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A Study of the Dynamic Difference Approximations on Time Scales

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Abstract

Various dynamic equations have been used extensively in modeling many important natural phenomena, such as the population or epidemic growth with unpredictable jump sizes, motion control of impulsive robot movements, and prediction of irregular option markets. Since dynamic derivatives are basic building blocks of most dynamic equations, it has been crucial to approximate the derivatives to yield computable discrete equations for numerical solutions. This motivates our investigations. This paper proposes a class of feasible approximation methods for the first and second order noncrossed dynamic derivatives. Applicable local error estimates are derived and discussed. Numerical experiments are given to illustrate our results.

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1 Introduction

A one-dimensional time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$ [2,7]. We denote $a = \sup \mathbb{T},\ b = \inf \mathbb{T}$ and $a,\ b \in \mathbb{T}$. Thus, $\mathbb{T}$ can be viewed as a closed set of real numbers superimposed over the interval $[a, b]$ from an approximation point-of-view. Based on $\mathbb{T}$, we may define the *forward-jump* and *backward-jump* functions $\sigma,\ \rho$ for $t \in \mathbb{T}$. We may write $f^\sigma(t) = f(\sigma(t)),\ f^\rho(t) = f(\rho(t))$, where f is a function defined on $\mathbb{T}$. We may further define the *forward-step* and *backward-step* functions μ and η. Denote

$$\lambda(t) = \mu(t)/\eta(t)$$

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whenever $\eta(t) \neq 0$. We say that a time scale $\mathbb{T}$ is *uniform* if for all $t \in \mathbb{T}^{\kappa}_{\kappa},\ \mu(t) = \eta(t)$ [14, 15]. A uniform time scale is either an interval if $\mu(t) = 0$ or a uniform difference grid if $\mu(t) > 0$ for all $t \in \mathbb{T}$. In our study, we also need the following sets [1, 14–17]:

$$\begin{aligned} \mathbb{A} : &= \{t \in \mathbb{T} : t \text{ is left-dense}\}, \\ \mathbb{B} : &= \{t \in \mathbb{T} : t \text{ is left-scattered}\}, \\ \mathbb{C} : &= \{t \in \mathbb{T} : t \text{ is right-dense}\}, \\ \mathbb{D} : &= \{t \in \mathbb{T} : t \text{ is right-scattered}\}. \end{aligned}$$

Without loss of generality, we may assume that $a \in \mathbb{A}$ and $b \in \mathbb{C}$.

Different types of dynamic derivatives, including Δ, ∇ and the combined $\diamondsuit_\alpha$ derivatives, have been introduced on different time scales [3, 4, 17]. Based on them, linear and nonlinear dynamic equations become possible. The most distinguished features of the dynamic derivatives from an application point-of-view include their mixed continuous and discrete structures, flexibilities in approximating hybrid natural processes [9, 11–13, 18, 19], and great potentials in adaptive simulations [6, 17]. Numerous recent publications can be found in the literature. For details, the reader is referred to [4, 6, 8, 14–17] and references therein. It has been difficult to establish executable numerical formulas for approximating the dynamic derivatives due to their hybrid features.

Let the functions f and g be defined on $\mathbb{S} \subseteq \mathbb{T}$, and g be an approximation of f. If

$$|f(t) - g(t)| = O\left(\max\{\mu^{\gamma}(t), \eta^{\gamma}(t)\}\right), \quad t \in \mathbb{S}, \tag{1.1}$$

where $0 \leq \mu,\ \eta < 1$, then we say that the approximation is accurate to the *order* γ with respect to the step functions on the time scale $\mathbb{S} \subseteq \mathbb{T}$. From an approximation point of view, the approximation g is *consistent* if and only if $\gamma > 0$ [11, 14].

This paper intends to discuss feasible numerical treatments of the most frequently used dynamic derivatives on time scales, including the first order and second order Δ, ∇, and $\diamondsuit_\alpha$ derivatives. Our discussions will be organized as follows. In Section 2, we will focus on approximations of the first order dynamic derivatives. Section 3 will be devoted to the study of the second order noncrossed dynamic derivatives approximations. Convergence properties of the approximation formulas will be obtained. The method of asymptotic expansions are utilized to construct acceptable estimates of the local numerical errors. Finally, in Section 4, numerical experiments will be carried out to illustrate our conclusions. A minimal experience with the time scales and approximation theories are assumed.

