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Periodic Solutions of Nonlinear Dynamic Systems with Feedback Control

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Abstract

In this paper, sufficient criteria for the existence of multiple positive periodic solutions of a certain nonlinear dynamic system with feedback control are established. This is done by the Avery–Henderson fixed point theorem and the Leggett–Williams fixed point theorem. By using the method of coincidence degree, sufficient conditions are derived ensuring the existence of at least one periodic solution of a more general nonlinear dynamic system with feedback control on time scales.

AMS Subject Classifications: Dynamic equations, periodic solution, nonlinear, feedback control, time scales, fixed point theorem, coincidence degree.
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1 Introduction

It is well known that the diversity of biological phenomena determines the complexity of biological and mathematical models. In investigating biological phenomena, most natural environments are physically highly variable in time. Theoretical evidence to date suggests that many population and community patterns represent intricate interactions between biology and variation in the physical environment, which are a major driver of population fluctuations (see [8] and other papers in the same issue). When the environmental fluctuations are taken into account, a model must be nonautonomous, and hence, of course, more difficult to analyze in general. But, in doing so, one can and should also take advantage of the properties of those varying parameters. Some periodically varying parameters are important choices in simulating intricate interactions between population change and its periodicity physical environment (such as seasonal effects of weather, food supplies, mating habits and so on). Moreover, as we know, the ecosystem in the real world is continuously distributed by unpredictable forces which can result in changes of the biological parameters such as survival rates. So it is necessary to study the question of wether or not an ecological system can withstand those unpredictable disturbances which persist for a finite period of time. Therefore, population models with feedback control have very strong real-world motivations and have been extensively explored by many authors ([7,12,16] and the references cited therein).

In this paper, we prove some theorems related to the existence of periodic solutions of nonlinear dynamic systems with feedback control on time scales. The theory of calculus on time scales (see [5] for more details) was initiated by Stefan Hilger in his PhD thesis [11] in order to unify continuous and discrete analysis. A dynamic equation on a time scale is related not only to the set of real numbers (continuous time scale, differential equations) and the set of integers (discrete time scale, difference equations) but also to those pertaining to more general time scales. Recently, this area has received a lot of attention and has a tremendous potential applications in the study of population dynamics, wound healing, mathematical epidemiology [5, 13, 17]. In addition, there exist some papers in the study of periodic solutions of population dynamics on time scales [3,4,6,9,15,18].

This paper is organized as follows. In the next section, for the reader's convenience, we will present some basic results from the calculus on time scales [5] and some fixed point theorems. Section 3 and Section 4 focus on establishing some sufficient criteria for the existence of multiple periodic solutions of a kind of nonlinear dynamic system with feedback control. Finally, in Section 5, sufficient conditions are derived ensuring the existence of at least one periodic solution of a more general nonlinear dynamic system with feedback control on time scales.

2 Preliminaries

In this section, we first introduce some basic results of the calculus on time scales so that the paper is self contained. For more details, one can see [5, 11].

Let $\mathbb{T}$ be a *time scale*, i.e., an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Throughout this paper, the time scale $\mathbb{T}$ is assumed to be unbounded above and below. Define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

and

$$\mu(t) = \sigma(t) - t \quad \text{for} \quad t \in \mathbb{T},$$

respectively. If $\sigma(t) = t$, then t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered). Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

In this case, $f^{\Delta}(t)$ is called the *delta derivative* of f at t. Moreover, f is said to be *delta differentiable* on $\mathbb{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. Then we define

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for} \quad s, r \in \mathbb{T}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exists (finite) at all left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathrm{C_{rd}}(\mathbb{T})$. A function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of such regressive and rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.1. If $p \in \mathcal{R}$, then the exponential function is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \text{ with } \xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1 + hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0, \end{cases}$$

where $t, s \in \mathbb{T}$ and Log is the principal logarithm.

Lemma 2.2. (i) *If f is delta differentiable at $t \in \mathbb{T}$, then f is continuous at t and $f^{\sigma} = f + \mu f^{\Delta}$ at t, where $f^{\sigma} = f \circ \sigma$.*

(ii) *If $p \in \mathcal{R}$ and $t, s, r \in \mathbb{T}$, then*

$$e_p(t,t) \equiv 1, \quad e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t),$$

where $\ominus p = -1/(1+\mu p)$, and

$$e_p(t,s)e_p(s,r) = e_p(t,r), \quad e_p^{\Delta}(\cdot,s) = pe_p(\cdot,s), \quad e_{\ominus p}^{\Delta}(\cdot,s) = -pe_{\ominus p}^{\sigma}(\cdot,s).$$

In this paper, the time scale $\mathbb{T}$ is assumed to be ω-*periodic*, i.e., $t \in \mathbb{T}$ implies $t \pm \omega \in \mathbb{T}$. This implies that the graininess μ is also ω-periodic. To facilitate the discussion below, we now introduce some notations to be used throughout this paper. Let

$$\kappa = \min\{[0,\infty) \cap \mathbb{T}\}, \quad I_\omega = [\kappa, \kappa+\omega] \cap \mathbb{T}, \quad g^l = \sup_{t\in\mathbb{T}} g(t), \quad g^s = \inf_{t\in\mathbb{T}} g(t),$$
$$\overline{g} = \frac{1}{\omega}\int_{I_\omega} g(s)\Delta s = \frac{1}{\omega}\int_{\kappa}^{\kappa+\omega} g(s)\Delta s,$$

where $g \in \mathrm{C_{rd}}(\mathbb{T})$ is an ω-periodic real function, i.e., $g(t+\omega) = g(t)$ for all $t \in \mathbb{T}$.

Next, let us recall some basic concepts, the well-known Avery–Henderson fixed point theorem [1] and Leggett–Williams fixed point theorem [14].

Let X be a real Banach space and P be a cone in X. An order is introduced in P by $\leq$, i.e., $x \leq y$ if and only if $y - x \in P$. If a map $\varrho : P \to [0,\infty)$ is a nonnegative continuous functional, then ϱ is said to be increasing if $\varrho(x) \leq \varrho(y)$ for all $x, y \in P$ and $x \leq y$ and is said to be concave if $\varrho(tx + (1-t)y) \geq t\varrho(x) + (1-t)\varrho(y)$ for all $x, y \in P$ and $t \in [0,1]$. For three positive constant numbers d, r, R and $r < R$, we define the following sets:

$$P(\varrho_1, d) = \{x \in P : \varrho_1(x) < d\}, \quad \partial P(\varrho_1, d) = \{x \in P : \varrho_1(x) = d\},$$
$$\overline{P(\varrho_1, d)} = \{x \in P : \varrho_1(x) \leq d\}, P_r = \{x \in P : \|x\| < r\},$$
$$\overline{P_r} = \{x \in P : \|x\| \leq r\}, \quad P(\varrho_2, r, R) = \{x \in P : r \leq \varrho_2(x), \|x\| \leq R\},$$

where ϱ_1 is a nonnegative continuous increasing functional and ϱ_2 is a nonnegative continuous concave functional.

