psi(u) - \inf_{u \in \overline{\Phi^{-1}((-\infty,r))}^w} \Psi(u)}{r - \Phi(u)} \\
&\leq \frac{-\inf_{u \in \overline{\Phi^{-1}((-\infty,r))}^w} \Psi(u)}{r} \\
&\leq \frac{\sup_{\frac{\|u\|^2}{2} \leq r} \sum_{k=a}^{b} F(k, u(k))}{r} \\
&\leq \frac{\max_{|\xi| \leq \sqrt{\frac{r(b+1)}{2}}} F(k, \xi)}{r}, \\
\varphi_2(r) &= \inf_{u \in \Phi^{-1}((-\infty,r))} \sup_{v \in \Phi^{-1}([r,\infty))} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)} \\
&\geq 2 \inf_{u \in \Phi^{-1}((-\infty,r))} \frac{\sum_{k=a}^{b} F(k, v(k)) - \sum_{k=a}^{b} F(k, u(k))}{\|v\|^2 - \|u\|^2} \\
&\geq 2 \inf_{\frac{\|u\|^2}{2} \leq r} \frac{\sum_{k=a}^{b} F(k, v(k)) - \sum_{k=a}^{b} F(k, u(k))}{\|v\|^2 - \|u\|^2} \\
&\geq 2 \inf_{\frac{\|u\|^2}{2} \leq r} \frac{\sum_{k=a}^{b} F(k, v(k)) - \max_{|\xi| \leq \sqrt{\frac{r(b+1)}{2}}} F(k, \xi)}{\|v\|^2 - \|u\|^2} \\
&\geq 2 \frac{\sum_{k=a}^{b} F(k, v(k)) - \max_{|\xi| \leq \sqrt{\frac{r(b+1)}{2}}} F(k, \xi)}{\|v\|^2},
\end{aligned}$$

since $\sum_{k=a}^{b} F(k, v(k)) \geq \max_{|\xi| \leq \sqrt{\frac{r(b+1)}{2}}} F(k, \xi)$. Hence, (2.6) and (2.7) hold. Now combining with (2.6) and (2.7), it suffices to find $r > 0$ and $v \in H$ with $\frac{1}{2}\|v\|^2 \geq r$ such that

$$\max_{|\xi| \leq \sqrt{\frac{r(b+1)}{2}}} F(k, \xi) < \frac{r\|v\|^2}{r + \|v\|^2} \sum_{k=a}^{b} F(k, v(k)). \tag{2.8}$$

Set

$$v(k) = \begin{cases} 0, & k = a - 1, \\ 2\beta, & k \in \mathbb{N}(a, b), \\ 0, & k = b + 1, \end{cases} \qquad r = \frac{2\alpha^2}{b + 1}.$$

It is clear that $v \in H$ and $\|v\|^2 = 8\beta^2$, $\Phi(v) = 4\beta^2$, $\Psi(v) = \sum_{k=a}^{b} F(k, 2\beta)$. Since

$\alpha < (2(b+1))^{\frac{1}{2}}\beta$, for $v \in H$ with $\dfrac{1}{2}\|v\|^2 \geq r$, by (2.8), we obtain that

$$\max_{|\xi|\leq\alpha} F(k,\xi) < \frac{\alpha^2}{\alpha^2 + 2(b+1)\beta^2} \sum_{k=a}^{b} F(k, 2\beta).$$

From the above inequality, we notice that

$$\frac{b+1}{\alpha^2} \max_{|\xi|\leq\alpha} F(k,\xi) \leq \frac{\sum_{k=a}^{b} F(k,2\beta) - \max_{|\xi|\leq\alpha} F(k,\xi)}{2\beta^2}. \tag{2.9}$$

On the other hand, we have

$$\varphi_1(r) \leq \frac{\max_{|\xi|\leq\sqrt{\frac{r(b+1)}{2}}} F(k,\xi)}{r} = \frac{b+1}{2\alpha^2} \max_{|\xi|\leq\alpha} F(k,\xi),$$

and

$$\begin{aligned}\varphi_2(r) &\geq 2\frac{\sum_{k=a}^{b} F(k,v(k)) - \max_{|\xi|\leq\sqrt{\frac{r(b+1)}{2}}} F(k,\xi)}{\|v\|^2} \\ &= \frac{\sum_{k=a}^{b} F(k,2\beta) - \max_{|\xi|\leq\alpha} F(k,\xi)}{4\beta^2}.\end{aligned}$$

It follows from (2.9) that $\varphi_1(r) < \varphi_2(r)$ for any $r > \inf_X \Phi$. Hence, the conditions of Theorem 1.1 are fulfilled, thus for each $\lambda \in \Big(\dfrac{4\beta^2}{\sum_{k=a}^{b} F(k,2\beta) - \max_{|\xi|\leq\alpha} F(k,\xi)},$ $\dfrac{2\alpha^2}{(b+1)\max_{|\xi|\leq\alpha} F(k,\xi)}\Big)$, the equation $\Phi'(u) + \lambda\Psi'(u) = 0$ has at least three nontrivial critical points in H, and so the problem (P_λ) has at least three nontrivial solutions. □

Example 2.3. Consider the following discrete Dirichlet boundary value problems

$$\begin{cases} \Delta^2 u(k-1) + \lambda f(k,u(k),\Delta u(k-1)) = 0, & k \in \mathbb{N}(1,7), \\ u(0) = u(8) = 0. \end{cases} \tag{2.10}$$

