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Stability of Nonlinear Stochastic Volterra Difference Equations with Respect to a Fading Perturbation

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Abstract

The paper concerns studies the stochastic stability and stochastic asymptotic stability of the equilibrium solution of a nonlinear Volterra difference equation which is subject to stochastic state independent disturbances. It is shown that if the linearized deterministic equation has summable solutions, then the nonlinear stochastic equation will be stable or asymptotically stable, provided that the initial condition, and the intensity of the stochastic disturbances are sufficiently small. The smallness of the intensity follows closely the conditions required for the stability of the stochastically perturbed linear Volterra difference equation.

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1 Introduction

In this note we generalize results of the paper [4] for system of nonlinear stochastic Volterra difference equations

$$X(n+1) = X(n) + \sum_{k=0}^{n} A(n-k)G(X(k)) + \sigma(n)\xi(n+1), \quad n \in \mathbb{N}; \quad X(0) = \zeta.$$

The paper [4] concerns necessary and sufficient conditions on the fading intensity of a state-independent stochastic perturbation for the asymptotic stability of a linear stochastic Volterra difference equation. It was shown there that the results obtained in the deterministic case are robust to fading stochastic perturbations which are independent of the state, once it is known that these perturbations fade more rapidly than an identifiable critical rate. The paper [4] extends the continuous-time approaches and results of the paper [3] to the discrete case, in which necessary and sufficient conditions for the stability of linear Itô–Volterra equations with deterministically fading noise intensity was obtained. Similar conditions on the noise intensity were established in the continuous case in [5].

Among others, global almost sure asymptotic stability for nonlinear stochastic difference equation without delays was considered in [2, 6, 12], for equation with delays and with the Volterra diffusion term in [10], for equation with the Volterra main term and coefficients satisfying linear growth conditions in [11].

To the best of our knowledge, local stability of the solutions of nonlinear stochastic difference equation was discussed only in [1], where the following equation was considered:

$$X_{n+1} = X_n - f(X_n) + \sigma_n \xi_{n+1}, \quad n \geq 0. \tag{1.1}$$

The function $f : \mathbb{R} \to \mathbb{R}$ was supposed to be continuous and obey the properties:

$$uf(u) > 0,\ u \neq 0,\ \text{ for all } u \in \mathbb{R}, \quad f(0) = 0; \quad \inf_{u>c} |f(u)| > 0 \text{ for all } c > 0. \tag{1.2}$$

It was not assumed that f admits a nontrivial linearisation at 0, as well as that f obeys a global linear bound. Thus no linearisation techniques was used in [1] to analyze the stability close to equilibrium. However analysis of the paper [1] holds for linear equations, or equations with global linear bounds as well as for the equations with nonhyperbolic equilibrium.

The present note can be considered as a next step in investigating local stability, now for the Volterra stochastic difference equation. In this note we suppose that the nonlinearity, presented in the Volterra stochastic difference equation, has the equilibrium (which is in our case equal to 0) of the hyperbolic type. So we can linearize the original nonlinear difference equation in a neighborhood of zero and work with the obtained linear equation. However this nonlinearity is not supposed to have linear growth bound outside of any zero neighborhood, and therefore we cannot expect that the solution converges to zero almost surely for each initial value (see reasoning about it in [1]).

Instead of the global a.s. stability we establish results about local boundedness and local asymptotic stability with probability close to 1.

The outline of the note is as follows. Section 2 deals with notations and main assumptions. In Section 3 we state some of the results on the almost sure asymptotic stability of linear Volterra difference equations proved in [4]. In Section 4 we state the main results of this note along with the lemmas about the stochastic boundedness of the noise term. Section 5 contains the proofs of the main results. Finally, Section 6 contains the proofs of lemmas about the stochastic boundedness of the noise term.

2 Notation and Main Assumptions

The following standard notation is employed. If E is a subset of $\mathbb{R}$, then the characteristic function $\chi_E : \mathbb{R} \to \mathbb{R}$ is defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

The standard basis vectors in $\mathbb{R}^r$ are denoted by $\mathbf{e}_1, \ldots, \mathbf{e}_r$. The standard inner product of x and y in $\mathbb{R}^r$ is denoted by $\langle x, y\rangle$. The $d \times d$ real-valued identity matrix is denoted by I_d. We say that a real sequence $a = \{a(n) : n \in \mathbb{N}\}$ obeys $a \in \ell^1(\mathbb{N};\mathbb{R})$ if $\sum_{n\in\mathbb{N}} |a(n)| < \infty$. We say $d \times r$ matrix-valued sequence $a = \{a(n) = (a(n))_{ij} : n \in \mathbb{N}\}$ is in $\ell^1(\mathbb{N};\mathbb{R}^d \times \mathbb{R}^r)$ if each entry $a_{ij} \in \ell^1(\mathbb{N};\mathbb{R})$. The convolution of two sequences $f = \{f(n) : n \in \mathbb{N}\}$ and $g = \{g(n) : n \in \mathbb{N}\}$, $f * g$, is a sequence defined by

$$(f * g)(n) = \sum_{k=0}^{n} f(n-k)g(k), \quad n \in \mathbb{N}.$$

The $*$ notation is also used here for the convolution of distribution functions; however, the type of convolution being used will be clear from the context.

We consider the nonlinear stochastic difference Volterra equation

$$X(n+1) = X(n)+\sum_{k=0}^{n} A(n-k)G(X(k))+\sigma(n)\xi(n+1), \quad n \in \mathbb{N}; \quad X(0) = \zeta \tag{2.1}$$

together with the linearized stochastic Volterra difference equation

$$Y(n+1) = Y(n) + \sum_{k=0}^{n} A(n-k)DG(0)Y(k) + \sigma(n)\xi(n+1), \quad n \in \mathbb{N}; \quad Y(0) = \zeta. \tag{2.2}$$

Here $A \in \ell^1(\mathbb{N};\mathbb{R}^d\times\mathbb{R}^d)$, $\sigma(n) \in \mathbb{R}^d\times\mathbb{R}^r$ and $DG(0)$ is the matrix of partial derivatives of G at 0. The following assumptions are made about G so that the linearization in (2.2) is legitimate, and which ensures that zero is a solution of the stochastically unperturbed Volterra difference equation.

$$G \in C(\mathbb{R}^d; \mathbb{R}^d); \tag{2.3}$$

$$\text{there is an open set } S \subseteq \mathbb{R}^d \text{ with } 0 \in S \text{ such that } G \in C^1(S; \mathbb{R}^d); \tag{2.4}$$

$$G(0) = 0. \tag{2.5}$$

Each $\xi(n)$ is a random vector in $\mathbb{R}^r$. We make the following standing assumption about ξ through the paper:

$$\xi = \{\xi(n)\}_{n\in\mathbb{N}} \text{ is a sequence of independent } \mathbb{R}^r\text{-valued random vectors;} \tag{2.6}$$

$$\xi_j(n) := \langle \xi(n), \mathbf{e}_j \rangle \text{ have distribution function } F, \text{ for all } n \text{ and } j; \tag{2.7}$$

$$\mathbb{E}[\xi_j(n)] = 0 \quad \mathbb{E}[\xi_j(n)^2] = 1, \quad j = 1, \dots, r,\ n \in \mathbb{N}; \tag{2.8}$$

$$\text{for fixed } n,\ \{\xi_j(n)\}_{j=1}^r \text{ are independent random variables.} \tag{2.9}$$

