International Journal of Difference Equations
ISSN 0973-6069, Volume 5, Number 2, pp. 149–174 (2010)
http://campus.mst.edu/ijde

The Linear Quadratic Regulator on Time Scales

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Abstract

In this paper, we unify and extend the linear quadratic regulator to dynamic equations on time scales associated with a linear time-invariant system. Here we seek to find an optimal control that minimizes a cost functional. We show that when the final state is fixed, the optimal (open-loop) input is in terms of a final state difference. On the other hand, when the final state is free, we have a closed-loop optimal control.

AMS Subject Classifications: 34N05, 39A06, 39A10, 39A13, 49N05, 49N10, 49N40, 49N90.
Keywords: Time scale, dynamic equation, optimal control, regulator problem, cost functional, Riccati equation.

1 Introduction

In the late 1950s and early 1960s, the emergence of state-space methods gave way to the formulation and solution of a new kind of optimal control problem. The goal of such problems was to find an optimal control that minimizes a quadratic cost functional associated with a linear system. In 1958, Kalman and Koepcke [8] first used this method to find an optimal control for a sampled-data system. Shortly thereafter, Kalman extended these results to continuous time (see [6, 7]). Since then, what is now called the linear quadratic regulator (LQR) plays a central rôle in control engineering. For more details on the linear quadratic regulator in the continuous and discrete cases, one may see the books by Lewis and Syrmos [10] and Kwakernaak and Sivan [9].

Received October 6, 2010; Accepted November 30, 2010
Communicated by Mehmet Ünal

In this work, we introduce the concept of the linear quadratic regulator on time scales, hence unifying the discrete and continuous cases and extending them to cases "in between." We consider the linear time-invariant state equation

$$x^{\Delta}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \tag{1.1}$$

associated with the quadratic performance index

$$J = \frac{1}{2}x^T(t_{\mathrm{f}})S(t_{\mathrm{f}})x(t_{\mathrm{f}}) + \frac{1}{2}\int_{t_0}^{t_{\mathrm{f}}} (x^T Qx + u^T Ru)(\tau)\Delta\tau, \tag{1.2}$$

where $S(t_{\mathrm{f}})$, $Q \geq 0$ and $R > 0$.

In Section 2, we offer a brief introduction to calculus on time scales. We also consider the derivative and integral of functions on an arbitrary time scale, along with some basic properties of each. Also, since we are mainly concerned with linear systems, we examine the matrix exponential and its properties. In Section 3, we consider the variational properties needed such that a minimum control exists. In Section 4, we find the form of cost functional (1.2) in the absence of an input. In this setting, we use a generalized Lyapunov equation to rewrite the cost functional. For Sections 5 and 6, we determine an optimal control u that minimizes (1.2) over the interval $[t_0, t_{\mathrm{f}}]$. We assume that $x(t_0)$ is given and that the final time $t_{\mathrm{f}} \in \mathbb{T}$ is a fixed known number. Depending on the final state, this optimal control can take on two completely different forms. In Section 5, the final state is fixed resulting in an open-loop control. This means that the optimal input is not in terms of the current state, but a final state difference instead. For Section 6, the final state is free resulting in a closed-loop control. This means that the optimal input is in terms of the current state (i.e., state feedback). Here, we use the "simple useful formula" to find a Riccati equation whose solution gives us an optimal control. In Section 7, we include scalar examples for both cases. This work is from the second author's dissertation [11].

2 Time Scales Preliminaries

Before we present our linear system, a brief introduction to the theory of dynamic equations on time scales is in order. For a more in-depth study of time scales, see Bohner and Peterson's books [2, 3].

Definition 2.1. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We let $\mathbb{T}^{\kappa} = \mathbb{T} \setminus \{\max \mathbb{T}\}$ if $\max \mathbb{T}$ exists; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$.

Example 2.2. The most common examples of time scales are $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, and $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$.

Definition 2.3. We define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ and the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by

$$\sigma(t) := \inf \{s \in \mathbb{T} : \ s > t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

Definition 2.4. For any function $f : \mathbb{T} \to \mathbb{R}$, we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma = f \circ \sigma$.

Next, we define the delta (or Hilger) derivative as follows.

Definition 2.5. Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. The *delta derivative* $f^\Delta(t)$ is the number (when it exists) such that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

In the next two theorems, we consider some properties of the delta derivative.

Theorem 2.6 (See [2, Theorem 1.16]). *Suppose $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

a. If f is differentiable at t, then f is continuous at t.

b. If f is continuous at t, where t is right-scattered, then f is differentiable at t and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

c. If f is differentiable at t, where t is right-dense, then

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

d. If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t). \tag{2.1}$$

Note that (2.1) is sometimes called the "simple useful formula."

Remark 2.7. Note the following examples.

a. When $\mathbb{T} = \mathbb{R}$, then (if the limit exists)

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

b. When $\mathbb{T} = \mathbb{Z}$, then

$$f^\Delta(t) = f(t+1) - f(t) =: \Delta f(t).$$

c. When $\mathbb{T} = q^{\mathbb{Z}}$ for $q > 1$, then

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t} =: D_q f(t).$$

Next we consider the linearity property as well as the product rules.

Theorem 2.8 (See [2, Theorem 1.20]). *Let $f, g : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^\kappa$. Then we have the following:*

a. For any constants α and β, the sum $(\alpha f + \beta g) : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f + \beta g)^\Delta(t) = \alpha f^\Delta(t) + \beta g^\Delta(t).$$

b. The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

Definition 2.9. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* on $\mathbb{T}$ when f is continuous in points $t \in \mathbb{T}$ with $\sigma(t) = t$ and it has finite left-sided limits in points $t \in \mathbb{T}$ with $\sup\{s \in \mathbb{T} : \ s < t\} = t$. The class of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}) = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $\mathrm{C^1_{rd}}$.

Theorem 2.10 (See [2, Theorem 1.74]). *Any rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ has an antiderivative F, i.e., $F^\Delta = f$ on $\mathbb{T}^\kappa$.*

Definition 2.11. Let $f \in \mathrm{C_{rd}}$ and let F be any function such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. Then the Cauchy integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all} \quad a, b \in \mathbb{T}.$$

Example 2.12. Let $a, b \in \mathbb{T}$ with $a < b$ and assume that $f \in \mathrm{C_{rd}}$. When $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,\mathrm{d}t.$$

When $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\,\Delta t = \sum_{t=a}^{b-1} f(t).$$

Next, we present the matrix exponential and some of its properties.

Definition 2.13. An $m \times n$ matrix-valued function A on $\mathbb{T}$ is rd-continuous if each of its entries are rd-continuous. Furthermore, if $m = n$, A is said to be *regressive* (we write $A \in \mathcal{R}$) if

$$I + \mu(t)A(t) \quad \text{is invertible for all} \quad t \in \mathbb{T}^\kappa.$$

Theorem 2.14 (See [2, Theorem 5.8]). *Suppose that A is regressive and rd-continuous. Then the initial value problem*

$$X^{\Delta}(t) = A(t)X(t), \quad X(t_0) = I,$$

where I is the identity matrix, has a unique $n \times n$ matrix-valued solution X.

Definition 2.15. The solution X from Theorem 2.14 is called the matrix exponential function on $\mathbb{T}$ and is denoted by $e_A(\cdot, t_0)$.

Theorem 2.16 (See [2, Theorem 5.21]). *Let A be regressive and rd-continuous. Then for $r, s, t \in \mathbb{T}$,*

a. $e_A(t, s)e_A(s, r) = e_A(t, r)$, *hence* $e_A(t, t) = I$,

b. $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$,

c. $e_A(t, \sigma(s)) = e_A(t, s)(I + \mu(s)A(s))^{-1}$,

d. $(e_A(\cdot, s))^{\Delta} = Ae_A(\cdot, s)$,

e. $(e_A(t, \cdot))^{\Delta} = -e_A^{\sigma}(t, \cdot)A(s) = -e_A(t, \cdot)(I + \mu(s)A(s))^{-1}A(s)$.

Next we give the solution (state response) to our linear system using variation of parameters.