Lemma 2.3 (Avery and Henderson [1]). *Let P be a cone in a Banach space X. Let α and γ be nonnegative continuous increasing functionals on P, and let β be a nonnegative continuous functional on P with $\beta(0) = 0$ such that for some $c > 0$ and $M > 0$,*

$$\gamma(x) \leq \beta(x) \leq \alpha(x) \quad \text{and} \quad \|x\| \leq M\gamma(x) \quad \text{for all} \quad x \in \overline{P(\gamma, c)}.$$

Suppose there exists a completely continuous operator $T : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that

$$\beta(\pi x) \leq \pi\beta(x) \quad \text{for} \quad 0 \leq \pi \leq 1 \quad \text{and} \quad x \in \partial P(\beta, b)$$

and

(i) $\gamma(Tx) > c$ *for all* $x \in \partial P(\gamma, c)$*;*

(ii) $\beta(Tx) < b$ *for all* $x \in \partial P(\beta, b)$*;*

(iii) $P(\alpha, a) \neq \emptyset$ *and* $\alpha(Tx) > a$ *for* $x \in \partial P(\alpha, a)$.

Then T *has at least two fixed points* x_1 *and* x_2 *belonging to* $\overline{P(\gamma, c)}$ *such that*

$$a < \alpha(x_1), \;\; \beta(x_1) < b, \;\; b < \beta(x_2), \;\; \gamma(x_2) < c.$$

The following lemma can be found in [16].

Lemma 2.4. *Let* P *be a cone in a Banach space* X*. Let* α *and* γ *be nonnegative continuous increasing functionals on* P*, and let* β *be a nonnegative continuous functional on* P *with* $\beta(0) = 0$ *such that for some* $c > 0$ *and* $M > 0$*,*

$$\gamma(x) \leq \beta(x) \leq \alpha(x) \quad \textit{and} \quad \|x\| \leq M\gamma(x) \quad \textit{for all} \quad x \in \overline{P(\gamma, c)}.$$

Suppose there exists a completely continuous operator $T : \overline{P(\gamma, c)} \to P$ *and* $0 < a < b < c$ *such that*

$$\beta(\pi x) \leq \pi\beta(x) \quad \textit{for} \quad 0 \leq \pi \leq 1 \quad \textit{and} \quad x \in \partial P(\beta, b)$$

and

(i) $\gamma(Tx) < c$ *for all* $x \in \partial P(\gamma, c)$*;*

(ii) $\beta(Tx) > b$ *for all* $x \in \partial P(\beta, b)$*;*

(iii) $P(\alpha, a) \neq \emptyset$ *and* $\alpha(Tx) < a$ *for* $x \in \partial P(\alpha, a)$.

Then T *has at least two fixed points* x_1 *and* x_2 *belonging to* $\overline{P(\gamma, c)}$ *such that*

$$a < \alpha(x_1), \;\; \beta(x_1) < b, \;\; b < \beta(x_2), \;\; \gamma(x_2) < c.$$

Finally we state the Leggett–Williams fixed point theorem.

Lemma 2.5 (Leggett and Williams [14])**.** *Let* $T : \overline{P_R} \to \overline{P_R}$ *be completely continuous and* ϕ *be a nonnegative continuous concave functional on* P *such that* $\phi(x) \leq \|x\|$ *for all* $x \in \overline{P_R}$*. Suppose there exist positive constants* r, r_1, r_2, R *with* $0 < r < r_1 < r_2 \leq R$ *such that*

(i) $\{x \in P(\phi, r_1, r_2) : \phi(x) > r_1\} \neq \emptyset$ *and* $\phi(Tx) > r_1$ *for* $x \in P(\phi, r_1, r_2)$*;*

(ii) $\|Tx\| < r$ *for* $x \in \overline{P_r}$*;*

(iii) $\phi(Tx) > r_1$ *for* $x \in P(\phi, r_1, R)$ *with* $\|Tx\| > r_2$.

Then T *has at least three fixed points* x_1, x_2, x_3 *satisfying*

$$x_1 \in P_r, \; x_2 \in \{x \in P(\phi, r_1, R) : \phi(x) > r_1\} \quad \textit{and} \quad x_3 \in \overline{P_R} \setminus (P(\phi, r_1, R) \cup \overline{P_r}).$$

3 Two Positive Periodic Solutions

The purpose of this section is to study the periodicity of the nonlinear dynamic system with feedback control on a general time scale

$$\begin{aligned} x^\Delta(t) &= r(t)x(t) - f(t, x(t), u(t)), \\ u^\Delta(t) &= -\delta(t)u^\sigma(t) + \eta(t)x(t), \end{aligned} \tag{3.1}$$

where $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ and $r, \delta, \eta : \mathbb{T} \to (0, \infty)$ are all rd-continuous and ω-periodic for $t \in \mathbb{T}$, and $\omega > 0$ is called the period of (3.1).

In order to obtain our main conclusion in this section, we now make some necessary preparations.

Lemma 3.1. *If* (x, u) *is any solution of* (3.1) *and* u *is* ω*-periodic, then*

$$u(t) = \int_t^{t+\omega} K(t, s)\eta(s)x(s)\Delta s =: (\Psi x)(t), \tag{3.2}$$

where

$$K(t, s) = \frac{e_\delta(s, t)}{e_\delta(\kappa + \omega, \kappa) - 1} \quad \textit{for} \quad t \le s \le t + \omega.$$

Proof. Suppose (x, u) is a solution of (3.1) such that u is ω-periodic. Multiply the equation

$$u^\Delta(t) + \delta(t)u^\sigma(t) = \eta(t)x(t)$$

on both sides by $e_\delta(t, \kappa)$ and use the product rule on time scales (see [5, Theorem 1.20(iii)]) to obtain

$$(ue_\delta^\Delta(\cdot, \kappa)(t) = e_\delta(t, \kappa)\eta(t)x(t).$$

Integrating from t to $t + \omega$ provides

$$u(t + \omega)e_\delta(t + \omega, \kappa) - u(t)e_\delta(t, \kappa) = \int_t^{t+\omega} e_\delta(s, \kappa)\eta(s)x(s)\Delta s.$$