Let R be the resolvent defined by

$$R(n+1) = R(n) + \sum_{k=0}^{n} A(n-k)DG(0)R(k), \quad n \in \mathbb{N}; \quad R(0) = I_d, \tag{2.10}$$

so that each $R(n) \in \mathbb{R}^d \times \mathbb{R}^d$. The significance of the resolvent in this context is that it allows Y to be written purely in terms of the perturbation, according to the variation of parameters formula

$$Y(n) = R(n)\zeta + \sum_{k=1}^{n} R(n-k)\sigma(k-1)\xi(k), \quad n \in \mathbb{N}. \tag{2.11}$$

Remark 2.1. In what is following we assume that $A \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$ and R defined by (2.10) obeys $R \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$. We do not present here any conditions which guarantee that $R \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$, since they are quite complicated. Conditions of this sort could be found in numerous papers of V. Kolmanovskii, e.g., [9] (see also [8, 11]). For example, we can show (see e.g., [11] and also [9]) that $R(n) \to 0$ as $n \to \infty$, if $\left|1 + A(0) + \sum_{i=1}^{\infty} |A(i)|\right| < 1$.

We note that in the 1-dimensional case, $d = 1$, from condition $R \in \ell^1(\mathbb{N}; \mathbb{R}^1 \times \mathbb{R}^1)$, we can conclude that $DG(0)$ cannot be degenerate. Otherwise, equation (2.10) implies that $R(n) \equiv \text{const}$, which means that $R \notin \ell^1(\mathbb{N}; \mathbb{R}^1 \times \mathbb{R}^1)$. For example, $G(u) = -au + u^2$, for some $a > 0$, is a suitable choice, while when $G(u) = -u^3$, equation (2.10) has only constant solution. We note that $G(u) = -u^3$ has a nonhyperbolic type of equilibrium at zero.

3 Results for the Linear Equation (2.2)

In this section, we state some of the results obtained in Appleby, Riedle and Rodkina [4] for the almost sure asymptotic stability of linear Volterra difference equations. In this paper, we will prove direct nonlinear analogues of all the results quoted here. However, the results will be for stochastic local stability and stochastic local asymptotic stability.

The crucial stochastic sequence is the $\mathbb{R}^d$-valued sequence $U = \{U(n) : n \in \mathbb{N}\}$ defined by

$$U(n) = \sigma(n)\xi(n+1), \quad n \in \mathbb{N}. \tag{3.1}$$

For (2.2), it has been shown that the condition $U(n) \to 0$ a.s. is equivalent to $Y(n) \to 0$ as $n \to \infty$. Therefore, for the stability of the linear equation, it is important to characterise the circumstances under which $U(n) \to 0$, a.s. A simple sufficient condition is given as follows.

Proposition 3.1. *Suppose that $\xi = \{\xi(n) : n \in \mathbb{N}\}$ is an $\mathbb{R}^r$-valued sequence of random vectors which obeys* (2.8)*, and $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ is an $\mathbb{R}^d \times \mathbb{R}$-valued sequence. Let U be defined by* (3.1)*. If $\sum_{n=1}^{\infty} |\sigma(n)|^2 < \infty$, then $U(n) \to 0$ as $n \to \infty$, a.s.*

Necessary and sufficient conditions for $U(n) \to 0$ a.s. are given under independence restrictions in the next result.

Lemma 3.2. *Let $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ be an $\mathbb{R}^d \times \mathbb{R}^r$-valued sequence. Suppose that the sequence of random variables $\xi = \{\xi(n) : n \in \mathbb{N}\}$ obeys* (2.6)–(2.9)*. Let U be defined by* (3.1)*. Define the distribution function $F_{n,i,j}$ by*

$$F_{n,i,j}(x) = \begin{cases} F(x/|\sigma_{ij}(n)|), & \sigma_{ij}(n) > 0, \\ 1 - F(-x/|\sigma_{ij}(n)|), & \sigma_{ij}(n) < 0, \\ \chi_{[0,\infty)}(x), & \sigma_{ij}(n) = 0 \end{cases} \tag{3.2}$$

and $F_{n,i}$ as the convolution of the distributions $F_{n,i,j}$ for $j = 1, \ldots, r$. Then

$$\sum_{i=1}^{d} \sum_{n=1}^{\infty} [1 - F_{n,i}(\varepsilon) + F_{n,i}(-\varepsilon)] < \infty \quad \textit{for all } \varepsilon \in \mathbb{Q}^+ \tag{3.3}$$

is equivalent to $U(n) \to 0$ as $n \to \infty$ a.s.

For the general linear equation (2.2), we get the following necessary and sufficient conditions on F and $\sigma(n)$ such that $Y(n) \to 0$ a.s. as $n \to \infty$.

Theorem 3.3. *Let $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ be an $\mathbb{R}^d \times \mathbb{R}^r$-valued sequence. Suppose the sequence of random variables $\xi = \{\xi(n) \in \mathbb{R}^r : n \in \mathbb{N}\}$ obeys* (2.6)–(2.9)*. Let $A \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^r)$ and Y be the process given by* (2.2)*. Let R defined by* (2.10) *obey $R \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$. Then the following are equivalent:*

(a) (3.3) *holds, where* $F_{n,i}$ *and* $F_{n,i,j}$ *are defined as in Lemma 3.2;*

(b) $\lim_{n\to\infty} Y(n) = 0$ *a.s.*

An important special case is for Gaussian sequences as stated in the following lemma, which can be considered as generalization of [4, Corollary 3.1] to the multi-dimensional case.

Corollary 3.4. *Let* $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ *be an* $\mathbb{R}^d \times \mathbb{R}^r$*-valued sequence. Let* $A \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$ *and let* R *defined by* (2.10) *obey* $R \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$. *Let* $\xi = \{\xi(n) \in \mathbb{R}^r : n \in \mathbb{N}\}$ *be a sequence of random vectors obeying* (2.6)–(2.9) *such that* $\xi_j(n) = \langle \xi(n), \mathbf{e}_j \rangle$ *are standard normal random variables. Define* $\sigma_i(n) \geq 0$ *by* $\sigma_i^2(n) = \sum_{j=1}^{r} \sigma_{ij}^2(n)$. *Let* Y *be the process defined by* (2.2).