Theorem 2.17 (See [2, Theorem 5.24]). *Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then the solution of the initial value problem*

$$x^{\Delta}(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0$$

is given by

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

3 Optimization of Linear Systems on Time Scales

Definition 3.1. Let $a, b \in \mathbb{T}$ with $a < b$ and $\alpha, \beta \in \mathbb{R}^n$. A function $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ with $\hat{y}(a) = \alpha$, $\hat{y}(b) = \beta$ is said to be a (weak) *local minimum* to the variational problem

$$\mathcal{J}(y) = \int_a^b L(t, y^{\sigma}(t), y^{\Delta}(t))\Delta t \to \min, \quad y(a) = \alpha, \quad y(b) = \beta, \tag{3.1}$$

where $L : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$, if there exists $\delta > 0$ such that $||y - \hat{y}|| < \delta$ and $\mathcal{J}(\hat{y}) \leq \mathcal{J}(y)$ for all $y \in \mathrm{C}^1_{\mathrm{rd}}$ satisfying $\hat{y}(a) = \alpha$ and $\hat{y}(b) = \beta$. If $\mathcal{J}(\hat{y}) < \mathcal{J}(y)$ for all $\hat{y} \neq y$, then

$\hat{y}$ is said to be *proper*. An $\eta \in \mathrm{C}^1_{\mathrm{rd}}$ is called an *admissible variation* of (3.1) provided $\eta(a) = \eta(b) = 0$. Let $\eta \in \mathrm{C}^1_{\mathrm{rd}}$ be an admissible variation. We define the function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(\varepsilon) = \Phi(\varepsilon; y, \eta) = \mathcal{J}(y + \varepsilon\eta), \quad \varepsilon \in \mathbb{R}.$$

Then the *first variation* of $\mathcal{J}$ is defined by $\mathcal{J}_1(y, \eta) = \dot{\Phi}(0; y, \eta)$, while the *second variation* of $\mathcal{J}$ is defined by $\mathcal{J}_2(y, \eta) = \ddot{\Phi}(0; y, \eta)$.

In the next two theorems, we provide necessary and sufficient conditions for a local minimum.

Theorem 3.2 (See [1, Theorem 3.2]). *If $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ is a local minimum of* (3.1)*, then $\mathcal{J}_1(\hat{y}, \eta) = 0$ and $\mathcal{J}_2(\hat{y}, \eta) \geq 0$ for all admissible variations η.*

Theorem 3.3 (See [1, Theorem 3.3]). *Let $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ with $\hat{y}(a) = \alpha$ and $\hat{y}(b) = \beta$. If $\mathcal{J}_1(\hat{y}, \eta) = 0$ and $\mathcal{J}_2(\hat{y}, \eta) > 0$ for all nontrivial admissible variations η, then $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ is a proper weak local minimum to* (3.1).

Now we consider the linear time-invariant system (plant)

$$x^{\Delta}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \tag{3.2}$$

where $x \in \mathbb{R}^n$ represents the state and $u \in \mathbb{R}^m$ represents the input. Associated with (3.2) is the quadratic cost functional

$$J = \frac{1}{2}x^T(t_{\mathrm{f}})S(t_{\mathrm{f}})x(t_{\mathrm{f}}) + \frac{1}{2}\int_{t_0}^{t_{\mathrm{f}}} (x^TQx + u^TRu)(\tau)\Delta\tau, \tag{3.3}$$

where $S(t_{\mathrm{f}})$, $Q \geq 0$ and $R > 0$. Now we determine the form of the input (control law) that minimizes (3.3). To minimize (3.3), we introduce the augmented cost functional

$$\begin{aligned} J^+ &= \frac{1}{2}x^T(t_{\mathrm{f}})S(t_{\mathrm{f}})x(t_{\mathrm{f}}) \\ &\quad + \int_{t_0}^{t_{\mathrm{f}}} \left[\frac{1}{2}(x^TQx + u^TRu) + (\lambda^\sigma)^T(Ax + Bu - x^{\Delta})\right](\tau)\Delta\tau \\ &= \frac{1}{2}x^T(t_{\mathrm{f}})S(t_{\mathrm{f}})x(t_{\mathrm{f}}) + \int_{t_0}^{t_{\mathrm{f}}} [H(x, u, \lambda^\sigma) - (\lambda^\sigma)^T x^{\Delta}](\tau)\Delta\tau, \end{aligned}$$

where the so-called *Hamiltonian* H is given by

$$H(x, u, \lambda) = \frac{1}{2}(x^TQx + u^TRu) + \lambda^T(Ax + Bu), \tag{3.4}$$

and $\lambda \in \mathbb{R}^n$ is a multiplier to be determined in later subsections.

Next, we provide necessary conditions for an optimal control. We assume that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{t_0}^{t_{\mathrm{f}}} f(\tau, \varepsilon)\Delta\tau = \int_{t_0}^{t_{\mathrm{f}}} \frac{\partial}{\partial\varepsilon}f(\tau, \varepsilon)\Delta\tau \tag{3.5}$$

for all $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that $f(\cdot, \varepsilon), \partial f(\cdot, \varepsilon)/\partial\varepsilon \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T})$.

Lemma 3.4. *Assume* (3.5). *Then the first variation of J^+ is zero provided that x, u, and λ satisfy*

$$x^\Delta = Ax + Bu, \tag{3.6a}$$

$$-\lambda^\Delta = Qx + A^T\lambda^\sigma, \tag{3.6b}$$

$$0 = Ru + B^T\lambda^\sigma. \tag{3.6c}$$

Proof. First note that

$$\begin{aligned}
\Phi(\varepsilon) &= J^+((x,u,\lambda)+\varepsilon(\eta_1,\eta_2,\eta_3)) \\
&= \frac{1}{2}(x+\varepsilon\eta_1)^T(t_\mathrm{f})S(t_\mathrm{f})(x+\varepsilon\eta_1)(t_\mathrm{f}) \\
&\quad +\frac{1}{2}\int_{t_0}^{t_\mathrm{f}}[(x+\varepsilon\eta_1)^TQ(x+\varepsilon\eta_1)+(u+\varepsilon\eta_2)^TR(u+\varepsilon\eta_2)](\tau)\Delta\tau \\
&\quad +\int_{t_0}^{t_\mathrm{f}}[(\lambda^\sigma+\varepsilon\eta_3^\sigma)^TA(x+\varepsilon\eta_1)+(\lambda^\sigma+\varepsilon\eta_3^\sigma)^TB(u+\varepsilon\eta_2)](\tau)\Delta\tau \\
&\quad -\int_{t_0}^{t_\mathrm{f}}(\lambda^\sigma+\varepsilon\eta_3^\sigma)^T(x+\varepsilon\eta_1)^\Delta(\tau)\Delta\tau.
\end{aligned}$$

Then

$$\begin{aligned}
\dot\Phi(\varepsilon) &= \frac{1}{2}\eta_1^T(t_\mathrm{f})S(t_\mathrm{f})(x+\varepsilon\eta_1)(t_\mathrm{f})+\frac{1}{2}(x+\varepsilon\eta_1)^T(t_\mathrm{f})S(t_\mathrm{f})\eta_1(t_\mathrm{f}) \\
&\quad +\frac{1}{2}\int_{t_0}^{t_\mathrm{f}}[\eta_1^TQ(x+\varepsilon\eta_1)+(x+\varepsilon\eta_1)^TQ\eta_1](\tau)\Delta\tau \\
&\quad +\frac{1}{2}\int_{t_0}^{t_\mathrm{f}}[\eta_2^TR(u+\varepsilon\eta_2)+(u+\varepsilon\eta_2)^TR\eta_2](\tau)\Delta\tau \\
&\quad +\int_{t_0}^{t_\mathrm{f}}[(\eta_3^\sigma)^TA(x+\varepsilon\eta_1)+(\lambda^\sigma+\varepsilon\eta_3^\sigma)^TA\eta_1](\tau)\Delta\tau \\
&\quad +\int_{t_0}^{t_\mathrm{f}}[(\eta_3^\sigma)^TB(u+\varepsilon\eta_2)+(\lambda^\sigma+\varepsilon\eta_3^\sigma)^TB\eta_2](\tau)\Delta\tau \\
&\quad -\int_{t_0}^{t_\mathrm{f}}[(\eta_3^\sigma)^T(x+\varepsilon\eta_1)^\Delta+(\lambda^\sigma+\varepsilon\eta_3^\sigma)^T\eta_1^\Delta](\tau)\Delta\tau.
\end{aligned}$$

Thus the first variation can be written as

$$\begin{aligned}
\dot\Phi(0) &= [S(t_\mathrm{f})x(t_\mathrm{f})-\lambda(t_\mathrm{f})]^T\eta_1(t_\mathrm{f})+\lambda^T(t_0)\eta_1(t_0) \\
&\quad +\int_{t_0}^{t_\mathrm{f}}[(A^T\lambda^\sigma+Qx+\lambda^\Delta)^T\eta_1+(Ru+B^T\lambda^\sigma)^T\eta_2](\tau)\Delta\tau \\
&\quad +\int_{t_0}^{t_\mathrm{f}}[(Ax+Bu-x^\Delta)^T\eta_3^\sigma](\tau)\Delta\tau.
\end{aligned}$$

Now in order for $\dot{\Phi}(0) = 0$, we set each coefficient of independent increments η_1, η_2, η_3^σ equal to zero. This yields the necessary conditions for a minimum of J^+. Using the Hamiltonian (3.4), we have state and costate equations

$$x^\Delta = H_\lambda(x, u, \lambda^\sigma) = Ax + Bu$$

and

$$-\lambda^\Delta = H_x(x, u, \lambda^\sigma) = Qx + A^T\lambda^\sigma.$$

Similarly, we have the stationary condition

$$0 = H_u(x, u, \lambda^\sigma) = Ru + B^T\lambda^\sigma.$$

This concludes the proof. □

Remark 3.5. We note that x, u, and λ solve (3.6) if and only if they solve

$$x^\Delta = Ax - BR^{-1}B^T\lambda^\sigma, \tag{3.7a}$$

$$-\lambda^\Delta = Qx + A^T\lambda^\sigma, \tag{3.7b}$$

$$u = -R^{-1}B^T\lambda^\sigma. \tag{3.7c}$$

Note that in order to find an optimal control, one must determine a value for the costate.