According to Lemma 2.2(ii), we obtain

$$u(t) = \int_t^{t+\omega} \frac{e_\delta(s, t)}{e_\delta(t + \omega, t) - 1}\eta(s)x(s)\Delta s.$$

By [2, Theorem 2.1], $e_\delta(t + \omega, t) - 1$ does not depend on $t \in \mathbb{T}$, so (3.2) follows. □

For further use, note also

$$\begin{aligned} A_2 := \frac{\eta^s}{e_\delta(\kappa + \omega, \kappa) - 1} &\le K(t, s)\eta(s) \\ &\le \frac{e_\delta(t + \omega, t)\eta^l}{e_\delta(\kappa + \omega, \kappa) - 1} = \frac{e_\delta(\kappa + \omega, \kappa)\eta^l}{e_\delta(\kappa + \omega, \kappa) - 1} =: A_1. \end{aligned}$$

By Lemma 3.1, the existence of ω-periodic solutions of (3.1) is equivalent to the existence of ω-periodic solutions for the equation

$$x^{\Delta}(t) = r(t)x(t) - f(t, x(t), (\Psi x)(t)). \tag{3.3}$$

In this section, we always assume that the following conditions are satisfied:

(H_1) $f(t, v, \Psi v) \geq 0$ for $(t, v, \Psi v) \in \mathbb{T} \times \mathbb{R}^2$.

(H_2) For any $\varepsilon > 0$, there exists $\lambda > 0$ such that for any $v_1, v_2 \in \mathbb{R}$, $|v_1 - v_2| \leq \lambda$ implies

$$|f(t, v_1, \Psi v_1) - f(t, v_2, \Psi v_2)| < \varepsilon \quad \text{for all} \quad t \in I_\omega.$$

In order to explore the existence of periodic solutions of (3.1), we first embed our problem in the frame of Lemma 2.3 and Lemma 2.4. Define

$$X = \{x \in \mathrm{C}(\mathbb{T}, \mathbb{R}) : \ x(t + \omega) = x(t) \text{ for all } t \in \mathbb{T}\}.$$

It is not difficult to show that X is a Banach space when it is endowed with the norm $\|x\| = \sup_{t \in I_\omega} |x(t)|$. Let x be an ω-periodic solution of (3.3). Multiply (3.3) on both sides by $e_{\ominus r}(\sigma(t), \kappa)$ and use Lemma 2.2(ii) and the product rule on time scales to obtain

$$(x e_{\ominus r}(\cdot, \kappa))^{\Delta}(t) = -e_{\ominus r}(\sigma(t), \kappa) f(t, x(t), (\Psi x)(t)).$$

Then $x e_{\ominus r}(\cdot, \kappa)$ is a nonincreasing function on $\mathbb{T}$ (since r is nonnegative, use [5, Theorem 2.48(i)]). For $x \in X$, integrating the above equality from t to $t + \omega$ provides

$$\begin{aligned} x(t) &= -\int_t^{t+\omega} \frac{e_{\ominus r}(\sigma(s), t)}{e_{\ominus r}(t + \omega, t) - 1} f(s, x(s), (\Psi x)(s)) \Delta s \\ &= \int_t^{t+\omega} G(t, \sigma(s)) f(s, x(s), (\Psi x)(s)) \Delta s, \end{aligned}$$

where (applying again [2, Theorem 2.1])

$$G(t, \sigma(s)) = \frac{e_{\ominus r}(\sigma(s), t)}{1 - e_{\ominus r}(\kappa + \omega, \kappa)} \quad \text{for} \quad t \leq s \leq t + \omega$$

and

$$B_2 := \frac{e_{\ominus r}(\kappa + \omega, \kappa)}{(1 + r^l \mu^l)(1 - e_{\ominus r}(\kappa + \omega, \kappa))} \leq G(t, \sigma(s)) \leq \frac{1}{1 - e_{\ominus r}(\kappa + \omega, \kappa)} =: B_1.$$

Set

$$P = \{x \in X : \ x(t) \geq \theta \|x\|, \ t \in I_\omega \text{ and } x e_{\ominus r}(\cdot, \kappa) \text{ is nonincreasing on } \mathbb{T}\},$$

where

$$\theta = \frac{e_{\ominus r}(\kappa+\omega,\kappa)}{1+r^l\mu^l}.$$

Obviously, P is a cone in X. For $x \in P$ and $t \in \mathbb{T}$, define an operator T by

$$(Tx)(t) = \int_t^{t+\omega} G(t,\sigma(s))f(s,x(s),(\Psi x)(s))\Delta s.$$

Lemma 3.2. *$T : P \to P$ is well defined.*

Proof. It is clear that $(Tx) : \mathbb{T} \to \mathbb{R}$ is continuous such that (use again [2, Theorem 2.1]) $(Tx)(t+\omega) = (Tx)(t)$. Moreover, we have

$$\|Tx\| \le B_1 \int_\kappa^{\kappa+\omega} f(s,x(s),(\Psi x)(s))\Delta s$$

and

$$(Tx)(t) \ge B_2 \int_\kappa^{\kappa+\omega} f(s,x(s),(\Psi x)(s))\Delta s \ge \frac{B_2}{B_1}\|Tx\| = \theta\|Tx\|.$$

In addition, we have

$$\begin{aligned}
&((Tx)e_{\ominus r}(\cdot,\kappa))^\Delta(t) \\
&= f(t,x(t),(\Psi x)(t))\left[\frac{e_{\ominus r}(\sigma(t+\omega),\kappa)}{1-e_{\ominus r}(\kappa+\omega,\kappa)} - \frac{e_{\ominus r}(\sigma(t),\kappa)}{1-e_{\ominus r}(\kappa+\omega,\kappa)}\right] \\
&= -e_{\ominus r}(\sigma(t),\kappa)f(t,x(t),(\Psi x)(t)).
\end{aligned}$$

Therefore, $Tx \in P$. □

It is not difficult to show that x is a positive ω-periodic solution of (3.3) if and only if x is a fixed point of the operator T on P. Let $\xi, \zeta \in \mathbb{T}$ be such that $\kappa \le \xi < \zeta \le \kappa+\omega$. Then we define the increasing, nonnegative, continuous functionals α, β and γ on P by

$$\begin{aligned}
\gamma(x) &= \max_{\zeta \le t \le \kappa+\omega} e_{\ominus r}(t,\kappa)x(t) = e_{\ominus r}(\zeta,\kappa)x(\zeta); \\
\beta(x) &= \min_{\xi \le t \le \zeta} e_{\ominus r}(t,\kappa)x(t) = e_{\ominus r}(\zeta,\kappa)x(\zeta); \\
\alpha(x) &= \min_{\kappa \le t \le \xi} e_{\ominus r}(t,\kappa)x(t) = e_{\ominus r}(\xi,\kappa)x(\xi).
\end{aligned}$$

Obviously, we have

$$\gamma(x) = \beta(x) \le \alpha(x) \quad \text{for all} \quad x \in P.$$

In addition, for each $x \in P$, we have $\gamma(x) = e_{\ominus r}(\zeta,\kappa)x(\zeta) \ge e_{\ominus r}(\zeta,\kappa)\theta\|x\|$. Thus,

$$\|x\| \le e_r(\zeta,\kappa)\frac{1}{\theta}\gamma(x) \quad \text{for all} \quad x \in P.$$

Finally, it is easy to show that

$$\beta(\pi x) = \pi\beta(x) \quad \text{for} \quad 0 \le \pi \le 1 \quad \text{and} \quad x \in P.$$

Now we impose conditions on f such that (3.3) has at least two positive periodic solutions.