(a) *If*

$$\sum_{i=1}^{d} \sum_{n=1}^{\infty} \sigma_i(n) \exp\left(-\frac{1}{2}\frac{\varepsilon^2}{\sigma_i^2(n)}\right) < \infty \quad \textit{for all } \varepsilon \in \mathbb{Q}^+, \tag{3.4}$$

then $\mathbb{P}\left[\lim_{n\to\infty} Y(n) = 0\right] = 1.$

(b) *If* (3.4) *does not hold, then* $\mathbb{P}\left[\lim_{n\to\infty} Y(n) = 0\right] = 0.$

If, moreover, each of the maps $n \mapsto \sigma_i^2(n)$ *is nonincreasing, then the following are equivalent:*

(a) $\lim_{n\to\infty} \|\sigma(n)\|^2 \log n = 0$*;*

(b) $\mathbb{P}\left[\lim_{n\to\infty} Y(n) = 0\right] = 1.$

Remark 3.5. The logarithmic decay condition (a) was shown to be necessary for a class of nondelay gradient dynamical system in Chan and Williams [7]. The necessary and sufficient conditions on the rate of decay of the noise intensity σ given in (a) is the same as that required in [3]. The rate of decay $\sigma^2(n) \log n \to 0$ has also been shown to be necessary and sufficient for almost sure asymptotic stability in a class of nonlinear stochastic delay-differential equations studied in [5]. In [1] it was also proved that when ξ is a sequence of iid standard normal random variables, then $\sigma_n^2 \log n \to 0$ is a necessary and sufficient condition to guarantee the asymptotic stability for the equation (1.1).

4 Main Results

We will prove the following analogues of these results. The fundamental result is the following: If the stochastic sequence $(\sigma(n)\xi(n+1))_{n\geq 0}$ is uniformly bounded on an event, then the solution of the nonlinear equation (2.1) is uniformly bounded on that event; moreover, the bound on both sequences can be made arbitrarily small.

Everywhere in this section we suppose that

$$A \in \ell^1(\mathbb{N};\mathbb{R}^d \times \mathbb{R}^d) \text{ and } R \text{ defined by (2.10) obeys } R \in \ell^1(\mathbb{N};\mathbb{R}^d \times \mathbb{R}^d). \tag{4.1}$$

We also define

$$S = 1 \vee \sum_{n=0}^{\infty} |(R*A)(n)|_1, \quad \bar{R} = \sum_{n=0}^{\infty} |R(n)|_1. \tag{4.2}$$

Proposition 4.1. *Let condition* (4.1) *hold. Suppose that G obeys* (2.3)–(2.5)*. Let $\delta_0 > 0$ be such that*

$$|G(x) - DG(0)x|_1 < \frac{1}{4S}|x|_1 \quad \textit{for all } |x|_1 < \delta_0.$$

Let $\varepsilon \in (0, \delta_0)$ and

$$|\zeta|_1 < \varepsilon\left(1 \wedge \frac{1}{4\bar{R}}\right).$$

If X is the solution of (2.1) *and U is defined by* (3.1)*, then*

$$\left\{\omega : \sup_{n\geq 0} |X(n,\omega)|_1 < \varepsilon\right\} \supseteq \left\{\omega : \sup_{n\geq 0} |U(n,\omega)|_1 < \varepsilon/(4\bar{R})\right\}.$$

Moreover, if $\bar{A} = \sum_{k=0}^{\infty} |A(k)DG(0)|_1$, then

$$\left\{\omega : \sup_{n\geq 0} |X(n,\omega)|_1 < \varepsilon\right\} \subseteq \left\{\omega : \sup_{n\geq 0} |U(n,\omega)|_1 < 3\varepsilon(1 + \bar{A}/2)\right\}.$$

A key technical ingredient in this proof is to show that the solution to linear equation (2.2) is uniformly bounded once $(U(n))_{n\geq 0}$ is uniformly bounded.

Lemma 4.2. *Let condition* (4.1) *hold. If Y is defined by* (2.2) *with $Y(0) = 0$ and U is defined by* (3.1)*, then for any $\varepsilon > 0$ we have*

$$\left\{\omega : \sup_{n\geq 0} |Y(n,\omega)|_1 < \varepsilon/4\right\} \supseteq \left\{\omega : \sup_{n\geq 0} |U(n,\omega)|_1 < \varepsilon/(4\bar{R})\right\}.$$

The property of X derived in Proposition 4.1 enables us to show that the asymptotic convergence of $(U(n))_{n\geq 0}$ implies the stochastic asymptotic stability of the solution of (2.1).

Proposition 4.3. *Let condition* (4.1) *hold. Suppose that* G *obeys* (2.3)–(2.5). *Let* $\delta_0 > 0$ *be such that*

$$|G(x) - DG(0)x|_1 < \frac{1}{4S}|x|_1 \quad \textit{for all } |x|_1 < \delta_0.$$

Let $\varepsilon \in (0, \delta_0)$, *and*

$$|\zeta|_1 < \varepsilon\left(1 \wedge \frac{1}{4\bar{R}}\right).$$

If X *is the solution of* (2.1) *and* U *is defined by* (3.1), *then*

$$\left\{\omega : \lim_{n\to\infty} X(n,\omega) = 0\right\} \supseteq \left\{\omega : \sup_{n\geq 0} |U(n,\omega)|_1 < \varepsilon/(4\bar{R})\right\} \cap \left\{\omega : \lim_{n\to\infty} U(n,\omega) = 0\right\}.$$

Moreover

$$\left\{\omega : \lim_{n\to\infty} X(n,\omega) = 0\right\} \subseteq \left\{\omega : \lim_{n\to\infty} U(n,\omega) = 0\right\}.$$

The second part of both Propositions 4.1 and 4.3 shows that the boundedness or convergence of U are not merely convenient hypotheses that ensure the boundedness or asymptotic stability of the solution of (2.1), but are in fact essential. This is important because necessary and sufficient conditions are available for the asymptotic convergence of U (see e.g., [1]).

Neither Proposition 4.3 nor 4.3 imposes *deterministic* conditions on the intensity σ nor the distributions of the disturbances ξ which ensure that the solution is stable or asymptotically stable. However, we have seen that this amounts merely to determining deterministic conditions under which the events $\left\{\omega : \sup_{n\geq 0} |U(n,\omega)|_1 < \varepsilon\right\}$ and $\left\{\lim_{n\to\infty} U(n,\omega) = 0\right\}$ have sufficiently large probabilities. We can then show that the solution of (2.1) is stochastically stable under the appropriate conditions: First, when $\sigma \in \ell^2(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^r)$ has sufficiently small ℓ^2 norm.

Theorem 4.4. *Let condition* (4.1) *hold. Suppose that* $\sigma \in \ell^2(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^r)$ *and that the sequence of* $\mathbb{R}^r$*-valued random vectors* $\xi = \{\xi(n) : n \in \mathbb{N}\}$ *obeys* (2.8). *Suppose that* G *obeys* (2.3)–(2.5). *Let* X *be the solution of* (2.1). *Then, for every* $\varepsilon > 0$ *and* $\eta \in (0,1)$, *there exists* $\delta = \delta(\eta, \varepsilon) > 0$ *such that*

$$|\zeta|_1 < \delta, \quad \sum_{n=0}^{\infty} |\sigma(n)|_F^2 < \delta$$

implies $\mathbb{P}\left[\sup_{n\geq 0} |X(n)| < \varepsilon\right] > 1 - \eta$, *and moreover that* $\mathbb{P}\left[\lim_{n\to\infty} X(n) = 0\right] > 1 - \eta$.