In Figure 3.1, we present the block diagram that describes the formulation of the state and costate equations in our linear quadratic optimal controller.

Finally, we give sufficient conditions for a local optimal control.

Lemma 3.6. *Assume* (3.5)*. Then the second variation of* J^+ *is positive provided that* η_1 *and* η_2 *satisfy the constraints* $\eta_1^\Delta = A\eta_1 + B\eta_2$ *and* $\eta_2 \neq 0$*.*

Proof. Taking the second derivative of $\Phi(\varepsilon)$, we have

$$\begin{aligned}
\ddot{\Phi}(\varepsilon) &= \frac{1}{2}\eta_1^T(t_f)S(t_f)\eta_1(t_f) + \frac{1}{2}\eta_1^T(t_f)S(t_f)\eta_1(t_f) \\
&\quad + \frac{1}{2}\int_{t_0}^{t_f} [\eta_1^T Q\eta_1 + \eta_1^T Q\eta_1 + \eta_2^T R\eta_2 + \eta_2^T R\eta_2](\tau)\Delta\tau \\
&\quad + \int_{t_0}^{t_f} [(\eta_3^\sigma)^T A\eta_1 + (\eta_3^\sigma)^T A\eta_1 + (\eta_3^\sigma)^T B\eta_2 + (\eta_3^\sigma)^T B\eta_2](\tau)\Delta\tau \\
&\quad - \int_{t_0}^{t_f} [(\eta_3^\sigma)^T \eta_1^\Delta + (\eta_3^\sigma)^T \eta_1^\Delta](\tau)\Delta\tau \\
&= \eta_1^T(t_f)S(t_f)\eta_1(t_f) + \int_{t_0}^{t_f} [\eta_1^T Q\eta_1 + \eta_2^T R\eta_2](\tau)\Delta\tau \\
&\quad + 2\int_{t_0}^{t_f} [(A\eta_1 + B\eta_2 - \eta_1^\Delta)^T \eta_3^\sigma](\tau)\Delta\tau.
\end{aligned}$$

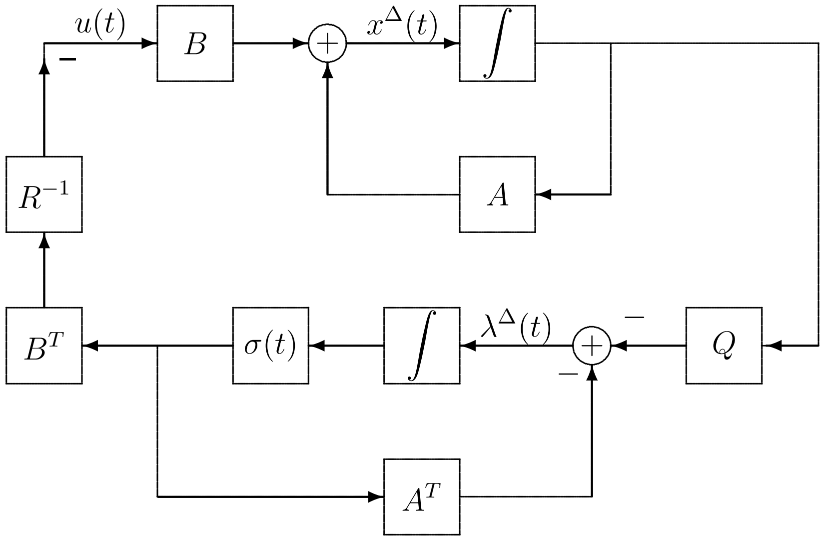

Figure 3.1: Relationship of the state and costate equations with the optimal control in the LQR

If we assume that η_1 and η_2 satisfy the constraint

$$\eta_1^{\Delta} = A\eta_1 + B\eta_2,$$

then the second variation is given by

$$\ddot{\Phi}(0) = \eta_1^T(t_{\mathrm{f}})S(t_{\mathrm{f}})\eta_1(t_{\mathrm{f}}) + \int_{t_0}^{t_{\mathrm{f}}} [\eta_1^T Q\eta_1 + \eta_2^T R\eta_2](\tau)\Delta\tau. \tag{3.8}$$

Note that $S(t_{\mathrm{f}})$ and $Q \geq 0$ while $R > 0$. Thus if $\eta_2 \neq 0$, then (3.8) is guaranteed to be positive. □

4 Zero Input and the Observability Lyapunov Equation

In this section, we find the form of our cost functional (3.3) in the absence of a given input. Here we make no assumptions as to whether or not the final state is fixed. We use the notion of Lyapunov and Riccati equations on time scales to obtain our results. We now consider the autonomous dynamic equation (3.6a) with $B = 0$, i.e.,

$$x^{\Delta} = Ax, \tag{4.1}$$

where A is assumed to be regressive.

Definition 4.1. Let $S \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$. Then $V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$V(t,x) = x^T S(t) x$$

is called a time scales *Lyapunov function*.

Lemma 4.2. *Let $S : \mathbb{T} \to \mathbb{R}^n$ be differentiable. If x solves* (4.1)*, then*

$$(x^T S x)^\Delta = x^T[A^T S + (I+\mu A^T)SA + (I+\mu A^T)S^\Delta (I+\mu A)]x. \tag{4.2}$$

Proof. Using the product rule, we have

$$\begin{aligned}(x^T S x)^\Delta &= (x^T S)^\Delta x^\sigma + (x^T S)x^\Delta \\ &= [(x^T)^\Delta S + (x^T)^\sigma S^\Delta] x^\sigma + (x^T S)Ax.\end{aligned}$$

Now using (2.1) and (4.1), we have

$$\begin{aligned}(x^T S x)^\Delta &= [x^T A^T S + (x+\mu x^\Delta)^T S^\Delta](x+\mu x^\Delta) + x^T SAx \\ &= [x^T A^T S + x^T (I+\mu A)^T S^\Delta](I+\mu A)x + x^T SAx \\ &= x^T[A^T S + (I+\mu A^T)SA + (I+\mu A^T)S^\Delta (I+\mu A)]x.\end{aligned}$$

This gives (4.2) as desired. □

Definition 4.3. Let $S \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ and Q be symmetric. Then a Lyapunov equation is defined to be

$$-S^\Delta = (I+\mu A^T)^{-1}[A^T S + (I+\mu A^T)SA + Q](I+\mu A)^{-1}. \tag{4.3}$$

As (4.3) describes the interaction between the plant and cost in the absence of u, it can be called an "observability" Lyapunov equation.

Lemma 4.4. *An S solves* (4.3) *if and only if it solves*

$$-S^\Delta = A^T S^\sigma + (I+\mu A^T)S^\sigma A + Q. \tag{4.4}$$

Proof. Multiplying (4.3) by $(I+\mu A^T)$ on the left and by $(I+\mu A)$ on the right, we arrive at

$$-(I+\mu A^T)S^\Delta (I+\mu A) = A^T S + (I+\mu A^T)SA + Q.$$

Now expanding the left-hand side, we have

$$-S^\Delta - \mu A^T S^\Delta - \mu(I+\mu A^T)S^\Delta A = A^T S + (I+\mu A^T)SA + Q.$$

Moving the last two terms over to the right-hand side and using (2.1) yields

$$\begin{aligned}-S^\Delta &= A^T(S+\mu S^\Delta) + (I+\mu A^T)(S+\mu S^\Delta)A + Q \\ &= A^T S^\sigma + (I+\mu A^T)S^\sigma A + Q.\end{aligned}$$

This gives (4.4) as desired. □

Theorem 4.5. *A solution to* (4.3) *is given by*

$$S(t) = e_A^T(t_{\mathrm{f}}, t)S(t_{\mathrm{f}})e_A(t_{\mathrm{f}}, t) + \int_t^{t_{\mathrm{f}}} e_A^T(\tau, t)Qe_A(\tau, t)\Delta\tau.$$