2 First Order Dynamic Difference Approximations

Let the function f be defined on $\mathbb{T}$. For values $h_1 \geq \mu(t),\ h_2 \geq \eta(t)$ we define the *forward, backward* and *central dynamic differences of* f as

$$\begin{aligned} f^F(t) &= \frac{f(t+h_1)-f(t)}{h_1}, \quad \text{if } t,\ t+h_1 \in \mathbb{T}, &(2.1)\\ f^B(t) &= \frac{f(t)-f(t-h_2)}{h_2}, \quad \text{if } t-h_2,\ t \in \mathbb{T}, &(2.2)\\ f^C(t) &= \frac{f(t+h_1)-f(t-h_2)}{h_1+h_2}, \quad \text{if } t-h_2,\ t,\ t+h_1 \in \mathbb{T}, &(2.3)\end{aligned}$$

respectively.

Theorem 2.1. *If f is Δ differentiable and t is right-dense, then*

$$\lim_{h_1\to\mu(t)} f^F(t) = f^\Delta(t), \quad t \in \mathbb{T}^\kappa.$$

Further, if f is twice differentiable on (a,b) then we have the local error estimate

$$\left|f^F(t)-f^\Delta(t)\right| \leq h_1\left|f''(\xi)\right|, \quad t<\xi<t+h_1,\ t\in\mathbb{T}^\kappa. \tag{2.4}$$

Proof. According to (2.1), for any $t \in \mathbb{T}^\kappa$,

$$\begin{aligned}\lim_{h_1\to\mu(t)} f^F(t) &= \lim_{t+h_1\to\sigma(t)} \frac{f(t+h_1)-f(t)}{t+h_1-t}\\ &= \left\{\begin{array}{cc} f'(t^+), & \text{if } \mu(t)=0,\\ \dfrac{f(\sigma(t))-f(t)}{\mu(t)}, & \text{otherwise,}\end{array}\right\} = f^\Delta(t).\end{aligned}$$

Further, let $t \in \mathbb{C}\cap\mathbb{T}^\kappa$. According to the above discussion, $f^F(t)$ approaches $f'(t^+)$ as $h_1 \to 0$. This indicates that $f^F(t)$ is a conventional forward difference formula for the directional derivative in $\mathbb{T}^\kappa$ and therefore (2.4) is true. On the other hand, if $t \in \mathbb{D}\cap\mathbb{T}^\kappa$, by Taylor's remainder theorem,

$$\begin{aligned} f^F(t)-f^\Delta(t) &= \frac{f(t+h_1)-f(t)}{h_1} - \frac{f(\sigma(t))-f(t)}{\mu(t)}\\ &= f'(\xi_1)-f'(\xi_2) = (\xi_1-\xi_2)f''(\xi),\end{aligned}$$

in which $|\xi_1-\xi_2| \leq h_1$ since $\xi_1,\ \xi_2 \in (t,t+h_1),\ \sigma(t)\leq h_1$; and

$$\xi \in (\min\{\xi_1,\xi_2\},\max\{\xi_1,\xi_2\}) \subseteq (t,t+h_1).$$

Therefore (2.4) holds. We note that $\xi_1,\ \xi_2,\ \xi$ may not necessarily be in $\mathbb{T}^\kappa$. □

Similarly, we have the following.

Theorem 2.2. *If f is ∇ differentiable and t is left-dense, then*

$$\lim_{h_2\to\eta(t)} f^B(t) = f^\nabla(t), \quad t\in\mathbb{T}_\kappa.$$

Further, if f is twice differentiable on (a,b) then we have the local error estimate

$$\left|f^B(t) - f^\nabla(t)\right| \le h_2\left|f''(\zeta)\right|,\ t-h_2<\zeta<t,\ t\in\mathbb{T}_\kappa. \tag{2.5}$$

Proof. According to (2.2), for any $t\in\mathbb{T}_\kappa$,

$$\begin{aligned}\lim_{h_2\to\eta(t)} f^B(t) &= \lim_{t-h_2\to\rho(t)}\frac{f(t)-f(t-h_2)}{t-(t-h_2)}\\ &= \left\{\begin{array}{cc} f'(t^-), & \text{if } \eta(t)=0,\\ \dfrac{f(t)-f(\rho(t))}{\eta(t)}, & \text{otherwise,}\end{array}\right\} = f^\nabla(t).\end{aligned}$$