Next, we consider the general case, when the conditions of Theorem 3.3 hold.

Theorem 4.5. *Let condition* (4.1) *hold. Let* $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ *be an* $\mathbb{R}^d \times \mathbb{R}^r$*-valued sequence. Suppose that* G *obeys* (2.3)–(2.5)*. Suppose the sequence of random variables* $\xi = \{\xi(n) \in \mathbb{R}^r : n \in \mathbb{N}\}$ *obeys* (2.6)–(2.9)*. Suppose that* $F_{n,i}$ *and* $F_{n,i,j}$ *are as defined in Lemma 3.2. Let* X *be the solution of* (2.1)*. Then, for every* $\varepsilon > 0$ *and* $\eta \in (0,1)$*, there exists* $\delta = \delta(\eta, \varepsilon) > 0$ *such that*

$$|\zeta|_1 < \delta, \quad \sum_{n=0}^{\infty}\sum_{i=1}^{d} 1 - F_{n,i}(\varepsilon/d) + F_{n,i}(-\varepsilon/d) < \eta \tag{4.3}$$

implies $\mathbb{P}\left[\sup_{n\geq 0} |X(n)| < \varepsilon\right] > 1 - \eta$.

Finally, we consider the case when the disturbances are Gaussian. This is the nonlinear analogue of Corollary 3.4 above for the linear equation.

Theorem 4.6. *Let condition* (4.1) *hold and let* σ *be a sequence in* $\mathbb{R}^d \times \mathbb{R}^r$*. Suppose that* G *obeys* (2.3)–(2.5)*. Suppose* $\xi = \{\xi(n) : n \in \mathbb{N}\}$ *is a Gaussian sequence of* $\mathbb{R}^r$*-valued random vectors such that* (2.6)–(2.9) *hold. Let* X *be the solution of* (2.1)*. Then for every* $\varepsilon > 0$ *and* $\eta \in (0,1)$*, there exists* $\delta = \delta(\eta, \varepsilon) > 0$ *such that*

$$|\zeta|_1 < \delta, \quad \sup_{n\geq 0} |\sigma(n)|_F^2 \log(n+e) < \delta$$

implies $\mathbb{P}\left[\sup_{n\geq 0} |X(n)| < \varepsilon\right] > 1 - \eta$*. If moreover* $\lim_{n\to\infty} |\sigma(n)|_F^2 \log n = 0$*, then*

$$\mathbb{P}\left[\lim_{n\to\infty} X(n) = 0\right] > 1 - \eta.$$

In each case, these results are consequences of the following results on the stochastic boundedness of $(U(n))_{n\geq 0}$. The proofs of these results are postponed to the last section.

Lemma 4.7. *For every* $\varepsilon > 0$ *and* $\eta \in (0,1)$ *define*

$$\delta_1(\eta, \varepsilon) = \frac{\eta\varepsilon^2}{rd^2}.$$

Let U *be defined by* (3.1) *and* ξ *obey* (2.8)*. If* $\sum_{n=0}^{\infty} |\sigma(n)|_F^2 < \delta_1(\eta, \varepsilon)$*, then*

$$\mathbb{P}\left[\sup_{n\geq 0} |U(n)|_1 < \varepsilon\right] \geq 1 - \eta, \quad \textit{and moreover} \quad \mathbb{P}\left[\lim_{n\to\infty} U(n) = 0\right] = 1.$$

Lemma 4.8. *Let* $\sigma = \{\sigma(n) : n \in \mathbb{N}\}$ *be an* $\mathbb{R}^d \times \mathbb{R}^r$*-valued sequence. Suppose the sequence of random variables* $\xi = \{\xi(n) \in \mathbb{R}^r : n \in \mathbb{N}\}$ *obeys* (2.6)–(2.9)*. Let* U *be defined by* (3.1)*. If* (4.3) *holds, where* $F_{n,i}$ *and* $F_{n,i,j}$ *are as defined in Lemma 3.2, then* $\mathbb{P}\left[\sup_{n\geq 0} |U(n)|_1 < \varepsilon\right] > 1 - \eta$.

Lemma 4.9. *Let* $\varepsilon > 0$ *and* $\eta \in (0,1)$*. Suppose* $\xi = \{\xi(n) : n \in \mathbb{N}\}$ *is a Gaussian sequence of* $\mathbb{R}^r$*-valued random vectors such that* (2.6)–(2.9) *hold. Let* U *be defined by* (3.1)*. Define*

$$\nu(\eta) = 2 \vee \left(\frac{2}{\sqrt{2\pi}} \frac{d}{\eta} \sum_{n=0}^{\infty} (n+e)^{-2} \right), \quad \delta_1(\eta, \varepsilon) = \frac{\varepsilon^2}{\nu^2(\eta) d^2}.$$

If $\sup_{n \geq 0} |\sigma(n)|_F^2 \log(n+e) < \delta_1(\eta, \varepsilon)$*, then* $\mathbb{P}\left[\sup_{n\geq 0} |U(n)|_1 < \varepsilon\right] > 1 - \eta$*. If moreover* $\lim_{n\to\infty} |\sigma(n)|_F^2 \log(n) = 0$*, then* $\mathbb{P}\left[\lim_{n\to\infty} U(n) = 0\right] = 1$.

5 Proofs of Main Results

5.1 Proof of Lemma 4.2

We recall that since $R \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$, we may define $\bar{R} = \sum_{n=0}^{\infty} |R(n)|_1 < \infty$. Define $U(n) = \sigma(n)\xi(n+1)$, $n \geq 0$. Since $Y(0) = 0$, we have

$$Y(n) = \sum_{k=1}^{n} R(n-k) U(k-1), \quad n \geq 1.$$

Define

$$B_U = \left\{ \omega : \sup_{n \geq 0} |U(n, \omega)|_1 < \varepsilon/(4\bar{R}) \right\}. \tag{5.1}$$

For $\omega \in B_U$

$$|Y(n,\omega)|_1 \leq \sum_{k=1}^{n} |R(n-k)|_1 |U(k-1,\omega)|_1 < \varepsilon/4.$$

Thus, if $B_Y := \omega\{\omega : \sup_{n\geq 0} |Y(n,\omega)|_1 < \varepsilon/4\}$, we have $B_U \subseteq B_Y$.