Proof. First note that Theorem 2.16(a) allows the representation

$$S(t) = e_A^T(t_{\mathrm{f}}, t)\tilde{S}(t)e_A(t_{\mathrm{f}}, t),$$

where

$$\tilde{S}(t) = S(t_{\mathrm{f}}) + \int_t^{t_{\mathrm{f}}} e_A^T(\tau, t_{\mathrm{f}})Qe_A(\tau, t_{\mathrm{f}})\Delta\tau.$$

Now using product rule, we have

$$\begin{aligned} S^\Delta(t) &= (e_A^\Delta(t_{\mathrm{f}}, t))^T\tilde{S}(\sigma(t))e_A(t_{\mathrm{f}}, \sigma(t)) + e_A^T(t_{\mathrm{f}}, t)\tilde{S}^\Delta(t)e_A(t_{\mathrm{f}}, t) \\ &\quad + e_A^T(t_{\mathrm{f}}, t)\tilde{S}(\sigma(t))e_A^\Delta(t_{\mathrm{f}}, t). \end{aligned}$$

Since by Theorem 2.16(e)

$$\begin{aligned} (e_A^\Delta(t_{\mathrm{f}}, t))^T\tilde{S}(\sigma(t))e_A(t_{\mathrm{f}}, \sigma(t)) &= -A^T e_A(t_{\mathrm{f}}, \sigma(t))\tilde{S}(\sigma(t))e_A(t_{\mathrm{f}}, \sigma(t)) \\ &= -A^T S(\sigma(t)), \end{aligned}$$

by Theorem 2.16(a)

$$\begin{aligned} e_A^T(t_{\mathrm{f}}, t)\tilde{S}^\Delta(t)e_A(t_{\mathrm{f}}, t) &= -e_A^T(t_{\mathrm{f}}, t)e_A^T(t, t_{\mathrm{f}})Qe_A(t, t_{\mathrm{f}})e_A(t_{\mathrm{f}}, t) \\ &= -Q, \end{aligned}$$

and by Theorem 2.16(c) and (e)

$$\begin{aligned} e_A^T(t_{\mathrm{f}}, t)\tilde{S}(\sigma(t))e_A^\Delta(t_{\mathrm{f}}, t) &= -(I + \mu(t)A^T)e_A^T(t_{\mathrm{f}}, \sigma(t))\tilde{S}(\sigma(t))e_A(t_{\mathrm{f}}, \sigma(t))A \\ &= -(I + \mu(t)A^T)S(\sigma(t))A, \end{aligned}$$

S solves (4.4). By Lemma 4.4, S also solves (4.3). This concludes the proof. □

Now we rewrite the quadratic performance index (3.3) as follows.

Theorem 4.6. *Suppose that S solves* (4.3). *If x solves* (4.1) *and $u = 0$, then the cost functional* (3.3) *can be rewritten as*

$$J = \frac{1}{2}x^T(t_0)S(t_0)x(t_0).$$

Proof. We use Lemma 4.2 to rewrite the cost functional (3.3) as

$$\begin{aligned} J &= \frac{1}{2}x^T(t_\mathrm{f})S(t_\mathrm{f})x(t_\mathrm{f}) + \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} (x^TQx)(t)\Delta t \\ &= \frac{1}{2}x^T(t_\mathrm{f})S(t_\mathrm{f})x(t_\mathrm{f}) + \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} (x^TQx)(t)\Delta t + \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} (x^TSx)^\Delta(t)\Delta t \\ &\quad -\frac{1}{2}x^T(t_\mathrm{f})S(t_\mathrm{f})x(t_\mathrm{f}) + \frac{1}{2}x^T(t_0)S(t_0)x(t_0) \\ &= \frac{1}{2}x^T(t_0)S(t_0)x(t_0) + \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} [x^TQx + (x^TSx)^\Delta](t)\Delta t \\ &= \frac{1}{2}x^T(t_0)S(t_0)x(t_0) \\ &\quad +\frac{1}{2}\int_{t_0}^{t_\mathrm{f}} x^T[Q + (I+\mu A^T)S^\Delta(I+\mu A) + A^TS(I+\mu A) + SA]x(t)\Delta t \\ &= \frac{1}{2}x^T(t_0)S(t_0)x(t_0), \end{aligned}$$

where (4.3) is used in the last step. □

5 Fixed-Final-State and Open-Loop Control

In this section, we seek an optimal control when the final state is fixed. We assume $Q = 0$ and solve (3.7) subject to

$$x(t_0) = x_0 \quad \text{and} \quad x(t_\mathrm{f}) = x_\mathrm{f}, \tag{5.1}$$

where x_f is a given fixed reference value. Since $x(t_\mathrm{f})$ is fixed, it is redundant to have a final-state weighting in the performance index, so we let $S(t_\mathrm{f}) = 0$. Then the cost functional (3.3) becomes

$$J = \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} (u^TRu)(\tau)\Delta\tau. \tag{5.2}$$

Definition 5.1. The *final state difference*, d_f, is the difference between the zero input solution and the desired final state, i.e.,

$$d_\mathrm{f} = x(t_\mathrm{f}) - e_A(t_\mathrm{f}, t_0)x(t_0). \tag{5.3}$$

Next, we find an optimal control proportional to (5.3). Note that the following theorem mirrors Kalman's generalized controllability criterion as found in [4, Theorem 3.2].

Theorem 5.2. *Let $Q = 0$ and suppose that x, u, and λ solve* (3.7) *such that* (5.1) *hold. If*

$$G_C := \int_{t_0}^{t_\mathrm{f}} e_A(t_\mathrm{f}, \sigma(\tau))BR^{-1}B^Te_A^T(t_\mathrm{f}, \sigma(\tau))\Delta\tau \quad \textit{is invertible,} \tag{5.4}$$

then

$$u(t) = R^{-1}B^T e_A^T(t_\mathrm{f}, \sigma(t))G_C^{-1}d_\mathrm{f}. \tag{5.5}$$

Proof. Since $Q = 0$, (3.7b) becomes $\lambda^\Delta = -A^T\lambda^\sigma$, whose solution, using Theorem 2.16(e), is

$$\lambda(t) = e_A^T(t_\mathrm{f}, t)\lambda(t_\mathrm{f}). \tag{5.6}$$

Using (5.6), (3.7a) becomes

$$x^\Delta(t) = Ax(t) - BR^{-1}B^T e_A^T(t_\mathrm{f}, \sigma(t))\lambda(t_\mathrm{f}). \tag{5.7}$$

Now solving (5.7) with Theorem 2.17 at time $t = t_\mathrm{f}$, we get

$$\begin{aligned} x(t_\mathrm{f}) &= e_A(t_\mathrm{f}, t_0)x(t_0) - \int_{t_0}^{t_\mathrm{f}} e_A(t_\mathrm{f}, \sigma(\tau))BR^{-1}B^T e_A^T(t_\mathrm{f}, \sigma(\tau))\lambda(t_\mathrm{f})\Delta\tau \\ &= e_A(t_\mathrm{f}, t_0)x(t_0) - G_C\lambda(t_\mathrm{f}). \end{aligned}$$

Then solving for $\lambda(t_\mathrm{f})$ and using (5.1), we have

$$\lambda(t_\mathrm{f}) = -G_C^{-1}[x(t_\mathrm{f}) - e_A(t_\mathrm{f}, t_0)x(t_0)] = -G_C^{-1}d_\mathrm{f}.$$

Using this and (5.6), (3.7c) becomes

$$\begin{aligned} u(t) &= -R^{-1}B^T\lambda(\sigma(t)) \\ &= -R^{-1}B^T e_A^T(t_\mathrm{f}, \sigma(t))\lambda(t_\mathrm{f}) \\ &= R^{-1}B^T e_A^T(t_\mathrm{f}, \sigma(t))G_C^{-1}d_\mathrm{f}. \end{aligned}$$

Hence (5.5) holds. □

Note that (5.4) represents a weighted controllability Gramian. The control (5.5) represents the minimum-energy control that drives the given initial state $x(t_0)$ to the desired final reference value $x(t_\mathrm{f})$. It is called an *open-loop* control since while it depends on both the initial and final states, it does not rely on the current state.

Next, we determine the optimal cost.