We further notice that when $t\in\mathbb{A}\cap\mathbb{T}_\kappa$, $f^B(t)$ approaches $f'(t^-)$ as $h_2\to 0$. This indicates that $f^B(t)$ is a standard backward difference approximation of the directional derivative. Therefore (2.5) must be true in this circumstance. On the other hand, if $t\in\mathbb{B}\cap\mathbb{T}_\kappa$,

$$\begin{aligned} f^B(t)-f^\nabla(t) &= \frac{f(t)-f(t-h_2)}{h_2} - \frac{f(t)-f(\rho(t))}{\eta(t)}\\ &= f'(\zeta_1)-f'(\zeta_2) = (\zeta_1-\zeta_2)f''(\zeta),\end{aligned}$$

in which $|\zeta_1-\zeta_2|\le h_2,\ \zeta_1,\ \zeta_2\in(t-h_2,t),\ \rho(t)\le h_2$; and

$$\zeta\in(\min\{\zeta_1,\zeta_2\},\max\{\zeta_1,\zeta_2\})\subseteq(t-h_1,t).$$

Therefore (2.5) holds. We note again that $\zeta_1,\ \zeta_2,\ \zeta$ may not necessarily be in $\mathbb{T}_\kappa$. □

Theorem 2.3. *Let f be $\diamondsuit_\alpha$ differentiable and t be left-dense and right-dense. If*

$$\lim_{\substack{h_1\to\mu(t)\\ h_2\to\eta(t)}} \frac{h_1}{h_1+h_2} = \alpha, \quad 0<\alpha<1,$$

then

$$\lim_{\substack{h_1\to\mu(t)\\ h_2\to\eta(t)}} f^C(t) = f^{\diamondsuit_\alpha}(t), \quad t\in\mathbb{T}_\kappa^\kappa.$$

Further, if f is differentiable on (a,b) then we have the local error estimate

$$\begin{aligned}\left|f^C(t)-f^{\diamondsuit_\alpha}(t)\right| &\le \max\{h_1,h_2\}\max\{|f''(\xi)|,|f''(\zeta)|\},\\ &\qquad t<\xi<t+h_1,\ t-h_2<\zeta<t,\ t\in\mathbb{T}_\kappa^\kappa.\end{aligned} \tag{2.6}$$

Proof. Recall by (2.3) and Theorems 2.1 and 2.2, for $t \in \mathbb{T}^{\kappa}_{\kappa}$ we have

$$\begin{aligned}
\lim_{\substack{h_1 \to \mu(t) \\ h_2 \to \eta(t)}} f^C(t) &= \lim_{\substack{h_1 \to \mu(t) \\ h_2 \to \eta(t)}} \frac{f(s+h_1) - f(s) + f(s) - f(s-h_2)}{h_1 + h_2} \\
&= \lim_{\substack{h_1 \to \mu(t) \\ h_2 \to \eta(t)}} \left[\frac{h_1}{h_1+h_2} \, \frac{f(t+h_1)-f(t)}{h_1} + \frac{h_2}{h_1+h_2} \, \frac{f(t)-f(t-h_2)}{h_2} \right] \\
&= \alpha \lim_{t+h_1 \to \sigma(t)} \frac{f(t+h_1)-f(t)}{t+h_1-t} + (1-\alpha) \lim_{t-h_2 \to \rho(t)} \frac{f(t)-f(t-h_2)}{t-(t-h_2)} \\
&= f^{\diamondsuit_\alpha}(t),
\end{aligned}$$

provided that the limit α exists. The error estimate (2.6) can be obtained readily by combining (2.4), (2.5), and utilizing a triangular inequality. □

Remark 2.4. Theorem 2.3 implies that, by choosing different ratios of the nonuniform grid steps h_1/h_2, (2.3) converges to any desired $\diamondsuit_\alpha$ derivative value. A sensible choice of such step ratios in practical computations may be $h_2 = sh_1$ with a scaling factor $s > 0$. In the circumstance we have

$$0 < \alpha = \lim_{\substack{h_1 \to \mu(t) \\ h_2 \to \eta(t)}} \frac{h_1}{h_1+h_2} = \lim_{h_1 \to \mu(t)} \frac{h_1}{h_1 + sh_1} = \frac{1}{1+s} < 1.$$

In a further special case when $s \equiv 1$, we have $h_1 = h_2 = h$, and $f^C(t)$ reduces to the conventional central difference formula which approximates the arithmetic average of $f^{\Delta}(t),\ f^{\nabla}(t)$ as $t + h \to \sigma(t)$ and $t - h \to \rho(t)$.