Theorem 3.3. *Assume that there exist constant numbers a, b and c with $0 < a < b < c$ such that*

$$0 < a < \frac{\Upsilon_\xi}{\Lambda_\zeta} b < \frac{A_2\theta^2\Upsilon_\xi}{A_1\Lambda_\zeta} c \quad \textit{or} \quad 0 < a < \frac{A_2\theta^2}{A_1} e_r(\zeta,\xi) b < \left(\frac{A_2\theta^2}{A_1}\right)^2 e_r(\zeta,\xi) c.$$

Suppose f satisfies the following conditions:

(V$_1$) $f(t, x(t), (\Psi x)(t)) > \dfrac{c}{\Lambda_\zeta}$ *for*

$$\theta c e_r(\zeta,\kappa) \le x(t) \le \frac{c}{\theta} e_r(\zeta,\kappa),$$
$$A_2\omega\theta c e_r(\zeta,\kappa) \le (\Psi x)(t) \le A_1\omega\frac{c}{\theta} e_r(\zeta,\kappa), \quad t \in [\zeta, \kappa+\omega];$$

(V$_2$) $f(t, x(t), (\Psi x)(t)) < \dfrac{b}{\Gamma_\zeta}$ *for*

$$\theta b e_r(\zeta,\kappa) \le x(t) \le \frac{b}{\theta} e_r(\zeta,\kappa),$$
$$A_2\omega\theta b e_r(\zeta,\kappa) \le (\Psi x)(t) \le A_1\omega\frac{b}{\theta} e_r(\zeta,\kappa), \quad t \in [\kappa, \kappa+\omega];$$

(V$_3$) $f(t, x(t), (\Psi x)(t)) > \dfrac{a}{\Upsilon_\xi}$ *for*

$$\theta a e_r(\xi,\kappa) \le x(t) \le \frac{a}{\theta} e_r(\xi,\kappa),$$
$$A_2\omega\theta a e_r(\xi,\kappa) \le (\Psi x)(t) \le A_1\omega\frac{a}{\theta} e_r(\xi,\kappa), \quad t \in [\xi, \kappa+\omega],$$

where

$$\Lambda_\zeta = e_{\ominus r}(\zeta,\kappa)\int_\zeta^{\kappa+\omega} G(\zeta,\sigma(s))\Delta s, \quad \Upsilon_\xi = e_{\ominus r}(\xi,\kappa)\int_\xi^{\kappa+\omega} G(\xi,\sigma(s))\Delta s,$$
$$\Gamma_\zeta = e_{\ominus r}(\zeta,\kappa)\left[\int_\zeta^{\kappa+\omega} G(\zeta,\sigma(s))\Delta s + \int_\kappa^\zeta G(\zeta-\omega,\sigma(s))\Delta s\right].$$

Then (3.3) *has at least two positive ω-periodic solutions.*

Proof. We embed our problem in the frame of Lemma 2.3. This proof is divided into the following four steps.
Step 1. The operator $T : \overline{P(\gamma, c)} \to P$ is completely continuous.

Proof of Step 1. By (H$_2$), for any $\varepsilon > 0$, there exists $\lambda > 0$ such that for any $v_1, v_2 \in \mathbb{R}$, $|v_1 - v_2| \le \lambda$ implies

$$|f(t, v_1, \Psi v_1) - f(t, v_2, \Psi v_2)| < \frac{\varepsilon}{B_1 \omega} \quad \text{for all} \quad t \in I_\omega.$$

For the above $\varepsilon > 0$ and $\lambda > 0$, if $x, y \in P$ and $\|x - y\| < \lambda$, then we have

$$|(Tx)(t) - (Ty)(t)| \le B_1 \int_\kappa^{\kappa+\omega} |f(s, x(s), (\Psi x)(s)) - f(s, y(s), (\Psi y)(s))| \Delta s < \varepsilon$$

for $t \in I_\omega$. This implies that T is continuous. Next, we show that T is uniformly bounded and equicontinuous. For $x \in \overline{P(\gamma, c)}$, we have $\gamma(x) = e_{\ominus r}(\zeta, \kappa) x(\zeta) \le c$. Then $\|x\| \le \theta x(\zeta) \le \theta e_r(\zeta, \kappa) c =: L$. By (H$_2$), for $\varepsilon = 1$ and $x, y \in \overline{P(\gamma, c)}$, there exists $\lambda > 0$ such that $\|x-y\| < \lambda$ implies $|f(t, x(t), (\Psi x)(t)) - f(t, y(t), (\Psi y)(t))| < 1$ for $t \in I_\omega$. Choose $N > 0$ such that $L/N < \lambda$. For $x \in \overline{P(\gamma, c)}$, we define $x^i(t) = (x(t)i)/N$ for $i = 0, 1, \ldots, N$. Then we have

$$\|x^i - x^{i-1}\| = \sup_{t \in \mathbb{T}} \left| \frac{x(t)i}{N} - \frac{x(t)(i-1)}{N} \right| = \|x\| \frac{1}{N} \le \frac{L}{N} < \lambda$$

and

$$|f(t, x^i(t), (\Psi x^i)(t)) - f(t, x^{i-1}(t), (\Psi x^{i-1})(t))| < 1 \quad \text{for} \quad t \in I_\omega.$$

Therefore, for $t \in I_\omega$, one can reach

$$\begin{aligned} |f(t, x(t), (\Psi x)(t))| &\le \sum_{i=1}^N |f(t, x^i(t), (\Psi x^i)(t)) - f(t, x^{i-1}(t), (\Psi x^{i-1})(t))| \\ &\quad + |f(t, 0, 0)| \\ &< N + \sup_{t \in I_\omega} |f(t, 0, 0)| =: Q. \end{aligned}$$