Theorem 5.3. *If u is given by* (5.5)*, then the cost functional* (5.2) *can be rewritten as*

$$J = \frac{1}{2}d_\mathrm{f}^T G_C^{-1}d_\mathrm{f}. \tag{5.8}$$

Proof. Let u be as given by (5.5). Then (5.2) can be rewritten as

$$\begin{aligned} J &= \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} (u^T R u)(\tau)\Delta\tau \\ &= \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} d_\mathrm{f}^T G_C^{-1} e_A(t_\mathrm{f},\sigma(\tau)) B R^{-1} R R^{-1} B^T e_A^T(t_\mathrm{f},\sigma(\tau)) G_C^{-1} d_\mathrm{f}\Delta\tau \\ &= \frac{1}{2} d_\mathrm{f}^T G_C^{-1} \int_{t_0}^{t_\mathrm{f}} e_A(t_\mathrm{f},\sigma(\tau)) B R^{-1} B^T e_A^T(t_\mathrm{f},\sigma(\tau))\Delta\tau G_C^{-1} d_\mathrm{f} \\ &= \frac{1}{2} d_\mathrm{f}^T G_C^{-1} G_C G_C^{-1} d_\mathrm{f} \\ &= \frac{1}{2} d_\mathrm{f}^T G_C^{-1} d_\mathrm{f}. \end{aligned}$$

Hence (5.8) holds. □

6 Free-Final-State and Closed-Loop Control

In this section, we develop an optimal control law in the form of state feedback. In considering the boundary conditions, note that $x(t_0)$ is known (meaning $\eta_1(t_0) = 0$) while $x(t_\mathrm{f})$ is free (meaning $\eta_1(t_\mathrm{f}) \neq 0$). Thus the coefficient on $\eta_1(t_\mathrm{f})$ must be zero. This gives the terminal condition on the costate to be

$$\lambda(t_\mathrm{f}) = S(t_\mathrm{f})x(t_\mathrm{f}). \tag{6.1}$$

Remark 6.1. Now in order to solve the two-point boundary value problem, we make the assumption that x and λ satisfy a linear relationship similar to (6.1) for all $t \in [t_0, t_\mathrm{f}]$, i.e.,

$$\lambda(t) = S(t)x(t). \tag{6.2}$$

This condition (6.2) is called a “sweep condition,” a term used by Bryson and Ho in [5]. Since the terminal condition $S(t_\mathrm{f}) \geq 0$, it is natural to assume that $S \geq 0$ as well.

Theorem 6.2. *Assume that S solves*

$$-S^\Delta = Q + A^T S^\sigma + (I+\mu A^T)S^\sigma \left(I + \mu B R^{-1} B^T S^\sigma\right)^{-1} \left(A - B R^{-1} B^T S^\sigma\right). \tag{6.3}$$

If x satisfies

$$x^\Delta = \left(I + \mu B R^{-1} B^T S^\sigma\right)^{-1} \left(A - B R^{-1} B^T S^\sigma\right) x \tag{6.4}$$

and λ is given by (6.2), *then*

$$-\lambda^\Delta = Qx + A^T \lambda^\sigma. \tag{6.5}$$

Proof. Since λ is as given in (6.2), we may use the product rule, (6.3), (6.4), and (2.1) to arrive at

$$\begin{aligned} -\lambda^\Delta &= -S^\Delta x - S^\sigma x^\Delta \\ &= Qx + A^T S^\sigma x + (I + \mu A^T) S^\sigma x^\Delta - S^\sigma x^\Delta \\ &= Qx + A^T S^\sigma x + \mu A^T S^\sigma x^\Delta \\ &= Qx + A^T S^\sigma x^\sigma \\ &= Qx + A^T \lambda^\sigma, \end{aligned}$$

which gives (6.5) as desired. □

In order to rewrite the right-hand side of the Riccati equation (6.3), we now show the following matrix identity.

Lemma 6.3. *If $R + \mu B^T S^\sigma B$ is invertible, then*

$$A - BR^{-1}B^T S^\sigma = \left(I + \mu BR^{-1}B^T S^\sigma\right)[A - B(R + \mu B^T S^\sigma B)^{-1} B^T S^\sigma (I + \mu A)]. \tag{6.6}$$

Proof. The right-hand side of (6.6) is equal to

$$\begin{aligned} &A + \mu BR^{-1}B^T S^\sigma A - B(R + \mu B^T S^\sigma B)^{-1} B^T S^\sigma (I + \mu A) \\ &\quad -\mu BR^{-1}B^T S^\sigma B(R + \mu B^T S^\sigma B)^{-1} B^T S^\sigma (I + \mu A), \end{aligned} \tag{6.7}$$

and the last term of (6.7) is equal to

$$\begin{aligned} &-BR^{-1}(-R + R + \mu B^T S^\sigma B)(R + \mu B^T S^\sigma B)^{-1} B^T S^\sigma (I + \mu A) \\ &= BR^{-1}R(R + \mu B^T S^\sigma B)^{-1} B^T S^\sigma (I + \mu A) \\ &\quad -BR^{-1}(R + \mu B^T S^\sigma B)(R + \mu B^T S^\sigma B)^{-1} B^T S^\sigma (I + \mu A) \\ &= B(R + \mu B^T S^\sigma B)^{-1} B^T S^\sigma (I + \mu A) - BR^{-1}B^T S^\sigma (I + \mu A). \end{aligned}$$

Using this in (6.7) yields (6.6). □

Theorem 6.4. *If both $R + \mu B^T S^\sigma B$ and $I + \mu BR^{-1}B^T S^\sigma$ are invertible, then S solves* (6.3) *if and only if it solves*

$$-S^\Delta = Q + A^T S^\sigma + (I + \mu A^T) S^\sigma A - (I + \mu A^T) S^\sigma B(R + \mu B^T S^\sigma B)^{-1} B^T S^\sigma (I + \mu A). \tag{6.8}$$

Proof. The statement follows directly from Lemma 6.3. □

Next, we rewrite the right-hand side of the Riccati equation (6.8) in terms of the Kalman gain defined as follows.

Definition 6.5. Let $R+\mu B^TS^\sigma B$ be invertible. Then the matrix-valued function

$$K(t)=(R+\mu(t)B^TS^\sigma(t)B)^{-1}B^TS^\sigma(t)(I+\mu(t)A) \tag{6.9}$$

is called the *state feedback* or *Kalman gain*.

Lemma 6.6. *If $R+\mu B^TS^\sigma B$ is invertible and S is symmetric, then*

$$(I+\mu A^T)S^\sigma B(R+\mu B^TS^\sigma B)^{-1}B^TS^\sigma(I+\mu A)=K^T(R+\mu B^TS^\sigma B)K. \tag{6.10}$$

Proof. Using (6.9) twice, we have

$$\begin{aligned}K^T(R+\mu B^TS^\sigma B)K &= (I+\mu A^T)S^\sigma B(R+\mu B^TS^\sigma B)^{-1}(R+\mu B^TS^\sigma B)K\\ &= (I+\mu A^T)S^\sigma BK\\ &= (I+\mu A^T)S^\sigma B(R+\mu B^TS^\sigma B)^{-1}B^TS^\sigma(I+\mu A).\end{aligned}$$

This gives (6.10) as desired. □

Theorem 6.7. *If $R+\mu B^TS^\sigma B$ is invertible and S is symmetric, then S solves* (6.8) *if and only if it solves*

$$-S^\Delta=Q+A^TS^\sigma+(I+\mu A^T)S^\sigma A-K^T(R+\mu B^TS^\sigma B)K. \tag{6.11}$$

Proof. The statement follows directly from Lemma 6.6. □

In order to rewrite the right-hand side of the Riccati equation (6.11) in so-called (generalized) Joseph stabilized form (see [10]), we now show the following matrix identity.

Lemma 6.8. *If $R+\mu B^TS^\sigma B$ is invertible and S is symmetric, then*

$$\begin{aligned}&(I+\mu A^T)S^\sigma A-K^T(R+\mu B^TS^\sigma B)K\\ &\quad= -K^TB^TS^\sigma+(I+\mu(A-BK)^T)S^\sigma(A-BK)+K^TRK.\end{aligned} \tag{6.12}$$

Moreover, both sides of (6.12) *are equal to*

$$(I+\mu A^T)S^\sigma(A-BK).$$

Proof. We use (6.9) to rewrite the left-hand side of (6.12) as

$$\begin{aligned}&(I+\mu A^T)S^\sigma A-(I+\mu A^T)S^\sigma B(R+\mu B^TS^\sigma B)^{-1}(R+\mu B^TS^\sigma B)K\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad= (I+\mu A^T)S^\sigma(A-BK),\end{aligned}$$

and we use (6.9) again to rewrite the right-hand side of (6.12) as

$$\begin{aligned}&-K^TB^TS^\sigma+(I+\mu A^T)S^\sigma(A-BK)-\mu K^TB^TS^\sigma(A-BK)+K^TRK\\ &\quad= (I+\mu A^T)S^\sigma(A-BK)-K^TB^TS^\sigma(I+\mu A)+K^T(R+\mu B^TS^\sigma B)K\\ &\quad= (I+\mu A^T)S^\sigma(A-BK).\end{aligned}$$

Hence (6.12) holds. □

Theorem 6.9. *If $R + \mu B^T S^\sigma B$ is invertible and S is symmetric, then S solves* (6.11) *if and only if it solves*

$$-S^\Delta = Q + (A - BK)^T S^\sigma + (I + \mu(A - BK))^T S^\sigma (A - BK) + K^T RK \quad (6.13)$$

and if and only if it solves

$$-S^\Delta = Q + A^T S^\sigma + (I + \mu A^T) S^\sigma (A - BK). \quad (6.14)$$

Proof. This statement follows directly from Lemma 6.8. □

When an optimal control is in terms of the current state, it is said to be a *closed-loop* control. We now consider the form of an optimal control that minimizes (3.3).