We note that, however, such a limit α may not exist in general. A typical example is that when the step sequences are chosen as

$$h_{1,n} = \frac{1}{n},\ h_{2,n} = s_n h_{1,n} \ \text{ with variable scaling factors } \ s_n = 1 + (-1)^n.$$

In the case the limit of

$$\alpha_n = \frac{1}{2+(-1)^n}, \quad n \to \infty,$$

does not exist, although $\lim_{n\to\infty} h_1 = \lim_{n\to\infty} h_2 = 0$.

Remark 2.5. Theorems 2.1 and 2.2 ensure that the dynamic differences (2.1), (2.2) are not only first order approximations to the corresponding dynamic derivatives, respectively, but also first order approximations to the derivative function $f'(t)$ if it exists. On the other hand, although (2.3) may provide a first order approximation of the dynamic derivative $f^{\diamondsuit_\alpha}(t)$ and $f'(t)$, if they exist, on an uniform mesh superimposed on $\mathbb{T}$, it approximates neither the diamond-α derivative, nor $f'(t)$ on an arbitrary subset of $\mathbb{T}$.

3 Second Order Dynamic Difference Approximations

Let $t \in \mathbb{T}$ and $t-h_2-h_4,\ t-h_2,\ t,\ t+h_1,\ t+h_1+h_3$, where $h_\ell = h_\ell(t) \neq 0,\ \ell = 1,2,3,4$, are positive values which are distinct in general. Based on (2.1)–(2.3), we propose the following second order dynamic difference formulas:

$$\begin{aligned} f^{FF}(t) &= \frac{h_1 f(t+h_1+h_3) - (h_1+h_3) f(t+h_1) + h_3 f(t)}{h_1^2 h_3}, \\ &\quad \text{if } t,\ t+h_1, t+h_1+h_3 \in \mathbb{T}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} f^{BB}(t) &= \frac{h_4 f(t) - (h_2+h_4) f(t-h_2) + h_2 f(t-h_2-h_4)}{h_2^2 h_4}, \\ &\quad \text{if } t-h_2-h_4,\ t-h_2, t \in \mathbb{T}. \end{aligned} \tag{3.2}$$

Theorem 3.1. *If f is twice Δ differentiable and $t,\ t+h_1$ are right-dense, then*

$$\lim_{\substack{h_1 \to \mu(t) \\ h_3 \to \mu(\sigma(t))}} f^{FF}(t) = f^{\Delta\Delta}(t), \quad t \in \mathbb{T}^{\kappa^2}. \tag{3.3}$$

Further, if f is continuously differentiable on (a,b) then we have the local error estimate

$$\left| f^{FF}(t) - f^{\Delta\Delta}(t) \right| \le \phi_1 \frac{h_3}{h_1} + O\left(h_1 + h_3 + \frac{h_3^2}{h_1} \right), \quad t \in \mathbb{T}^{\kappa^2}, \tag{3.4}$$

where

$$\phi_1 = \begin{cases} \dfrac{(s_1 - s_3)M}{s_1}, & t \in \mathbb{D} \cap \mathbb{T}^{\kappa^2},\ \sigma(t) \in \mathbb{D} \cap \mathbb{T}^{\kappa}, \\ \left(\dfrac{1}{2} - \dfrac{\beta h_1}{h_3} \right) M, & t \in \mathbb{C} \cap \mathbb{T}^{\kappa^2}, \\ \dfrac{M}{2}, & t \in \mathbb{D} \cap \mathbb{T}^{\kappa^2},\ \sigma(t) \in \mathbb{C} \cap \mathbb{T}^{\kappa}, \end{cases}$$

$$\beta = \lim_{\substack{h_1 \to \mu(t) \\ h_3 \to \mu(\sigma(t))}} \frac{h_3}{2h_1}, \quad M = \sup_{a<t<b} \frac{|f''(t)|}{2}, \quad 0 < s_1,\ s_3 \le 1.$$