It follows that

$$\|Tx\| \le B_1 \int_\kappa^{\kappa+\omega} f(s, x(s), (\Psi x)(s)) \Delta s < B_1 \omega Q.$$

Moreover, we have (use [5, Theorem 1.117])

$$\begin{aligned} (Tx)^\Delta(t) &= \int_\kappa^{\kappa+\omega} G^\Delta(t, \sigma(s)) f(s, x(s), (\Psi x)(s)) \Delta s \\ &\quad + G(\sigma(t), \sigma(t+\omega)) f(t+\omega, x(t+\omega), (\Psi x)(t+\omega)) \\ &\quad - G(\sigma(t), \sigma(t)) f(t, x(t), (\Psi x)(t)) \\ &= r(t)(Tx)(t) - f(t, x(t), (\Psi x)(t)). \end{aligned}$$

Therefore, we obtain

$$|(Tx)^{\Delta}(t)| \le r^{l}\|Tx\| + |f(t,x(t),(\Psi x)(t))| \le r^{l}B_1\omega Q + Q.$$

This implies that T is uniformly bounded and equicontinuous. It follows from the Arzelà–Ascoli theorem that the operator T is completely continuous. □

Step 2. Condition (i) of Lemma 2.3 is satisfied.

Proof of Step 2. Let $x \in \partial P(\gamma, c)$. Then $\gamma(x) = e_{\ominus r}(\zeta,\kappa)x(\zeta) = c$. Since $\|x\| \le x(t)/\theta$ for all $t \in [\zeta, \kappa+\omega]$, we have

$$x(t) \ge \theta\|x\| \ge \theta x(\zeta) \ge \theta c e_r(\zeta,\kappa), \quad x(t) \le \|x\| \le e_r(\zeta,\kappa)\frac{1}{\theta}\gamma(x) = \frac{c}{\theta}e_r(\zeta,\kappa).$$

Moreover, it is easy to show that

$$A_2\omega\theta c e_r(\zeta,\kappa) \le (\Psi x)(t) \le A_1\omega\frac{c}{\theta}e_r(\zeta,\kappa), \;\; t \in [\zeta, \kappa+\omega].$$

In view of (V_1), we get

$$\begin{aligned}\gamma(Tx) &= e_{\ominus r}(\zeta,\kappa)(Tx)(\zeta) = e_{\ominus r}(\zeta,\kappa)\int_{\zeta}^{\zeta+\omega} G(\zeta,\sigma(s))f(s,x(s),(\Psi x)(s))\Delta s\\ &> e_{\ominus r}(\zeta,\kappa)\frac{c}{\Lambda_\zeta}\int_{\zeta}^{\kappa+\omega} G(\zeta,\sigma(s))\Delta s = c,\end{aligned}$$

which verifies (i) of Lemma 2.3. □

Step 3. Condition (ii) of Lemma 2.3 is satisfied.

Proof of Step 3. For $x \in \partial P(\beta, b)$ and $\beta(x) = e_{\ominus r}(\zeta,\kappa)x(\zeta) = b$, we easily obtain

$$x(t) \ge \theta\|x\| \ge \theta x(\zeta) \ge \theta b e_r(\zeta,\kappa), \quad x(t) \le \|x\| \le e_r(\zeta,\kappa)\frac{1}{\theta}\beta(x) = \frac{b}{\theta}e_r(\zeta,\kappa)$$

and

$$A_2\omega\theta b e_r(\zeta,\kappa) \le (\Psi x)(t) \le A_1\omega\frac{b}{\theta}e_r(\zeta,\kappa)$$

for $t \in [\kappa, \kappa+\omega]$. It follows from ($V_2$) that

$$\begin{aligned}\beta(Tx) &= e_{\ominus r}(\zeta,\kappa)(Tx)(\zeta) = e_{\ominus r}(\zeta,\kappa)\int_{\zeta}^{\zeta+\omega} G(\zeta,\sigma(s))f(s,x(s),(\Psi x)(s))\Delta s\\ &= e_{\ominus r}(\zeta,\kappa)\left[\int_{\zeta}^{\kappa+\omega} G(\zeta,\sigma(s))f(s,x(s),(\Psi x)(s))\Delta s\right.\\ &\quad \left.+\int_{\kappa+\omega}^{\zeta+\omega} G(\zeta,\sigma(s))f(s,x(s),(\Psi x)(s))\Delta s\right]\\ &< e_{\ominus r}(\zeta,\kappa)\left[\int_{\zeta}^{\kappa+\omega} G(t,\sigma(s))\Delta s + \int_{\kappa}^{\zeta} G(\zeta-\omega,\sigma(s))\Delta s\right]\frac{b}{\Gamma_\zeta} = b,\end{aligned}$$

which verifies (ii) of Lemma 2.3. □

Step 4. Condition (iii) of Lemma 2.3 is satisfied.

Proof of Step 4. Clearly, $P(\alpha, a) \neq \emptyset$. For $x \in \partial P(\alpha, a)$ and $\alpha(x) = e_{\ominus r}(\xi, \kappa)x(\xi) = a$, similar to the above arguments, we have

$$x(t) \geq \theta\|x\| \geq \theta x(\xi) \geq \theta a e_r(\xi, \kappa), \quad x(t) \leq \|x\| \leq e_r(\xi, \kappa)\frac{1}{\theta}\alpha(x) = \frac{a}{\theta}e_r(\xi, \kappa)$$

and

$$A_2\omega\theta a e_r(\xi, \kappa) \leq (\Psi x)(t) \leq A_1\omega\frac{a}{\theta}e_r(\xi, \kappa)$$

for $t \in [\xi, \kappa + \omega]$. In view of ($V_3$), we get

$$\begin{aligned} \alpha(Tx) &= e_{\ominus r}(\xi, \kappa)(Tx)(\xi) = e_{\ominus r}(\xi, \kappa)\int_{\xi}^{\xi+\omega} G(\xi, \sigma(s))f(s, x(s), (\Psi x)(s))\Delta s \\ &> e_{\ominus r}(\xi, \kappa)\frac{a}{\Upsilon_\xi}\int_{\xi}^{\kappa+\omega} G(\xi, \sigma(s))\Delta s = a, \end{aligned}$$

which verifies (iii) of Lemma 2.3. □

To sum up, all the hypotheses of Lemma 2.3 are satisfied. Then T has at least two fixed points, that is, (3.3) has at least two positive periodic solutions x_1 and x_2 in $\overline{P(\gamma, c)}$ such that

$$x_1(\xi) > a e_r(\xi, \kappa), \quad x_1(\zeta) < b e_r(\zeta, \kappa), \quad x_2(\zeta) > b e_r(\zeta, \kappa), \quad x_2(\zeta) < c e_r(\zeta, \kappa).$$