Theorem 6.10. *Let $R + \mu B^T S^\sigma B$ be invertible. Suppose that x, u, and λ solve* (3.7) *such that* (6.2) *holds. Then*

$$u(t) = -K(t)x(t). \quad (6.15)$$

Proof. Using (3.7c), (6.2), (2.1), and (3.6a), we have

$$\begin{aligned} u(t) &= -R^{-1} B^T \lambda^\sigma(t) \\ &= -R^{-1} B^T S^\sigma(t) x^\sigma(t) \\ &= -R^{-1} B^T S^\sigma(t) \left(x(t) + \mu(t) x^\Delta(t) \right) \\ &= -R^{-1} B^T S^\sigma(t)(I + \mu(t)A)x(t) - \mu(t) R^{-1} B^T S^\sigma(t) B u(t). \end{aligned}$$

Now combining like terms and pre-multiplying by R we have

$$(R + \mu(t) B^T S^\sigma(t) B) u(t) = -B^T S^\sigma(t)(I + \mu(t)A)x(t),$$

which implies

$$u(t) = -(R + \mu(t) B^T S^\sigma(t) B)^{-1} B^T S^\sigma(t)(I + \mu(t)A)x(t).$$

Hence (6.15) follows using (6.9). □

The input (6.15) is referred to as the closed-loop control law that minimizes the cost functional (3.3). Note that in order to determine this input, we need only K and the solution to the Riccati equation, S. But recall that we obtained S from the sweep condition, not from the Riccati equation. The Riccati equation itself is used to find an optimal cost. Now the closed-loop plant is given by

$$x^\Delta(t) = (A - BK(t))x(t), \quad (6.16)$$

which can be used to find an optimal trajectory for any given $x(t_0)$. A block diagram describing this control scheme appears in Figure 6.1.

Finally, we determine the optimal cost.

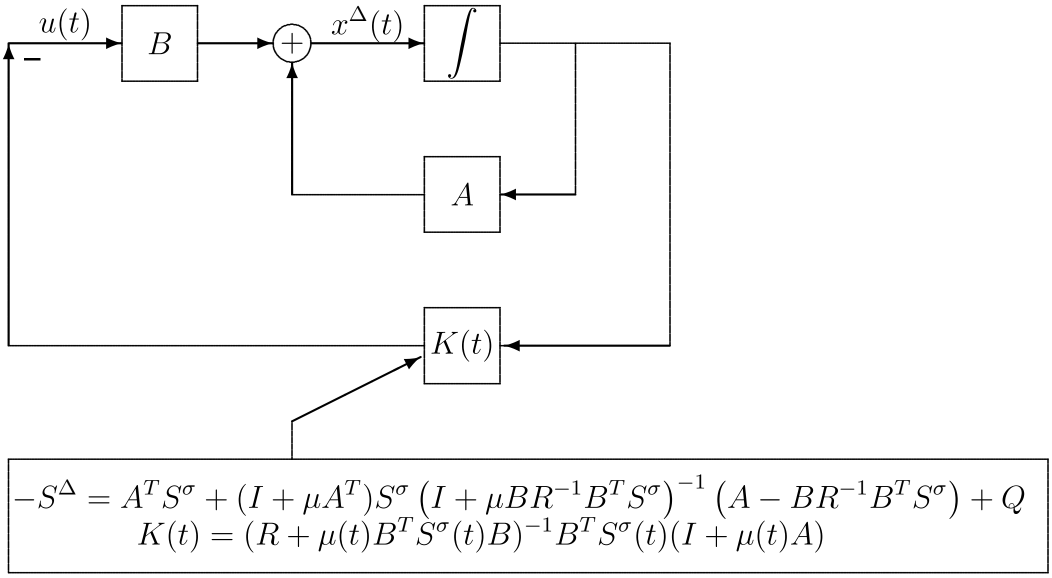

Figure 6.1: The free-final-state LQR

Theorem 6.11. *Suppose that S solves* (6.13)*. If x and u satisfy* (6.16) *and* (6.15)*, then the cost functional* (3.3) *can be rewritten as*

$$J = \frac{1}{2}x^T(t_0)S(t_0)x(t_0).$$

Proof. First note that we may use the product rule, (2.1), and (6.16) to find

$$\begin{aligned}
(x^TSx)^\Delta &= (x^TS)^\Delta x + (x^TS)^\sigma x^\Delta \\
&= (x^\Delta)^TS^\sigma x + x^TS^\Delta x + (x+\mu x^\Delta)^TS^\sigma x^\Delta \\
&= x^T[(A-BK)^TS^\sigma + S^\Delta + (I+\mu(A-BK))^TS^\sigma(A-BK)]x.
\end{aligned}$$

Using this and (6.15) in (3.3), we evaluate

$$\begin{aligned}
J = \frac{1}{2}x^T(t_0)S(t_0)x(t_0) &+ \frac{1}{2}\int_{t_0}^{t_f}[(x^TSx)^\Delta + x^TQx + u^TRu](\tau)\Delta\tau \\
= \quad & \frac{1}{2}x^T(t_0)S(t_0)x(t_0) + \frac{1}{2}\int_{t_0}^{t_f}[(x^TSx)^\Delta + x^TQx + x^TK^TRKx](\tau)\Delta\tau \\
= \quad & \frac{1}{2}x^T(t_0)S(t_0)x(t_0) + \frac{1}{2}\int_{t_0}^{t_f}\{x^T[S^\Delta + (A-BK)^TS^\sigma]x\}(\tau)\Delta\tau \\
& + \frac{1}{2}\int_{t_0}^{t_f}\left\{x^T[(I+\mu(A-BK))^TS^\sigma(A-BK) + Q + K^TRK]x\right\}(\tau)\Delta\tau.
\end{aligned}$$

Since S satisfies (6.13), the integrand is zero. This concludes the proof. □

From Theorem 6.11, if the current state and S are known, we can determine the optimal cost before we apply the optimal control or even calculate it.

7 Examples

Example 7.1. Let $\theta(t)$ represent the temperature of some object at time $t \in \mathbb{T}$. Assume that θ_m represents the temperature of the surrounding medium held constant. Let $u(t)$ be rate of heat supply to the medium. Then the rate of change in the object's temperature may be modelled by the dynamic equation

$$\theta^{\Delta}(t) = a(t)(\theta(t) - \theta_m) + b(t)u(t).$$

Suppose that we want to find a minimum input needed to heat the object over the interval $[t_0, t_{\mathrm{f}}]$. First, define the state to be the difference between the object's temperature and its surrounding environment, that is

$$x(t) = \theta(t) - \theta_m.$$

Then the state equation is given by

$$x^{\Delta}(t) = a(t)x(t) + b(t)u(t).$$

Next, define the associated cost functional

$$J = \frac{1}{2}\int_{t_0}^{t_{\mathrm{f}}} u^2(\tau)\Delta\tau.$$

Then from the general form of the Hamiltonian, we have

$$H(t, x, u, \lambda) = \frac{ru^2}{2} + \lambda(a(t)x + b(t)u), \tag{7.1}$$

where r is a constant. Finally, the state and costate equations and stationary condition are given by

$$x^{\Delta}(t) = H_{\lambda}(t, x(t), u(t), \lambda^{\sigma}(t)) = a(t)x(t) + b(t)u(t),$$

$$-\lambda^{\Delta}(t) = H_x(t, x(t), u(t), \lambda^{\sigma}(t)) = a(t)\lambda^{\sigma}(t),$$

and

$$0 = H_u(t, x(t), u(t), \lambda^{\sigma}(t)) = ru(t) + b(t)\lambda^{\sigma}(t). \tag{7.2}$$

Rewriting (7.2), we find the optimal control to be

$$u(t) = -\frac{b(t)\lambda^{\sigma}(t)}{r}. \tag{7.3}$$

Now plugging the optimal control into the state-costate equations, we have

$$x^{\Delta}(t) = a(t)x(t) - \frac{b^2(t)\lambda^{\sigma}(t)}{r}, \quad \lambda^{\Delta}(t) = -a(t)\lambda^{\sigma}(t). \tag{7.4}$$