Proof. First, we let $t \in \mathbb{D} \cap \mathbb{T}^{\kappa^2},\ \sigma(t) \in \mathbb{D} \cap \mathbb{T}^{\kappa}$. It is observed that

$$\begin{aligned} &\lim_{\substack{h_1 \to \mu(t) \\ h_3 \to \mu(\sigma(t))}} f^{FF}(t) \\ &= \lim_{\substack{t+h_1 \to \sigma(t) \\ t+h_1+h_3 \to \sigma^2(t)}} \frac{h_1 f(t+h_1+h_3) - (h_1+h_3) f(t+h_1) + h_3 f(t)}{h_1^2 h_3} \\ &= \frac{\mu(t) f^{\sigma^2}(t) - (\mu(t) + \mu^\sigma(t)) f^\sigma(t) + \mu^\sigma(t) f(t)}{\mu^2(t) \mu^\sigma(t)} = f^{\Delta\Delta}(t) \end{aligned} \tag{3.5}$$

according to [2]. Secondly, for $t \in \mathbb{C} \cap \mathbb{T}^{\kappa^2}$, we have $\sigma(t) = t$ and therefore $\sigma^2(t) = t$. It follows subsequently that

$$\begin{aligned}
&\lim_{\substack{h_1 \to \mu(t) \\ h_3 \to \mu(\sigma(t))}} f^{FF}(t) \\
&= \lim_{h_1,\, h_3 \to 0^+} \frac{[f(t+h_1+h_3) - f(t+h_1)]/h_3 - [f(t+h_1) - f(t)]/h_1}{h_1} \\
&= \lim_{h_1 \to 0^+} \frac{f^{\Delta}(t+h_1) - [f(t+h_1) - f(t)]/h_1}{h_1} = f^{\Delta\Delta}(t). \qquad (3.6)
\end{aligned}$$

Thirdly, in the case that $t \in \mathbb{D} \cap \mathbb{T}^{\kappa^2}$ and $\sigma(s) \in \mathbb{C} \cap \mathbb{T}^{\kappa}$, we find that

$$\begin{aligned}
&\lim_{\substack{h_1 \to \mu(t) \\ h_3 \to \mu(\sigma(t))}} f^{FF}(t) \\
&= \lim_{\substack{h_1 \to \mu(t) \\ h_3 \to 0^+}} \frac{[f(t+h_1+h_3) - f(t+h_1)]/h_3 - [f(t+h_1) - f(t)]/h_1}{h_1} \\
&= \frac{f'((\sigma(t))^+) - f^{\Delta}(t)}{\mu(t)} = f^{\Delta\Delta}(t) \qquad (3.7)
\end{aligned}$$

according to [14–16]. Combining (3.5)–(3.7), we acquire immediately (3.3).

To show the error estimate (3.4), we first expand $f^{FF}(t)$ at $t \in \mathbb{T}^{\kappa^2}$,

$$f^{FF}(t) = \frac{h_1 + h_3}{2h_1} f''(t) + \left(\frac{h_3^2}{6h_1} + \frac{h_1}{3} + \frac{h_3}{2} \right) f'''(t) + \cdots . \qquad (3.8)$$

For the case if $t \in \mathbb{D} \cap \mathbb{T}^{\kappa^2},\ \sigma(t) \in \mathbb{D} \cap \mathbb{T}^{\kappa}$, from (3.5) we deduce that

$$f^{\Delta\Delta}(t) = \frac{\mu + \mu^{\sigma}}{2\mu} f''(t) + \left(\frac{(\mu^{\sigma})^2}{6\mu} + \frac{\mu}{3} + \frac{\mu^{\sigma}}{2} \right) f'''(t) + \cdots . \qquad (3.9)$$

Subtracting (3.9) from (3.8) we obtain

$$\begin{aligned}
f^{FF}(t) - f^{\Delta\Delta}(t) \;=\; & \left(\frac{h_1 + h_3}{2h_1} - \frac{\mu + \mu^{\sigma}}{2\mu} \right) f''(t) \\
& + \left(\frac{h_3^2}{6h_1} + \frac{h_1}{3} + \frac{h_3}{2} - \frac{(\mu^{\sigma})^2}{6\mu} - \frac{\mu}{3} - \frac{\mu^{\sigma}}{2} \right) f'''(t) + \cdots . \qquad (3.10)
\end{aligned}$$