This completes the proof. □

Carrying out similar arguments as above, we let $\xi, \zeta \in \mathbb{T}$ be such that $\kappa \leq \xi < \zeta \leq \kappa + \omega$, and define the increasing, nonnegative, continuous functionals α, β and γ on P as follows:

$$\begin{aligned} \gamma(x) &= \min_{\xi \leq t \leq \zeta} e_{\ominus r}(t, \kappa)x(t) = e_{\ominus r}(\zeta, \kappa)x(\zeta); \\ \beta(x) &= \max_{\zeta \leq t \leq \kappa+\omega} e_{\ominus r}(t, \kappa)x(t) = e_{\ominus r}(\zeta, \kappa)x(\zeta); \\ \alpha(x) &= \max_{\xi \leq t \leq \kappa+\omega} e_{\ominus r}(t, \kappa)x(t) = e_{\ominus r}(\xi, \kappa)x(\xi). \end{aligned}$$

For each $x \in P$, it is easy to show that

$$\gamma(x) = \beta(x) \leq \alpha(x), \quad \|x\| \leq e_r(\zeta, \kappa)\frac{1}{\theta}\gamma(x), \quad \beta(\pi x) = \pi\beta(x) \quad \text{for} \quad 0 \leq \pi \leq 1.$$

By Lemma 2.4, we can easily obtain the following conclusion.

Theorem 3.4. *Assume that there exist constant numbers* a, b *and* c *with* $0 < a < b < c$ *such that*

$$0 < a < \frac{A_2\theta^2}{A_1} e_r(\zeta,\xi) b < \frac{A_2\theta^2\Gamma^*_\zeta}{A_1\Lambda^*_\zeta} e_r(\zeta,\xi) c$$

or

$$0 < a < \frac{A_2\theta^2}{A_1} e_r(\zeta,\xi) b < \left(\frac{A_2\theta^2}{A_1}\right)^2 e_r(\zeta,\xi) c.$$

Suppose f *satisfies the following conditions:*

(V_1^*) $f(t, x(t), (\Psi x)(t)) < \dfrac{c}{\Lambda^*_\zeta}$ *for*

$$\theta c e_r(\zeta,\kappa) \le x(t) \le \frac{c}{\theta} e_r(\zeta,\kappa),$$
$$A_2\omega\theta c e_r(\zeta,\kappa) \le (\Psi x)(t) \le A_1\omega\frac{c}{\theta} e_r(\zeta,\kappa), \quad t \in [\kappa, \kappa+\omega];$$

(V_2^*) $f(t, x(t), (\Psi x)(t)) > \dfrac{b}{\Gamma^*_\zeta}$ *for*

$$\theta b e_r(\zeta,\kappa) \le x(t) \le \frac{b}{\theta} e_r(\zeta,\kappa),$$
$$A_2\omega\theta b e_r(\zeta,\kappa) \le (\Psi x)(t) \le A_1\omega\frac{b}{\theta} e_r(\zeta,\kappa), \quad t \in [\zeta, \kappa+\omega];$$

(V_3^*) $f(t, x(t), (\Psi x)(t)) < \dfrac{a}{\Upsilon^*_\xi}$ *for*

$$\theta a e_r(\xi,\kappa) \le x(t) \le \frac{a}{\theta} e_r(\xi,\kappa),$$
$$A_2\omega\theta a e_r(\xi,\kappa) \le (\Psi x)(t) \le A_1\omega\frac{a}{\theta} e_r(\xi,\kappa), \quad t \in [\kappa, \kappa+\omega],$$

where

$$\Lambda^*_\zeta = e_{\ominus r}(\zeta,\kappa)\left[\int_\zeta^{\kappa+\omega} G(\zeta,\sigma(s)) + \int_\kappa^\zeta G(\zeta-\omega,\sigma(s))\Delta s\right],$$
$$\Gamma^*_\zeta = e_{\ominus r}(\zeta,\kappa)\int_\zeta^{\kappa+\omega} G(\zeta,\sigma(s))\Delta s,$$
$$\Upsilon^*_\xi = e_{\ominus r}(\xi,\kappa)\left[\int_\xi^{\kappa+\omega} G(\xi,\sigma(s))\Delta s + \int_\kappa^\xi G(\xi-\omega,\sigma(s))\Delta s\right].$$

Then (3.3) *has at least two positive* ω*-periodic solutions.*

Note that Theorem 3.3 and Theorem 3.4 also justify the statement that (3.1) has at least two positive periodic solutions.

4 Three Positive Periodic Solutions

In this section, we explore the existence of three positive ω-periodic solutions of (3.1). First of all, we assume that the following condition is satisfied.

(H$_3$) $f(t, v_1, v_2)$ is nondecreasing with respect to $(v_1, v_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $t \in \mathbb{T}$.

Set

$$P^* = \{x \in X : \ x(t) \geq \theta \|x\|\} .$$

It is clear that P^* is a cone in X.

Now we state and prove our main result.

Theorem 4.1. *Assume that* (H$_1$)–(H$_3$) *hold. Suppose there exist positive constants* r*,* r_1*, and* R *with* $0 < r < r_1 < R$ *such that*

$$B_1\omega \sup_{t\in I_\omega} f(t, R, A_1\omega R) \leq R,$$
$$B_1\omega \sup_{t\in I_\omega} f(t, r, A_1\omega r) < r,$$
$$B_2\omega \inf_{t\in I_\omega} f(t, r_1, A_2\omega r_1) > r_1.$$

Then (3.1) *has at least three positive* ω*-periodic solutions.*

Proof. Define a functional $\phi : P^* \to [0, \infty)$ by $\phi(x) = \min_{t\in I_\omega} x(t)$. Obviously, ϕ is a concave functional and $\phi(x) \leq \|x\|$ for all $x \in \overline{P^*_R}$. Meanwhile, define an operator T^* by

$$(T^*x)(t) = \int_t^{t+\omega} G(t, \sigma(s)) f(s, x(s), (\Psi x)(s))\Delta s \quad \text{for} \quad x \in \overline{P^*}.$$

For $x \in \overline{P^*_R}$, we have

$$\begin{aligned} \|Tx\| &\leq B_1 \int_\kappa^{\kappa+\omega} f(s, x(s), (\Psi x)(s))\Delta s \\ &\leq B_1 \int_\kappa^{\kappa+\omega} f(s, R, A_1\omega R)\Delta s \\ &\leq B_1\omega \sup_{t\in I_\omega} f(t, R, A_1\omega R) \leq R. \end{aligned}$$

Arguments similar to those in Section 3 show that $T^* : \overline{P^*_R} \to \overline{P^*_R}$ is completely continuous.