5.2 Proof of Proposition 4.1

Let $Z(n) = X(n) - Y(n)$, where $Y(0) = 0$. If we define

$$F(n+1) = \sum_{k=0}^{n} A(n-k)[G(X(k)) - DG(0)X(k)], \quad n \geq 0; \quad F(0) = 0,$$

we get

$$Z(n+1) = Z(n) + \sum_{k=0}^{n} A(n-k) DG(0) Z(k) + F(n+1).$$

Therefore $X(n) = Y(n) + Z(n) = Y(n) + R(n)\zeta + \sum_{k=1}^{n} R(n-k)F(k)$ and

$$\begin{aligned}
X(n) &= Y(n) + R(n)\zeta \\
&\quad + \sum_{k=0}^{n-1} \left\{ \sum_{l=k}^{n-1} R(n-1-l)A(l-k) \right\} [G(X(k)) - DG(0)X(k)] \\
&= Y(n) + R(n)\zeta \\
&\quad + \sum_{k=0}^{n-1} \left\{ \sum_{j=0}^{n-1-k} R(n-1-k-j)A(j) \right\} [G(X(k)) - DG(0)X(k)] \\
&= Y(n) + R(n)\zeta + \sum_{k=0}^{n-1} (R * A)(n-1-k)[G(X(k)) - DG(0)X(k)].
\end{aligned}$$

Recall that since $A \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$ and $R \in \ell^1(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^d)$, there exists $S := 1 \vee \sum_{n=0}^{\infty} |(R * A)(n)|_1 < \infty$. Since

$$\lim_{x \to 0} \frac{|G(x) - DG(0)x|_1}{|x|_1} = 0,$$

there is a $\delta > 0$ such that

$$|G(x) - DG(0)x|_1 < \frac{1}{4S}|x|_1 \quad \text{for all } x \text{ such that } |x|_1 < \delta.$$

Let $\varepsilon < \delta$. Suppose that $|\zeta|_1 < \varepsilon/(4\bar{R}) \wedge \varepsilon$. Thus $|X(0)|_1 < \varepsilon$. Let $\omega \in B_U$, where B_U is defined in (5.1). Then $|Y(n,\omega)|_1 < \varepsilon/4$ for all $n \in \mathbb{N}$. We consider the induction hypothesis at level $n-1$ that

$$|X(k,\omega)|_1 < \varepsilon \text{ for } k = 0, \dots, n-1, \omega \in B_U.$$

The hypothesis is true at level 0. Suppose that it is true at level $n-1$. As $\varepsilon < \delta$, and $|X(k,\omega)| < \varepsilon < \delta$ for $k = 0, \dots, n-1$, we have

$$|G(X(k,\omega)) - DG(0)X(k,\omega)|_1 < \frac{1}{4S}|X(k,\omega)|_1 < \frac{\varepsilon}{4S}.$$

Thus

$$\begin{aligned}
|X(n,\omega)|_1 &\le |Y(n,\omega)|_1 + |R(n)|_1|\zeta|_1 \\
&\quad + \sum_{k=0}^{n-1} |(R * A)(n-1-k)|_1 |G(X(k,\omega)) - DG(0)X(k,\omega)|_1.
\end{aligned}$$

Hence

$$\begin{aligned}|X(n,\omega)|_1 &< \frac{\varepsilon}{2}+\sum_{k=0}^{n-1}|(R*A)(n-1-k)|_1\frac{\varepsilon}{4S}\\ &= \frac{\varepsilon}{2}+\frac{\varepsilon}{4S}\sum_{k=0}^{n-1}|(R*A)(k)|_1\\ &\le \frac{\varepsilon}{2}+\frac{\varepsilon}{4S}\sum_{k=0}^{\infty}|(R*A)(k)|_1\\ &\le \frac{\varepsilon}{2}+\frac{\varepsilon}{4S}S=\frac{3\varepsilon}{4}<\varepsilon,\end{aligned}$$

and so the hypothesis holds at level n. Thus, if $\omega \in B_U$, we have $\sup_{n\ge 0}|X(n,\omega)|_1 < \varepsilon$, which completes the proof.

On the other hand,

$$Y(n)=X(n)-R(n)\zeta-\sum_{k=0}^{n-1}(R*A)(n-1-k)[G(X(k))-DG(0)X(k)].$$

Since $\omega \in \{\omega : \sup_{n\ge 0}|X(n,\omega)|_1<\varepsilon\}$ and $\varepsilon<\delta$, it follows that

$$|G(X(k,\omega))-DG(0)X(k,\omega)|_1<\frac{1}{4S}|X(k,\omega)|_1<\frac{\varepsilon}{4S},\quad k\in\mathbb{N}.$$

Therefore

$$\begin{aligned}|Y(n,\omega)|_1 &\le |X(n,\omega)|_1+|R(n)|_1|\zeta|_1+\frac{\varepsilon}{4S}\sum_{k=0}^{n-1}|(R*A)(k)|_1\\ &< \varepsilon+\bar{R}|\zeta|_1+\frac{\varepsilon}{4S}\cdot S\\ &< \varepsilon+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{3\varepsilon}{2}.\end{aligned}$$

Represent $U(n)=Y(n+1)-Y(n)-\sum_{k=0}^{n}A(n-k)DG(0)Y(k)$, so, with $\bar{A}=\sum_{k=0}^{\infty}|A(k)DG(0)|_1$, we have

$$\begin{aligned}|U(n,\omega)|_1 &\le |Y(n+1,\omega)|_1+|Y(n,\omega)|_1+\sum_{k=0}^{n}|A(n-k)DG(0)|_1|Y(k,\omega)|_1\\ &\le \frac{3\varepsilon}{2}+\frac{3\varepsilon}{2}+\frac{3\varepsilon}{2}\sum_{k=0}^{\infty}|A(k)DG(0)|_1=3\varepsilon\left(1+\frac{1}{2}\bar{A}\right).\end{aligned}$$

5.3 Proof of Proposition 4.3

The fundamental identity once again is

$$X(n) = Y(n) + R(n)\zeta + \sum_{k=0}^{n-1}(R * A)(n-1-k)[G(X(k)) - DG(0)X(k)].$$

We suppose that $\omega \in \left\{\sup_{n\geq 0}|U(n)|_1 < \varepsilon/(4\bar{R})\right\} \cap \left\{\lim_{n\to\infty} U(n) = 0\right\}$. Therefore, as $R \in \ell^1(\mathbb{N};\mathbb{R}^d \times \mathbb{R}^d)$, we have that $Y(n,\omega) \to 0$ as $n \to \infty$, and $R(n) \to 0$ as $n \to \infty$. Also, by Proposition 4.1, we have that $|X(n,\omega)|_1 < \varepsilon$ for all $n \in \mathbb{N}$, $\varepsilon \in (0, \delta_0)$. Hence

$$|G(X(k,\omega)) - DG(0)X(k,\omega)|_1 < \frac{1}{4S}|X(k,\omega)|_1, \quad k \in \mathbb{N},$$

and

$$|X(n,\omega)|_1 = |Y(n,\omega)|_1 + |R(n)|_1|\zeta|_1 + \sum_{k=0}^{n-1}\frac{1}{4S}|(R*A)(n-1-k)|_1|X(k,\omega)|_1, \quad n \in \mathbb{N}.$$