Take $F(k,u) = \dfrac{u^{2/3}}{2(1+k^2)} + 1$, where $F(k,\xi) = \displaystyle\int_0^{\xi} f(k,s,s')ds$ for $\xi \in \mathbb{R}$. Let $\alpha = \gamma = 1$, $\beta = \dfrac{1}{2}$ and $\delta = \dfrac{2}{3}$. It is easy to check that the assumptions (A_1) and (A_2) of Theorem 2.2 are satisfied, and so $\lambda \in \big(\dfrac{1}{6.225}, \dfrac{1}{5}\big) \subseteq \Big(\dfrac{1}{\sum_{k=1}^{7} F(k,1) - \max_{|\xi|\leq 1} F(k,\xi)},$ $\dfrac{1}{4\max_{|\xi|\leq 1} F(k,\xi)}\Big)$. Due to Theorem 1.1, problem (2.10) has at least three solutions.

For the case when $\lambda = 1$, we present the following result.

Theorem 2.4. *Assume that $f(k,u,v) \in C\big(\mathbb{N}(a,b)\times\mathbb{R}^2,\mathbb{R}\big)$, $f(t,u,v) \not\equiv 0$ on $\mathbb{N}(a,b)\times\mathbb{R}^2$, and there exist four positive constants $\alpha,\beta,\gamma,\delta$ with $\alpha < (2(b+1))^{\frac{1}{2}}\beta$ and $\delta < 2$ such that*

(B$_1$) $\max\limits_{|\xi|\le\alpha} F(k,\xi) < \min\big\{\dfrac{2\alpha^2}{b+1}, \sum\limits_{k=a}^{b} F(k,2\beta) - \dfrac{8\alpha^2\beta^2}{b+1}\big\}$, $(k,\xi)\in\mathbb{N}(a,b)\times[-\alpha,\alpha]$,

(B$_2$) $F(k,\xi) \le \gamma(1+|\xi|^{\delta})$, $(k,\xi)\in\mathbb{N}(a,b)\times\mathbb{R}$.

Then problem (P_1) has at least three nontrivial solutions.

Proof. It is easy to see that Theorem 2.2 can be used. We observe that, by assumption (B_1), $1 \in \big(\dfrac{4\beta^2}{\sum_{k=a}^{b} F(k,2\beta) - \max_{|\xi|\le\alpha} F(k,\xi)}, \dfrac{2\alpha^2}{(b+1)\max_{|\xi|\le\alpha} F(k,\xi)}\big)$. □

Example 2.5. Consider the following discrete Dirichlet boundary value problems

$$\begin{cases} \Delta^2 u(k-1) + f(k,u(k),\Delta u(k-1)) = 0, & k\in\mathbb{N}(1,7),\\ u(0)=u(8)=0. \end{cases} \tag{2.11}$$

Take $F(k,u) = \dfrac{3}{32}\big(1+u^{\frac{2}{3}}\big)$. Let $\alpha=\gamma=1$, $\beta=\dfrac{1}{2}$ and $\delta=\dfrac{2}{3}$, and satisfy $\alpha < (2(b+1))^{\frac{1}{2}}\beta$. It is easy to check that the assumptions (B_1) and (B_2) of Theorem 2.4 are satisfied, and so $1 \in (\dfrac{8}{9},\dfrac{4}{3}) \subseteq \big(\dfrac{4}{\sum_{k=1}^{7} F(k,1) - \max_{|\xi|\le 1} F(k,\xi)}, \dfrac{1}{4\max_{|\xi|\le1} F(k,\xi)}\big)$. Due to Theorem 1.1, problem (2.11) has at least three nontrivial solutions.

Remark 2.6. If $f(k,u,v)\in C\big(\mathbb{N}(a,b)\times\mathbb{R}^+\times\mathbb{R},\mathbb{R}^+\big)$ and $\lambda>0$, then solution of problem (P_λ) is concave, and there is a prior estimate

$$\max_{a\le k\le b} u(k) \le (b-a+2)\max\big\{\Delta u(a-1), -\Delta u(b)\big\}.$$

Corollary 2.7. *Assume that $f(k,u,v)\in C\big(\mathbb{N}(a,b)\times\mathbb{R}^+\times\mathbb{R},\mathbb{R}^+\big)$, $f(t,u,v)\not\equiv 0$ on $\mathbb{N}(a,b)\times\mathbb{R}^+\times\mathbb{R}$, and there exist four positive constants $\alpha,\beta,\gamma,\delta$ with $\alpha < (2(b+1))^{\frac{1}{2}}\beta$ and $\delta<2$ such that*

(A$_1$) $\max\limits_{0\le\xi\le\alpha} F(k,\xi) < \dfrac{\alpha^2}{\alpha^2+2(b+1)\beta^2}\sum\limits_{k=a}^{b} F(k,2\beta)$, $\quad (k,\xi)\in\mathbb{N}(a,b)\times[0,\alpha]$,

(A$_2$) $F(k,\xi)\le\gamma(1+\xi^{\delta})$, $\quad (k,\xi)\in\mathbb{N}(a,b)\times\mathbb{R}^+$.

Then for each $\lambda \in \big(\dfrac{4\beta^2}{\sum_{k=a}^{b} F(k,2\beta) - \max_{0\le\xi\le\alpha} F(k,\xi)}, \dfrac{2\alpha^2}{(b+1)\max_{0\le\xi\le\alpha} F(k,\xi)}\big)$, problem (P_λ) has at least three positive solutions.

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