Assuming that $a \in \mathcal{R}$ and solving for $\lambda(t)$, we have

$$\lambda(t) = e_a(t_{\mathrm{f}}, t)\lambda(t_{\mathrm{f}}), \tag{7.5}$$

where $\lambda(t_{\mathrm{f}})$ is yet to be determined. Plugging (7.5) into (7.4) yields

$$x^{\Delta}(t) = a(t)x(t) - \frac{b^2(t)}{r} e_a(t_{\mathrm{f}}, \sigma(t))\lambda(t_{\mathrm{f}}).$$

We use Theorem 2.17 to find

$$\begin{aligned} x(t) &= e_a(t,t_0)x(t_0) - \frac{1}{r}\int_{t_0}^{t} b^2(\tau)e_a(t,\sigma(\tau))e_a(t_{\mathrm{f}},\sigma(\tau))\Delta\tau\lambda(t_{\mathrm{f}}) \\ &= e_a(t,t_0)x(t_0) - \frac{e_a(t,t_{\mathrm{f}})}{r}\int_{t_0}^{t} b^2(\tau)e_a^2(t_{\mathrm{f}},\sigma(\tau))\Delta\tau\lambda(t_{\mathrm{f}}) \\ &= e_a(t,t_0)x(t_0) - \frac{e_a(t,t_{\mathrm{f}})}{r}\int_{t_0}^{t} b^2(\tau)e_{2\odot a}(t_{\mathrm{f}},\sigma(\tau))\Delta\tau\lambda(t_{\mathrm{f}}), \end{aligned} \tag{7.6}$$

where $2 \odot a = 2a + \mu a^2$. In order to find our state and costate, we must find $\lambda(t_{\mathrm{f}})$. Now we consider two different control schemes that can be used to determine $\lambda(t_{\mathrm{f}})$.

Case 1 (Fixed Final State): Assume that the temperature of the object is originally the same temperature as its surrounding medium, $\theta_m = 70^0$ F (i.e., $x(t_0) = 0$). Now we are looking for an optimal control such that the final temperature of the object is $\theta(t_{\mathrm{f}}) = 100^0$ F. This means that the value that the final state must take on is given by $x(t_{\mathrm{f}}) = 30^0$ F. Since both the initial and final states are fixed, the boundary conditions are given by $\eta_1(t_0) = \eta_1(t_{\mathrm{f}}) = 0$. Note that by (7.6), the final state can also be written as

$$x(t_{\mathrm{f}}) = 30 = -\frac{1}{r}\int_{t_0}^{t_{\mathrm{f}}} b^2(\tau)e_{2\odot a}(t_{\mathrm{f}},\sigma(\tau))\Delta\tau\lambda(t_{\mathrm{f}}). \tag{7.7}$$

Solving (7.7) for the final costate, we have

$$\lambda(t_{\mathrm{f}}) = \frac{-30r}{\displaystyle\int_{t_0}^{t_{\mathrm{f}}} b^2(\tau)e_{2\odot a}(t_{\mathrm{f}},\sigma(\tau))\Delta\tau}.$$

Now by (7.5), the final costate is given by

$$\lambda(t) = \frac{-30re_a(t_{\mathrm{f}},t)}{\displaystyle\int_{t_0}^{t_{\mathrm{f}}} b^2(\tau)e_{2\odot a}(t_{\mathrm{f}},\sigma(\tau))\Delta\tau}.$$

Using (7.3), the optimal control is given by

$$u(t) \;=\; \frac{30b(t)e_a(t_\mathrm{f},\sigma(t))}{\displaystyle\int_{t_0}^{t_\mathrm{f}} b^2(\tau)e_{2\odot a}(t_\mathrm{f},\sigma(\tau))\Delta\tau}. \tag{7.8}$$

Finally, the optimal trajectory is given by

$$x(t) = \frac{30e_a(t,t_\mathrm{f})\displaystyle\int_{t_0}^{t} b^2(\tau)e_{2\odot a}(t_\mathrm{f},\sigma(\tau))\Delta\tau}{\displaystyle\int_{t_0}^{t_\mathrm{f}} b^2(\tau)e_{2\odot a}(t_\mathrm{f},\sigma(\tau))\Delta\tau}. \tag{7.9}$$

<u>Case 2</u> (Free Final State): Now suppose that the final state is not quite 30^0 F. However, we would like the final state to be as close to 30^0 F as possible. Thus to make the difference $x(t_\mathrm{f}) - 30$ small, we include it in the cost functional

$$J = \frac{1}{2}s(x(t_\mathrm{f}) - 30)^2 + \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} u^2(\tau)\,\Delta\tau, \tag{7.10}$$

where s is some positive constant. Here, we would like to find a minimum input that minimizes both (7.10) and $|x(t_\mathrm{f}) - 30|$, later which can be thought of as an error term. Note that since the integrand remains unchanged, the Hamiltonian is still given by (7.1). Thus the equations for the state, costate, and control are the same as before. Next, we need to determine our boundary conditions. As before, $x(t_0) = 0$. On the other hand, $x(t_\mathrm{f})$ is not fixed, which means that $\eta_1(t_\mathrm{f}) \neq 0$. Therefore we require that

$$\lambda(t_\mathrm{f}) = s(x(t_\mathrm{f}) - 30). \tag{7.11}$$

Plugging (7.11) into (7.6), solving for $x(t_\mathrm{f})$ and using this again in (7.6), we find

$$\lambda(t_\mathrm{f}) = \frac{-30rs}{r + s\displaystyle\int_{t_0}^{t_\mathrm{f}} b^2(\tau)e_{2\odot a}(t_\mathrm{f},\sigma(\tau))\Delta\tau}.$$

Using this in (7.5), we have the optimal costate

$$\lambda(t) = \frac{-30rse_a(t_\mathrm{f},t)}{r + s\displaystyle\int_{t_0}^{t_\mathrm{f}} b^2(\tau)e_{2\odot a}(t_\mathrm{f},\sigma(\tau))\Delta\tau}.$$

Using this in (7.3), we have the optimal control

$$u(t) = \frac{30sb(t)}{r + s\displaystyle\int_{t_0}^{t_\mathrm{f}} b^2(\tau)e_{2\odot a}(t_\mathrm{f},\sigma(\tau))\Delta\tau}e_a(t_\mathrm{f},\sigma(t)). \tag{7.12}$$

By (7.6), the optimal state is given by

$$x(t)=\frac{30se_a(t,t_{\mathrm{f}})\displaystyle\int_{t_0}^{t}b^2(\tau)e_{2\odot a}(t_{\mathrm{f}},\sigma(\tau))\Delta\tau}{r+s\displaystyle\int_{t_0}^{t_{\mathrm{f}}}b^2(\tau)e_{2\odot a}(t_{\mathrm{f}},\sigma(\tau))\Delta\tau}. \tag{7.13}$$

In Example 7.1, we were unable to evaluate the integral for a general time scale due to the way the exponential is defined. In the next example, we consider a specific state coefficient that allows for evaluation of this integral.

Example 7.2. Let $\mathbb{T}$ be a general time scale and suppose there exists a constant c such that

$$c\equiv\frac{b^2(t)}{(2\odot a)(t)}\quad\text{for all}\quad t\in\mathbb{T}.$$

Then we may use Theorem 2.16(e) to evaluate the relevant integral

$$\begin{aligned}\int_{t_0}^{t}b^2(\tau)e_{2\odot a}(t_{\mathrm{f}},\sigma(\tau))\Delta\tau &= c\int_{t_0}^{t}(2\odot a)(\tau)e_{2\odot a}(t_{\mathrm{f}},\sigma(\tau))\Delta\tau\\ &= -c\int_{t_0}^{t}[e_{2\odot a}(t_{\mathrm{f}},\cdot)]^{\Delta}(\tau)\Delta\tau\\ &= c[e_{2\odot a}(t_{\mathrm{f}},t_0)-e_{2\odot a}(t_{\mathrm{f}},t)].\end{aligned}$$

From (7.6), we find, using $x(t_0)=0$,

$$\begin{aligned}x(t) &= e_a(t,t_0)x(t_0)-\frac{c}{r}e_a(t,t_{\mathrm{f}})[e_{2\odot a}(t_{\mathrm{f}},t_0)-e_{2\odot a}(t_{\mathrm{f}},t)]\lambda(t_{\mathrm{f}})\\ &= -\frac{c}{r}e_a(t,t_{\mathrm{f}})[e_a^2(t_{\mathrm{f}},t_0)-e_a^2(t_{\mathrm{f}},t)]\lambda(t_{\mathrm{f}})\\ &= \frac{c}{r}[e_a(t_{\mathrm{f}},t)-e_a(t,t_0)e_a(t_{\mathrm{f}},t_0)]\lambda(t_{\mathrm{f}}).\end{aligned}$$