Define $F(n,\omega) := |Y(n,\omega)|_1 + |R(n)|_1|\zeta|_1 \geq 0$. Then $F(n,\omega) \to 0$ as $n \to \infty$. Define $K(n) = \frac{1}{4S}|(R * A)(n)|_1 \geq 0$. Then, by the definition of S, we have

$$\sum_{n=0}^{\infty} K(n) = \frac{1}{4S}\sum_{n=0}^{\infty}|(R * A)(n)|_1 \leq \frac{1}{4S}\cdot S = \frac{1}{4}.$$

Finally, let $x(n,\omega) = |X(n,\omega)|_1 \geq 0$. Then

$$x(n,\omega) \leq F(n,\omega) + \sum_{k=0}^{n-1} K(n-1-k)x(k,\omega), \quad n \geq 1.$$

Now, define ρ by $\rho(n) = \sum_{k=0}^{n-1} K(n-1-k)\rho(k)$ for $n \geq 1$ and $\rho(0) = 1$. Define $y(n,\omega) = F(n,\omega) + \sum_{k=0}^{n-1} K(n-1-k)y(k,\omega)$ for $n \geq 1$ with $y(0,\omega) = x(0,\omega)$. Then $x(n,\omega) \leq y(n,\omega)$ for $n \in \mathbb{N}$ and

$$y(n,\omega) = \rho(n)y(0,\omega) + \sum_{k=0}^{n}\rho(n-k)F(k,\omega), \quad n \geq 1.$$

Since $\sum_{n=0}^{\infty} K(n) < 1$, it follows that $\sum_{n=0}^{\infty}\rho(n) < \infty$. Hence we also have $\rho(n) \to 0$ as $n \to \infty$. Because $F(n,\omega) \to 0$ as $n \to \infty$, it follows that $y(n,\omega) \to 0$ as $n \to \infty$, and so $|X(n,\omega)|_1 = x(n,\omega) \to 0$ as $n \to \infty$.

To prove the other part, suppose that $\omega \in \left\{\omega : \lim_{n\to\infty} X(n,\omega) = 0\right\}$. Since G is continuous and $G(0) = 0$, $G(X(n,\omega)) \to 0$ as $n \to \infty$. Therefore $(A*(G\circ X)(\omega))(n) \to 0$ as $n \to \infty$. Rearranging (2.1), we have

$$U(n) = X(n+1) - X(n) - \sum_{k=0}^{n} A(n-k)G(X(k)),$$

so $U(n,\omega) \to 0$ as $n \to \infty$, as required.

5.4 Proof of Theorem 4.4

Define $S = 1 \vee \sum_{n=0}^{\infty} |(R * A)(n)|_1$ and $\bar{R} = \sum_{n=0}^{\infty} |R(n)|_1$. Let $\delta_0 > 0$ be such that

$$|G(x) - DG(0)x|_1 < \frac{1}{4S}|x|_1 \quad \text{for all } |x|_1 < \delta_0.$$

Let $\varepsilon \in (0, \delta_0)$ and $|\zeta|_1 < \varepsilon\left(1 \wedge \frac{1}{4\bar{R}}\right)$. Then, by Proposition 4.1, we have

$$\left\{\omega : \sup_{n\geq 0} |X(n,\omega)|_1 < \varepsilon\right\} \supseteq \left\{\omega : \sup_{n\geq 0} |U(n,\omega)|_1 < \varepsilon/(4\bar{R})\right\}.$$

By Lemma 4.7, if

$$\sum_{n=0}^{\infty} |\sigma(n)|_F^2 < \frac{\eta(\varepsilon/4\bar{R})^2}{rd^2},$$

then $\mathbb{P}\left[\sup_{n\geq 0} |U(n)|_1 < \varepsilon/(4\bar{R})\right] > 1 - \eta$. Hence

$$\mathbb{P}\left[\sup_{n\geq 0} |X(n)|_1 < \varepsilon\right] > 1 - \eta.$$

We now see that defining

$$\delta(\eta, \varepsilon) = \frac{\eta(\varepsilon/4\bar{R})^2}{rd^2} \wedge \varepsilon\left(1 \wedge \frac{1}{4\bar{R}}\right)$$

suffices to prove the first part of the result. Since $\sigma \in \ell^2(\mathbb{N}; \mathbb{R}^d \times \mathbb{R}^r)$, Proposition 3.1 implies that $U(n) \to 0$ a.s. Thus Proposition 4.3 implies $\mathbb{P}\left[\lim_{n\to\infty} X(n) = 0\right] > 1 - \eta$.

5.5 Proof of Theorem 4.6

Define $S = 1 \vee \sum_{n=0}^{\infty} |(R * A)(n)|_1$ and $\bar{R} = \sum_{n=0}^{\infty} |R(n)|_1$. Let $\delta_0 > 0$ be such that

$$|G(x) - DG(0)x|_1 < \frac{1}{4S}|x|_1 \quad \text{for all } |x|_1 < \delta_0.$$

Let $\varepsilon \in (0, \delta_0)$ and $|\zeta|_1 < \varepsilon\left(1 \wedge \frac{1}{4\bar{R}}\right)$. Then, by Proposition 4.1, we have

$$\left\{\omega : \sup_{n\geq 0} |X(n,\omega)|_1 < \varepsilon\right\} \supseteq \left\{\omega : \sup_{n\geq 0} |U(n,\omega)|_1 < \varepsilon/(4\bar{R})\right\}.$$

Define

$$\nu(\eta) = 2 \vee \left(\frac{2}{\sqrt{2\pi}}\frac{d}{\eta}\sum_{n=0}^{\infty}(n+e)^{-2}\right), \quad \delta'(\eta,\varepsilon) = \frac{\varepsilon^2/(16\bar{R}^2)}{\nu^2(\eta)d^2}. \tag{5.2}$$

If $\sup_{n\geq 0} |\sigma(n)|_F^2 \log(n+e) < \delta'(\eta,\varepsilon)$, then by Lemma 4.9 we have

$$\mathbb{P}\left[\sup_{n\geq 0} |U(n)|_1 < \varepsilon/(4\bar{R})\right] > 1 - \eta.$$

Hence

$$\mathbb{P}\left[\sup_{n\geq 0} |X(n)|_1 < \varepsilon\right] > 1 - \eta.$$

We now see that defining

$$\delta(\eta,\varepsilon) = \frac{\varepsilon^2/(16\bar{R}^2)}{\nu(\eta)^2 d^2} \wedge \varepsilon\left(1 \wedge \frac{1}{4\bar{R}}\right),$$

where $\nu(\eta)$ is given by (5.2), suffices to prove the first part of the result.

