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A Note on the Asymptotic Solution of a Certain Rational Difference Equation

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Abstract

The aim of this paper is to show the existence of a solution of the difference equation in the title converging to zero as $n \to \infty$, and to determine its asymptotic behaviour.

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1 Introduction

In recent investigations of dynamical systems rational difference equations of higher order are of main importance, cf. Kulenovíc and Ladas [4] and the references therein. A special example is the equation

$$x_n = \frac{x_{n-3}}{1 + x_{n-1}x_{n-2}},\ n = 0, 1, 2, 3 \dots$$

about which in [5] it was shown that every solution converges as $n \to \infty$ to a 3-periodic solution $(\dots, p, q, r, p, q, r, \dots)$ which $pqr = 0$.

In [3] Berg investigated the difference equation

$$x_{n-3} = x_n(1 + x_{n-1}x_{n-2}),\ n = 0, 1, 2, 3 \dots. \tag{1.1}$$

This paper showed that there exists a solution of equation (1.1) converging to zero as $n \to \infty$, and the author determined its asymptotic behaviour.

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By a method similar as in [3], we shall now consider the difference equation

$$x_n = \frac{x_{n-3} - x_n^3 - x_{n-1}^3}{1 + x_{n-1}x_{n-2} + x_n x_{n-1}},\ n = 0, 1, 2, 3 \dots$$

or equivalently

$$x_n + x_n x_{n-1} x_{n-2} + x_n^2 x_{n-1} + x_n^3 + x_{n-1}^3 = x_{n-3},\ n = 0, 1, 2, 3 \dots. \tag{1.2}$$

2 Main Results

In this section, our purpose is to find the asymptotic behaviour of a solution of equation (1.2) tending to zero as $n \to \infty$, so we proceed as recommended in [2]. At first, we assume that a solution exists for a continuous argument $n = t$, and that it is continuously differentiable. Writing $x_n = x(t)$, approximating $x(t-1)$, $x(t-2)$ by x and $x(t-3)$ according to Taylor by $x - 3x'$, we approximate equation (1.2) by the differential equation

$$x(1 + 2x^2) = x - 3x' - 2x^3 \Leftrightarrow x' = -\frac{4}{3}x^3 \tag{2.1}$$

with the solution $x = \sqrt{3/(8t)}$, disregarding the constant of integration. Since we are interested in an asymptotic expansion of the solution x, we now look for the second term. For this reason we approximate $x(t-1)$ by $x - x'$, $x(t-2)$ by $x - 2x'$ and $x(t-3)$ by $x - 3x' + (9/2)x''$, and since equation (2.1) implies $x'' = -4x^2x'$, we approximate equation (1.2) by the differential equation

$$x + x(x - x')(x - 2x') + x^2(x - x') + x^3 + (x - x')^3 = x - 3x' + \frac{9}{2}x''. \tag{2.2}$$

After neglecting $5x(x')^2 - x'^3$, it turn into the equation

$$4x^3 = -(11x^2 + 3)x'$$

which can be integrated by

$$x = \sqrt{\frac{3}{8t + 22\ln x}}$$

disregarding again the constant of integration. Obviously, a solution x tending to zero satisfies $x \sim \sqrt{3/(8t)}$ as before, so that by iteration we obtain

$$x = \sqrt{\frac{3}{8t}}\left(1 + \frac{11}{16}\frac{\ln t}{t}\right) \tag{2.3}$$

up to smaller terms as $n \to \infty$. This result encourages us to expect a solution of equation (1.2) of the form

$$x = \frac{1}{\sqrt{n}}\left(a + \frac{b\ln n}{n} + \frac{c\ln^2 n + d\ln n + e}{n^2}\right) \tag{2.4}$$

up to smaller terms as $n \to \infty$. Replacing this as an ansatz into equation (1.2), we find by means of the DERIVE system in accordance with equation (2.3)

$$a = \frac{\sqrt{6}}{4},\ b = \frac{11\sqrt{6}}{64},\ c = \frac{363\sqrt{6}}{32.64} = \frac{363\sqrt{6}}{2048},\ d = -\frac{121\sqrt{6}}{512},\ e = 0. \tag{2.5}$$

In the terminology of [1] equation (2.4) with the coefficients (2.5) represents an asymptotic solution of equation (1.2). However, we shall show that it represents in fact the asymptotic behaviour of a real solution of equation (1.2). For this reason we use the following result of Stević [5, Theorem 2], which is a generalization of [1, Theorem 1] to equations of order $k \geq 1$.