<u>Case 1</u> (Fixed Final State): From (7.8), the optimal control is given by

$$u(t)=\frac{30(2\odot a)e_a(t_{\mathrm{f}},t)}{b(t)[e_{2\odot a}(t_{\mathrm{f}},t_0)-e_{2\odot a}(t_{\mathrm{f}},t)]}. \tag{7.14}$$

From (7.9), the optimal state is given by

$$x(t)=\frac{30[e_a(t_{\mathrm{f}},t)-e_a(t,t_0)e_a(t_{\mathrm{f}},t_0)]}{[e_{2\odot a}(t_{\mathrm{f}},t_0)-1]}. \tag{7.15}$$

<u>Case 2</u> (Free Final State): From (7.12), we have the optimal control

$$u(t)=\frac{30s(2\odot a)b(t)e_a(t_{\mathrm{f}},\sigma(t))}{r(2\odot a)(t)+sb^2(t)[e_{2\odot a}(t_{\mathrm{f}},t_0)-1]}. \tag{7.16}$$

From (7.13), the optimal state is given by

$$x(t) = \frac{30sb^2(t)[e_a(t_f, t) - e_a(t, t_0)e_a(t_f, t_0)]}{r(2 \odot a)(t) + sb^2(t)[e_{2\odot a}(t_f, t_0) - 1]}. \tag{7.17}$$

Example 7.3. Let $\mathbb{T} = \mathbb{R}$ with $c \equiv \dfrac{b^2(t)}{(2 \odot a)(t)} = \dfrac{b^2(t)}{2a(t)}$. Then the state and costate equations and stationary condition are given by

$$\dot{x}(t) = a(t)x(t) + b(t)u(t),$$

$$-\dot{\lambda}(t) = a(t)\lambda(t),$$

and

$$0 = ru(t) + b(t)\lambda(t).$$

Case 1 (Fixed Final State): By (7.14), the optimal control is given by

$$u(t) = \frac{60a(t)e^{a(t_f - t)}}{b(t)[e^{2a(t_f - t_0)} - e^{2a(t_f - t)}]},$$

while the optimal state by (7.15) is given by

$$x(t) = \frac{30[e^{a(t_f - t)} - e^{a(t - t_0)}e^{a(t_f - t_0)}]}{[e^{2a(t_f - t_0)} - 1]}.$$

Case 2 (Free Final State): From (7.16), we have the optimal control

$$u(t) = \frac{60sa(t)b(t)e^{a(t_f - t)}}{2ra(t) + sb^2(t)[e^{2a(t_f - t_0)} - 1]}$$

with the optimal state by (7.17) given by

$$x(t) = \frac{60sa(t)b^2(t)[e^{a(t_f - t)} - e^{a(t - t_0)}e^{a(t_f - t_0)}]}{2ra(t) + sb^2(t)[e^{2a(t_f - t_0)} - 1]}.$$

Example 7.4. Let $\mathbb{T} = \mathbb{Z}$ with $c \equiv \dfrac{b^2(t)}{(2 \odot a)(t)} = \dfrac{b^2(t)}{a(t)[2 + a(t)]}$. Then the state and costate equations and stationary condition are given by

$$\Delta x(t) = a(t)x(t) + b(t)u(t),$$

$$-\Delta\lambda(t) = a(t)\lambda(t + 1),$$

and

$$0 = ru(t) + b(t)\lambda(t + 1).$$

Case 1 (Fixed Final State): By (7.14), the optimal control is given by

$$u(t) = \frac{30(2 \odot a)(t)e_a(t_\mathrm{f}, t)}{b(t)[e_{2\odot a}(t_\mathrm{f}, t_0) - e_{2\odot a}(t_\mathrm{f}, t)]},$$

where

$$e_m(t_\mathrm{f}, t_0) = \prod_{\tau \in \mathbb{T} \cap [t_0, t_\mathrm{f})} (1 + m(\tau)).$$

From (7.15), the optimal state is given by

$$x(t) = \frac{30[e_a(t_\mathrm{f}, t) - e_a(t, t_0)e_a(t_\mathrm{f}, t_0)]}{[e_{2\odot a}(t_\mathrm{f}, t_0) - 1]}.$$

Case 2 (Free Final State): By (7.16), we have the optimal control

$$u(t) = \frac{30s(2 \odot a)(t)b(t)e_a(t_\mathrm{f}, t+1)}{r(2 \odot a)(t) + sb^2(t)[e_{2\odot a}(t_\mathrm{f}, t_0) - 1]}$$

with the optimal state by (7.17) given by

$$x(t) = \frac{30sb^2(t)[e_a(t_\mathrm{f}, t) - e_a(t, t_0)e_a(t_\mathrm{f}, t_0)]}{r(2 \odot a)(t) + sb^2(t)[e_{2\odot a}(t_\mathrm{f}, t_0) - 1]}.$$

Example 7.5. Let $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$ and $c \equiv \dfrac{b^2(t)}{(2 \odot a)(t)} = \dfrac{b^2(t)}{a(t)[2 + (q-1)ta(t)]}$. Then the state and costate equations and stationary condition are given by

$$D_q x(t) = a(t)x(t) + b(t)u(t),$$

$$-D_q \lambda(t) = a(t)\lambda(qt),$$

and

$$0 = ru(t) + b(t)\lambda(qt).$$

Case 1 (Fixed Final State): By (7.14), the optimal control is given by

$$u(t) = \frac{30(2 \odot a)(t)e_a(t_\mathrm{f}, t)}{b(t)[e_{2\odot a}(t_\mathrm{f}, t_0) - e_{2\odot a}(t_\mathrm{f}, t)]},$$

where

$$e_m(t_\mathrm{f}, t_0) = \prod_{\tau \in \mathbb{T} \cap [t_0, t_\mathrm{f})} (1 + (q-1)m\tau).$$

From (7.15), the optimal state is given by

$$x(t) = \frac{30[e_a(t_\mathrm{f}, t) - e_a(t, t_0)e_a(t_\mathrm{f}, t_0)]}{[e_{2\odot a}(t_\mathrm{f}, t_0) - 1]}.$$

Case 2 (Free Final State): By (7.16), we have the optimal control

$$u(t) = \frac{30s(2 \odot a)(t)b(t)e_a(t_{\rm f}, qt)(t)}{r(2 \odot a)(t) + sb^2(t)[e_{2\odot a}(t_{\rm f}, t_0) - 1]}.$$

From (7.17), the optimal state is given by

$$x(t) = \frac{30sb^2(t)[e_a(t_{\rm f}, t) - e_a(t, t_0)e_a(t_{\rm f}, t_0)]}{r(2 \odot a)(t) + sb^2(t)[e_{2\odot a}(t_{\rm f}, t_0) - 1]}.$$

Example 7.6. Here we rewrite the Riccati equation (6.13) for various time scales.

a. If $\mathbb{T} = \mathbb{R}$, then (6.13) becomes

$$-\dot{S}(t) = Q + (A - BK(t))^T S(t) + S(t)(A - BK(t)) + K^T(t)RK(t),$$

where $K(t) = R^{-1}B^T S(t)$.

b. If $\mathbb{T} = \mathbb{Z}$, then (6.13) is given by

$$\begin{aligned} S(t) - S(t+1) &= Q + (A - BK(t))^T S(t+1) + K^T(t)RK(t) \\ &\quad + (I + A - BK(t))^T S(t+1)(A - BK(t)), \end{aligned}$$

where $K(t) = (R + B^T S(t+1)B)^{-1} B^T S(t+1)(I + A)$.

c. If $\mathbb{T} = h\mathbb{Z}$, then (6.13) turns into

$$\begin{aligned} \frac{S(t) - S(t+h)}{h} &= Q + (A - BK(t))^T S(t+h) + K^T(t)RK(t) \\ &\quad + (I + h(A - BK(t))^T)S(t+h)(A - BK(t)), \end{aligned}$$

where $K(t) = (R + hB^T S(t+h)B)^{-1} B^T S(t+h)(I + hA)$.

d. If $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, then (6.13) is given by

$$\begin{aligned} D_q S(t) &= Q + (A - BK(t))^T S(qt) + K^T(t)RK(t) \\ &\quad + (I + (q-1)t(A - BK(t))^T)S(qt)(A - BK(t)), \end{aligned}$$

where $K(t) = (R + (q-1)tB^T S(qt)B)^{-1} B^T S(qt)(I + (q-1)tA)$.

Remark 7.7. Aside from the examples given, the LQR can also be extended to include applications in tracking and disturbance/rejection. We study such applications in a forthcoming paper. It should be noted that the LQR mirrors the controllability properties we studied in [4]. In future work, we seek an optimal state estimate that minimizes an error "covariance." This mirrors the observability properties we studied in [4]. Such an observer is called a Kalman filter. In the discrete and continuous cases, the Riccati equations for the LQR and the Kalman filter are mathematically dual to each other. We show that this duality is preserved in their unification and extension.

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