To prove the asymptotic stability, we notice that $|\sigma(n)|_F^2 \log n \to 0$ as $n \to \infty$ and Corollary 3.4 implies that $U(n) \to 0$ as $n \to \infty$ a.s., because the a.s. convergence to 0 of Y and U are equivalent. Thus, by Proposition 4.3 and the above analysis, we have that $\mathbb{P}\left[\lim_{n\to\infty} X(n) = 0\right] > 1 - \eta$.

6 Proofs of Boundedness of U

6.1 Proof of Lemma 4.7

Define the events

$$A_n(\varepsilon) = \{\omega : |\sigma(n)\xi(n+1),\omega)|_1 < \varepsilon\}, \quad n \geq 0. \tag{6.1}$$

Then

$$\mathbb{P}\left[\sup_{n\geq 0}|U(n)|_1<\varepsilon\right]=\mathbb{P}\left[\bigcap_{n=0}^{\infty}A_n(\varepsilon)\right].$$

Since $\mathbb{P}\left[\bigcup_{n=0}^{\infty}\bar{A}_n(\varepsilon)\right]\leq\sum_{n=0}^{\infty}\mathbb{P}[\bar{A}_n(\varepsilon)]$, it is obvious that

$$\mathbb{P}\left[\sup_{n\geq 0}|U(n)|_1\geq\varepsilon\right]\leq\sum_{n=0}^{\infty}\mathbb{P}[\bar{A}_n(\varepsilon)]. \tag{6.2}$$

The task in hand is now the following: First, we obtain an upper estimate on $\mathbb{P}[\bar{A}_n(\varepsilon)]$. Then, we show that not only this sequence of probabilities is summable, but that the upper bounds on the sums can be controlled in terms of δ and ε. From this, it can be shown that for a given η and ε, it is possible to specify a bound on the weighted sup-norm of σ such that $|U|$ always remains bounded by ε, with probability at least $1-\eta$.

Now we estimate $\mathbb{P}[\bar{A}_n(\varepsilon)]$. Define $U_i(n)=\langle U(n),\mathrm{e}_i\rangle$. So $|U(n)|_1=\sum_{i=1}^{d}|U_i(n)|$. Define $A_{n,i}(\varepsilon)=\{\omega:|U_i(n,\omega)|<\varepsilon/d\}$ for $i=1,\dots,d$. If $\omega\in\bigcap_{i=1}^{d}A_{n,i}(\varepsilon)$, then $|U(n,\omega)|_1<\varepsilon$, so $\omega\in A_n(\varepsilon)$. Hence $\bigcap_{i=1}^{d}A_{n,i}(\varepsilon)\subseteq A_n(\varepsilon)$, so

$$\mathbb{P}[\bar{A}_n(\varepsilon)]=1-\mathbb{P}[A_n(\varepsilon)]\leq 1-\mathbb{P}\left[\bigcap_{i=1}^{d}A_{n,i}(\varepsilon)\right]=\mathbb{P}\left[\bigcup_{i=1}^{d}\bar{A}_{n,i}(\varepsilon)\right]\leq\sum_{i=1}^{d}\mathbb{P}[\bar{A}_{n,i}(\varepsilon)].$$

Thus

$$\mathbb{P}\left[\sup_{n\geq 0}|U(n)|_1\geq\varepsilon\right]\leq\sum_{n=0}^{\infty}\sum_{i=1}^{d}\mathbb{P}[\bar{A}_{n,i}(\varepsilon)]=\sum_{n=0}^{\infty}\sum_{i=1}^{d}\mathbb{P}[|U_i(n)|\geq\varepsilon/d].$$

By Chebyshev's inequality we get

$$\mathbb{P}[|U_i(n)|\geq\varepsilon/d]\leq\frac{d^2}{\varepsilon^2}\mathbb{E}[U_i(n)^2],\quad n\geq 0.$$

Since $U_i(n)=\sum_{j=1}^{r}\sigma_{ij}(n)\xi_j(n+1)$, we have

$$U_i^2(n)=\left(\sum_{j=1}^{r}\sigma_{ij}(n)\xi_j(n+1)\right)^2\leq r\sum_{j=1}^{r}\sigma_{ij}^2(n)\xi_j^2(n+1).$$

Since $\mathbb{E}[\xi_j^2(n)] = 1$ for all $j = 1, \dots, r$ and all $n \in \mathbb{N}$, we also have

$$\mathbb{E}[U_i^2(n)] \le r \sum_{j=1}^{r} \sigma_{ij}^2(n).$$

Hence

$$\sum_{i=1}^{d} \mathbb{P}[|U_i(n)| \ge \varepsilon/d] \le \frac{rd^2}{\varepsilon^2} \sum_{i=1}^{d} \sum_{j=1}^{r} \sigma_{ij}^2(n) = \frac{rd^2}{\varepsilon^2} |\sigma(n)|_F^2,$$

and therefore

$$\mathbb{P}\left[\sup_{n\ge 0} |U(n)|_1 \ge \varepsilon\right] \le \frac{rd^2}{\varepsilon^2} \sum_{n=0}^{\infty} |\sigma(n)|_F^2 < \frac{rd^2}{\varepsilon^2} \cdot \frac{\eta\varepsilon^2}{rd^2} = \eta.$$

Thus $\mathbb{P}\left[\sup_{n\ge 0} |U(n)|_1 < \varepsilon\right] > 1 - \eta$, as required.

6.2 Proof of Lemma 4.8

Recall $U_i(n) = \sum_{j=1}^{r} \sigma_{ij}(n)\xi_j(n+1)$ and the estimate

$$\mathbb{P}\left[\sup_{n\ge 0} |U(n)|_1 \ge \varepsilon\right] \le \sum_{n=0}^{\infty} \sum_{i=1}^{d} \mathbb{P}[|U_i(n)| \ge \varepsilon/d]. \tag{6.3}$$

If $F_{n,i}$ is the distribution function of $U_i(n)$, then

$$\mathbb{P}[|U_i(n)| \ge \varepsilon/d] = 1 - F_{n,i}(\varepsilon/d) + F_{n,i}(-\varepsilon/d),$$

and $F_{n,i}$ is the convolution of the distribution functions $F_{n,i,j}$ for $j = 1, \dots, r$. Hence

$$\mathbb{P}\left[\sup_{n\ge 0} |U(n)|_1 \ge \varepsilon\right] \le \sum_{n=0}^{\infty} \sum_{i=1}^{d} \left[1 - F_{n,i}(\varepsilon/d) + F_{n,i}(-\varepsilon/d)\right],$$

and by (4.3) we have $\mathbb{P}\left[\sup_{n\ge 0} |U(n)|_1 \ge \varepsilon\right] > 1 - \eta$, as required.