First, we prove that condition (ii) of Lemma 2.5 is satisfied. For $x \in \overline{P^*_r}$, we obtain

$$\begin{aligned} \|Tx\| &\leq B_1 \int_\kappa^{\kappa+\omega} f(s, x(s), (\Psi x)(s))\Delta s \\ &\leq B_1 \int_\kappa^{\kappa+\omega} f(s, r, A_1\omega r)\Delta s \\ &\leq B_1\omega \sup_{t\in I_\omega} f(t, r, A_1\omega r) < r. \end{aligned}$$

Choose a positive constant r_2 such that $0 < r_1 < \theta r_2 < r_2 \le R$. Next, we show that the condition (i) of Lemma 2.5 holds. Obviously, $\{x \in P(\phi, r_1, r_2) : \phi(x) > r_1\} \neq \emptyset$. For $x \in P(\phi, r_1, r_2)$, we have $r_1 \le \phi(x) = \min_{t \in I_\omega} x(t) \le \|x\| \le r_2$. Then

$$\begin{aligned}\phi(Tx) &= \min_{t\in I_\omega}(Tx)(t) = \min_{t\in I_\omega}\int_t^{t+\omega} G(t,\sigma(s))f(s,x(s),(\Psi x)(s))\Delta s \\ &\ge B_2 \min_{t\in I_\omega}\int_t^{t+\omega} f(s,x(s),(\Psi x)(s))\Delta s \\ &\ge B_2\omega \inf_{t\in I_\omega} f(t,r_1,A_2\omega r_1) > r_1.\end{aligned}$$

Finally, we verify condition (iii) of Lemma 2.5. For $x \in P(\phi, r_1, R)$ and $\|Tx\| > r_2$, we have

$$\begin{aligned}\phi(Tx) &= \min_{t\in I_\omega}(Tx)(t) = \min_{t\in I_\omega}\int_t^{t+\omega} G(t,\sigma(s))f(s,x(s),(\Psi x)(s))\Delta s \\ &\ge B_2 \min_{t\in I_\omega}\int_t^{t+\omega} f(s,x(s),(\Psi x)(s))\Delta s \\ &\ge \frac{B_2}{B_1}\|Tx\| > \theta r_2 > r_1.\end{aligned}$$

Therefore, by Lemma 2.5, (3.3) has at least three positive ω-periodic solution. This implies that (3.1) has at least three positive ω-periodic solution. □

5 One Periodic Solution

In this section, we focus on periodicity of the more general nonlinear dynamic system with feedback control on a general time scale

$$\begin{aligned}&x^\Delta(t) = f(t,x(t),u(t)),\\ &u^\Delta(t) = -\delta(t)u^\sigma(t) + \eta(t)x(t),\end{aligned} \tag{5.1}$$

where $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ and $\delta, \eta : \mathbb{T} \to (0, \infty)$ are all rd-continuous and ω-periodic for $t \in \mathbb{T}$, and $\omega > 0$ is called the period of (5.1).

Let us recall the continuation theorem in coincidence degree theory, borrowing notations and terminology from [10], which will come into play later on.

Let X, Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$, then it follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N is called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 5.1 (Continuation Theorem)**.** *Let L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Suppose*

(a) *for each $\lambda \in (0,1)$, every solution z of $Lz = \lambda Nz$ is such that $z \notin \partial\Omega$;*

(b) *$QNz \neq 0$ for each $z \in \partial\Omega \cap \operatorname{Ker} L$ and the Brouwer degree $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*

Then the operator equation $Lz = Nz$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

In order to achieve an a-priori estimate of dynamic equations (5.1) on a time scale $\mathbb{T}$, we now give the following lemma.

Lemma 5.2. [3, Lemma 2.4] *Let $t_1, t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{R}$ is ω-periodic, then*

$$g(t) \leq g(t_1) + \int_\kappa^{\kappa+\omega} |g^\Delta(s)|\Delta s \quad \textit{and} \quad g(t) \geq g(t_2) - \int_\kappa^{\kappa+\omega} |g^\Delta(s)|\Delta s.$$

In order to explore the existence of periodic solutions of (5.1), we embed our problem in the frame of coincidence degree theory. Define

$$\mathcal{L}^\omega = \{y \in \mathrm{C}(\mathbb{T}, \mathbb{R}) : \ y(t+\omega) = y(t) \text{ for all } t \in \mathbb{T}\},$$
$$\|y\| = \max_{t \in I_\omega} |y(t)| \quad \text{for} \quad y \in \mathcal{L}^\omega.$$

It is not difficult to show that $(\mathcal{L}^\omega, \|\cdot\|)$ is a Banach space. Let

$$\mathcal{L}_0^\omega = \{y \in \mathcal{L}^\omega : \ \overline{y} = 0\}, \quad \mathcal{L}_c^\omega = \{y \in \mathcal{L}^\omega : \ y(t) \equiv h \in \mathbb{R} \text{ for } t \in \mathbb{T}\}.$$

Then it is easy to show that $\mathcal{L}_0^\omega$ and $\mathcal{L}_c^\omega$ are both closed linear subspaces of $\mathcal{L}^\omega$, $\mathcal{L}^\omega = \mathcal{L}_0^\omega \oplus \mathcal{L}_c^\omega$, and $\dim \mathcal{L}_c^\omega = 1$.

Theorem 5.3. *Assume*

(H$_4$) *there exists a constant $M_* > 0$ such that for any ω-periodic function $x, u : \mathbb{T} \to \mathbb{R}$,*

$$\int_\kappa^{\kappa+\omega} f(t, x(t), u(t))\Delta t = 0$$

implies

$$\int_\kappa^{\kappa+\omega} |f(t, x(t), u(t)|\Delta t \leq M_*;$$

(H$_5$) *there is a constant* $M^* > 0$ *such that if* $v_i \geq M^*$ *for* $i = 1$ *and* $i = 2$*, then*

$$f(t, v_1, v_2) > 0, \quad f(t, -v_1, -v_2) < 0, \quad t \in I_\omega$$

or

$$f(t, v_1, v_2) < 0, \quad f(t, -v_1, -v_2) > 0, \quad t \in I_\omega.$$

Then the system (5.1) *has at least one* ω*-periodic solution.*

Proof. According to Lemma 3.1, in order to obtain the existence of ω-periodic solutions of (5.1), we only need to consider the existence of ω-periodic solutions for the equation

$$x^\Delta(t) = f(t, x(t), (\Psi x)(t)). \tag{5.2}$$

Let $X = Z = \mathcal{L}^\omega$ and define

$$Nx = f(t, x(t), (\Psi x)(t)), \quad Lx = x^\Delta, Px = Qx = \overline{x}.$$

Then $\operatorname{Ker} L = \mathcal{L}_c^\omega$, $\operatorname{Im} L = \mathcal{L}_0^\omega$, and $\dim \operatorname{Ker} L = 1 = \operatorname{codim} \operatorname{Im} L$. Since $\mathcal{L}_0^\omega$ is closed in $\mathcal{L}^\omega$, it follows that L is a Fredholm mapping of index zero. It is not difficult to show that P and Q are continuous maps such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$K_P(x) = \hat{x} - \overline{\hat{x}}, \quad \text{where} \quad \hat{x}(t) = \int_\kappa^t x(s)\Delta s.$$