Theorem 2.1. *Let $f : \mathbb{R}_+^k \to \mathbb{R}_+$ be a continuous and nondecreasing function in each argument, and let $\{y_n\}$ and $\{z_n\}$ be sequences with $y_n < z_n$ for $n \geq n_0$ and such that*

$$y_{n-k} \leq f(y_n, y_{n-1}, \dots, y_{n-k+1}),\ f(z_n, z_{n-1}, \dots, z_{n-k+1}) \leq z_{n-k},\ \text{for } n \geq n_0 + k - 1. \tag{2.6}$$

Then the difference equation

$$x_{n-k} = f(x_n, x_{n-1}, \dots, x_{n-k+1}) \tag{2.7}$$

has a solution x_n such that

$$y_n \leq x_n \leq z_n \ \text{for } n \geq n_0. \tag{2.8}$$

Based on Theorem 2.1 we prove Theorem 2.2.

Theorem 2.2. *Equation* (1.2) *possesses a solution with the finite asymptotic expansion* (2.4) *as $n \to \infty$ and the coefficient* (2.5).

Proof. By means of abbreviation

$$F(x_n, x_{n-1}, x_{n-2}, x_{n-3}) = f(x_n, x_{n-1}, x_{n-2}) - x_{n-3}$$

the inequalities (2.6) turn into

$$F(z_n, z_{n-1}, z_{n-2}, z_{n-3}) \leq 0 \leq F(y_n, y_{n-1}, y_{n-2}, y_{n-3}) \tag{2.9}$$

with $f(x_n, x_{n-1}, x_{n-2}) = x_n + x_n x_{n-1} x_{n-2} + x_n^3 + x_{n-1}^3 + x_n^2 x_{n-1}$

These inequalities can be interpreted as a certain intermediate value property of the function $F(x_n, x_{n-1}, x_{n-2}, x_{n-3})$. Then the premisses concerning the arguments of f are satisfied. Inserting the ansatz (2.4) into

$$F(x_n, x_{n-1}, x_{n-2}, x_{n-3}) = x_n + x_n x_{n-1} x_{n-2} + x_n^3 + x_{n-1}^3 + x_n^2 x_{n-1} - x_{n-3}$$

we obtain again by means of the DERIVE system as $n \to \infty$

$$F \sim \frac{a}{2}(8a^2 - 3)\frac{1}{\sqrt{n^3}}$$

and taking into account successively the coefficients (2.5)

$$F \sim 3(b - \frac{11\sqrt{6}}{64})\frac{1}{\sqrt{n^5}}$$

$$F \sim 3(\frac{363\sqrt{6}}{2048} - c)\frac{\ln^2 n}{\sqrt{n^7}} \tag{2.10}$$

$$F \sim -3(d + \frac{121\sqrt{6}}{512})\frac{\ln n}{\sqrt{n^7}}$$

$$F \sim \frac{-5}{4}(e + 0)\frac{1}{\sqrt{n^7}}$$

as well as

$$F \sim \frac{219615\sqrt{6}}{32768}\frac{\ln^3 n}{\sqrt{n^9}}.$$

Choosing

$$y_n = x_n - \frac{p}{n^{\frac{5}{2}}},\ z_n = x_n + \frac{p}{n^{\frac{5}{2}}} \tag{2.11}$$

with some constant $p > 0$, we see from (2.11) with $0 - p$ respect $0 + p$ instead of $e = 0$ that the inequalities (2.9) are satisfied for sufficiently large n. Hence, using the coefficient (2.5) and considering that $p > 0$ can be chosen arbitrarily. The proof is complete. □

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