6.3 Proof of Lemma 4.9

Once again, we recall that $U_i(n) = \sum_{j=1}^{r} \sigma_{ij}(n)\xi_j(n+\ 1)$ and the estimate (6.3) holds. Since $\xi_j(n+1)$ is normally distributed, and for each n, $(\xi_j(n+1))_{j=1}^{r}$ is a collection of independent random variables, it follows that $U_i(n)$ is normally distributed with zero mean and variance $\sigma_i^2(n) := \sum_{j=1}^{r} \sigma_{ij}^2(n)$, $i = 1, 2, \dots, d$. Define

$$\Psi(x) = \int_x^\infty e^{-\frac{1}{2}u^2}\,du \le \frac{1}{x}e^{-x^2/2}, \quad x > 0.$$

Then

$$\mathbb{P}[|U_i(n)| \ge \varepsilon/d] = \frac{2}{\sqrt{2\pi}}\Psi(\varepsilon/(d\sigma_i(n))) \le \frac{2}{\sqrt{2\pi}}\frac{d\sigma_i(n)}{\varepsilon}\exp\left(-\frac{\varepsilon^2}{2d^2\sigma_i^2(n)}\right).$$

Hence

$$\mathbb{P}\left[\sup_{n\ge 0}|U(n)|_1 \ge \varepsilon\right] \le \frac{2}{\sqrt{2\pi}}\frac{d}{\varepsilon}\sum_{n=0}^{\infty}\sum_{i=1}^{d}\sigma_i(n)\exp\left(-\frac{\varepsilon^2}{2d^2\sigma_i^2(n)}\right).$$

The function $x \mapsto \varphi(x) := x\exp\left(-\frac{\varepsilon^2}{2d^2x^2}\right)$ increases on $[0,\infty)$ (we define $\varphi(0) = 0$). Since $\sigma_i(n) \le |\sigma(n)|_F$, we have that

$$\mathbb{P}\left[\sup_{n\ge 0}|U(n)|_1 \ge \varepsilon\right] \le \frac{2}{\sqrt{2\pi}}\frac{d^2}{\varepsilon}\sum_{n=0}^{\infty}|\sigma(n)|_F\exp\left(-\frac{\varepsilon^2}{2d^2|\sigma(n)|_F^2}\right).$$

Let $c = 1/(\nu d)$. Since $|\sigma(n)|_F^2\log(n+e) < \varepsilon^2/(\nu d)^2$ for all $n \in \mathbb{N}$, we get

$$|\sigma(n)|_F^2\log(n+e) < c^2\varepsilon^2,$$

which implies that

$$\exp\left(-\frac{\varepsilon^2}{2d^2|\sigma(n)|_F^2}\right) < \exp\left(-\frac{1}{2d^2c^2}\log(n+e)\right) = (n+e)^{-\frac{1}{2d^2c^2}},$$

and so

$$\begin{aligned}\mathbb{P}\left[\sup_{n\ge 0}|U(n)|_1 \ge \varepsilon\right] &\le \frac{2}{\sqrt{2\pi}}\frac{d^2}{\varepsilon}\sum_{n=0}^{\infty}\frac{c\varepsilon}{\sqrt{\log(n+e)}}(n+e)^{-\frac{1}{2d^2c^2}}\\ &< \frac{2}{\sqrt{2\pi}}cd^2\sum_{n=0}^{\infty}(n+e)^{-\frac{1}{2d^2c^2}}.\end{aligned}$$

Thus for all $n \in \mathbb{N}$ we have

$$\mathbb{P}\left[\sup_{n\ge 0}|U(n)|_1 \ge \varepsilon\right] < \frac{2}{\sqrt{2\pi}}\frac{d}{\nu}\sum_{n=0}^{\infty}(n+e)^{-\frac{\nu^2}{2}}.$$

Now, recall that

$$\nu = 2 \vee \frac{2}{\sqrt{2\pi}} \frac{d}{\eta} \sum_{n=0}^{\infty} (n+e)^{-2}.$$

Therefore, as $\nu \geq 2$, we have

$$\mathbb{P}\left[\sup_{n\geq 0} |U(n)|_1 \geq \varepsilon\right] < \frac{2}{\sqrt{2\pi}} \frac{d}{\nu} \sum_{n=0}^{\infty} (n+e)^{-2}.$$

But the definition of ν implies that $\frac{2}{\sqrt{2\pi}} \frac{d}{\nu} \sum_{n=0}^{\infty} (n+e)^{-2} \leq \eta$. Hence

$$\mathbb{P}\left[\sup_{n\geq 0} |U(n)|_1 \geq \varepsilon\right] < \eta,$$

which gives the result.

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References

[1] John A. D. Appleby, Gregory Berkolaiko, and Alexandra Rodkina. On local stability for a nonlinear difference equation with a non-hyperbolic equilibrium and fading stochastic perturbations. *J. Difference Equ. Appl.*, 14(9):923–951, 2008.

[2] John A. D. Appleby, D. Mackey, and Alexandra Rodkina. Almost sure polynomial asymptotic stability for stochastic difference equations. *J. Math. Sci. (N. Y.)*, 149(6):1629–1647, 2008.

[3] John A. D. Appleby and Markus Riedle. Almost sure asymptotic stability of stochastic Volterra integro-differential equations with fading perturbations. *Stoch. Anal. Appl.*, 24(4):813–826, 2006.

[4] John A. D. Appleby, Markus Riedle, and Alexandra Rodkina. On asymptotic stability of linear stochastic Volterra difference equations with respect to a fading perturbation. In *Proceedings of the 12th International Conference on Difference Equations and Applications, Kyoto, Japan*, volume 53 of *Adv. Stud. Pure Math.*, pages 281–292. Math. Soc. Japan, Tokyo, 2009.

[5] John A. D. Appleby and Alexandra Rodkina. Asymptotic stability of polynomial stochastic delay differential equations with damped perturbations. *Funct. Differ. Equ.*, 12(1-2):35–66, 2005.

[6] Gregory Berkolaiko and Alexandra Rodkina. Almost sure convergence of solutions to nonhomogeneous stochastic difference equation. *J. Difference Equ. Appl.*, 12(6):535–553, 2006.

[7] Terence Chan and David Williams. An "excursion" approach to an annealing problem. *Math. Proc. Cambridge Philos. Soc.*, 105(1):169–176, 1989.

[8] Saber Elaydi. *An introduction to difference equations*. Undergraduate Texts in Mathematics. Springer, New York, third edition, 2005.

[9] V. B. Kolmanovskiĭ. On the stability of some discrete-time Volterra equations. *Prikl. Mat. Mekh.*, 63(4):561–567, 1999.

[10] Alexandra Rodkina and Michael Basin. On delay-dependent stability for a class of nonlinear stochastic delay-differential equations. *Math. Control Signals Systems*, 18(2):187–197, 2006.

[11] Alexandra Rodkina, Xuerong Mao, and Vladimir Kolmanovskii. On asymptotic behaviour of solutions of stochastic difference equations with Volterra type main term. *Stochastic Anal. Appl.*, 18(5):837–857, 2000.

[12] Alexandra Rodkina and Henri Schurz. Global asymptotic stability of solutions of cubic stochastic difference equations. *Adv. Difference Equ.*, (3):249–260, 2004.