Thus

$$QNx = \frac{1}{\omega} \int_\kappa^{\kappa+\omega} f(s, x(s), (\Psi x)(s))\Delta s.$$

Obviously, QN and $K_P(I - Q)N$ are continuous. Since X is a Banach space, using the Arzelà–Ascoli theorem, it is easy to show that $\overline{K_P(I - Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$. Now we are in the position to search for an appropriate open, bounded subset Ω for the application of the continuation theorem (Lemma 5.1). For the operator equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$x^\Delta(t) = \lambda f(t, x(t), (\Psi x)(t)). \tag{5.3}$$

Assume that $x \in X$ is an arbitrary solution of equation (5.3) for a certain $\lambda \in (0, 1)$. Integrating both sides of (5.3) on the interval $[\kappa, \kappa + \omega]$, we have

$$\int_\kappa^{\kappa+\omega} f(t, x(t), (\Psi x)(t))\Delta t = 0 \tag{5.4}$$

By (H_4) and (H_5), (5.3) and (5.4), there exist three constants $M_* > 0$, $M_1 > 0$ and $M_2 > 0$, and $t_1, t_2 \in I_\omega$ such that

$$\int_\kappa^{\kappa+\omega} |x^\Delta(t)|\Delta t \le \int_\kappa^{\kappa+\omega} |f(t, x(t), (\Psi x)(t))|\Delta t \le M_*$$

and

$$x(t_1) < M_1, \quad (\Psi x)(t_1) < M_1, \quad - M_2 < x(t_2), \quad -M_2 < (\Psi x)(t_2).$$

It follows from Lemma 5.2 that

$$x(t) \le x(t_1) + \int_\kappa^{\kappa+\omega} |x^\Delta(t)|\Delta t < M_1 + M_*,$$
$$x(t) \ge x(t_2) - \int_\kappa^{\kappa+\omega} |x^\Delta(t)|\Delta t > -M_2 - M_*.$$

Now we define

$$\Omega := \{x \in X : \ |x(t)| < H, \ t \in I_\omega\},$$

where

$$H = M_* + M^* + M_1 + M_2 + \frac{M_* + M^* + M_1 + M_2}{\omega A_2} + \frac{M_* + M^* + M_1 + M_2}{\omega A_1}.$$

It is clear to show that Ω satisfies the requirement (a) in Lemma 5.1. If $x \in \partial\Omega \cap \operatorname{Ker} L$, then it is easy to show that $x(t) > M^*$, $(\Psi x)(t) > M^*$ or $x(t) < -M^*$, $(\Psi x)(t) < -M^*$ for all $t \in I_\omega$, and we have

$$QNx = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} f(s, x(s), (\Psi x)(s))\Delta s \ne 0.$$

Moreover, note that $J = I$ since $\operatorname{Im} Q = \operatorname{Ker} L$. In order to compute the Brouwer degree, let us consider the homotopy

$$H(\nu, x) = \nu x + (1-\nu)QNx, \ \ \nu \in [0, 1].$$

For any $x \in \partial\Omega \cap \operatorname{Ker} L$, $\nu \in [0, 1]$, we have $H(\nu, x) \ne 0$. By the homotopic invariance of topological degree, we have

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{QNx, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{x, \Omega \cap \operatorname{Ker} L, 0\} \ne 0,$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree. Now we have proved that Ω satisfies all requirements in Lemma 5.1. Thus $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$, that is, (5.1) has at least one ω-periodic solution in $\operatorname{Dom} L \cap \overline{\Omega}$. The proof is complete. □

In order to illustrate some features of our main theorem in this section, we explore the existence of periodic solutions of the following model with feedback controls.

Example 5.4. Consider the system

$$\begin{aligned} x^{\Delta}(t) &= r(t) - \frac{\exp\{x(t)\}}{K(t)} - \alpha(t)u(t), \\ u^{\Delta}(t) &= -\delta(t)u^{\sigma}(t) + \eta(t)\exp\{x(t)\}, \end{aligned} \tag{5.5}$$

where $r(t), K(t), \alpha(t), \delta(t), \eta(t) \in \mathrm{C_{rd}}(\mathbb{T}, (0, \infty))$ are all ω-periodic functions.

Theorem 5.5. (5.5) *has at least one ω-periodic solution.*

Proof. Set $(Px)(t) = \dfrac{\exp\{x(t)\}}{K(t)} + \alpha(t)(\Psi\exp\{x(t)\})(t)$. According to Theorem 5.3, we only need to prove that (H_4) and (H_5) are true. If $x(t)$ and $u(t)$ are ω-periodic functions and satisfy

$$\int_{\kappa}^{\kappa+\omega} (r(t) - (Px)(t))\Delta t = 0,$$

then we have

$$\int_{\kappa}^{\kappa+\omega} |r(t) - (Px)(t)|\Delta t \leq 2\int_{\kappa}^{\kappa+\omega} r(t)\Delta t > 0.$$

In addition, we can easily show that

$$\lim_{v\to\infty}(r(t) - Pv) = -\infty \quad \text{and} \quad \lim_{v\to-\infty}(r(t) - Pv) = r(t) > 0$$

hold uniformly in $t \in I_{\omega}$. By Theorem 5.3, (5.5) has at least one ω-periodic solution. Thus the proof is complete. □

Remark 5.6. Let $\mathbb{T} = \mathbb{R}$ and $\tilde{x}(t) = \exp\{x(t)\}$. Then (5.5) reduces to the continuous logistic model with feedback control system

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{x}(t)(r(t) - \dfrac{\tilde{x}(t)}{K(t)} - \alpha(t)u(t)), \\ \dot{u}(t) = -\delta(t)u(t) + \eta(t)\tilde{x}(t). \end{cases}$$

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