Recall that $0 < \mu \le h_1,\ 0 < \mu^{\sigma} \le h_3$. We define positive parameters $s_1 = \mu/h_1 \le 1,\ s_3 = \mu^{\sigma}/h_3 \le 1$. Substitute them into (3.10) to yield

$$\begin{aligned}
f^{FF}(t) - f^{\Delta\Delta}(t) \;=\; & \left(\frac{s_1 - s_3}{2s_1} \right) \frac{h_3}{h_1} f''(t) \\
& + \left[h_3 \left(\frac{s_1 - s_3^2}{6s_1} \right) \frac{h_3}{h_1} + \frac{1 - s_1}{3} h_1 + \frac{1 - s_3}{2} h_3 \right] f'''(t) + \cdots .
\end{aligned}$$

Thus, (3.4) is clear. Further, if $t \in \mathbb{C} \cap \mathbb{T}^{\kappa^2}$ then $\sigma(t) = \sigma^2(t) = t$. From (3.9), we observe that

$$f^{\Delta\Delta}(t) = \left(\frac{1}{2} + \beta\right) f''(t^+).$$

Thus,

$$\begin{aligned} f^{FF}(t) - f^{\Delta\Delta}(t) &= \frac{h_1 + h_3}{2h_1} f''(t^+) + \left(\frac{h_3^2}{6h_1} + \frac{h_1}{3} + \frac{h_3}{2}\right) f'''(t^+) + \cdots \\ &\quad - \left(\frac{1}{2} + \beta\right) f''(t^+) \\ &= \left(\frac{h_3}{2h_1} - \beta\right) f''(t) + \frac{1}{6}\left(\frac{h_3^2}{h_1} + 2h_1 + 3h_3\right) f'''(t^+) + \cdots . \end{aligned}$$

Therefore (3.4) is proved. Finally, if $t \in \mathbb{D} \cap \mathbb{T}^{\kappa^2}$, then $\sigma(t) \in \mathbb{C} \cap \mathbb{T}^{\kappa}$. Based on (3.7) and Theorem 2.1 we acquire that

$$\begin{aligned} f^{\Delta\Delta}(t) &= \frac{f'((t+\mu)^+) - f^{\Delta}(t)}{\mu(t)} = \frac{f'(t) + \mu f''(t) + \frac{\mu^2}{2} f'''(t) + \cdots - \frac{f(\sigma(t)) - f(t)}{\mu(t)}}{\mu(t)} \\ &= \frac{1}{2} f''(t) - \frac{\mu(t)}{3} f'''(t) + \cdots . \end{aligned}$$

Recalling (3.8), we have

$$f^{FF}(t) - f^{\Delta\Delta}(t) = \frac{h_3}{2h_1} f''(t) + \left(\frac{h_3^2}{6h_1} + \frac{h_1}{3} + \frac{h_3}{2} - \frac{\mu(t)}{3}\right) f'''(t) + \cdots$$

which completes our proof. □

By the same token, we may prove the following.

Theorem 3.2. *If f is twice ∇ differentiable and $t,\ t - h_2$ are left-dense, then*

$$\lim_{\substack{h_2 \to \eta(t) \\ h_4 \to \eta(\rho(t))}} f^{BB}(t) = f^{\nabla\nabla}(t), \quad t \in \mathbb{T}_{\kappa^2}.$$

Further, if f is continuously differentiable on (a, b) then we have the local error estimate

$$|f^{BB}(t) - f^{\nabla\nabla}(t)| \le \phi_2 \frac{h_4}{h_2} + O\left(h_2 + h_4 + \frac{h_4^2}{h_2}\right), \quad t \in \mathbb{T}_{\kappa^2}, \tag{3.11}$$

where

$$\phi_2 = \begin{cases} \dfrac{(s_2 - s_4)M}{s_2}, & t \in \mathbb{B} \cap \mathbb{T}_{\kappa^2},\ \sigma(t) \in \mathbb{B} \cap \mathbb{T}_{\kappa}, \\ \left(\dfrac{1}{2} - \dfrac{\tilde{\beta} h_2}{h_4}\right) M, & t \in \mathbb{A} \cap \mathbb{T}_{\kappa^2}, \\ \dfrac{M}{2}, & t \in \mathbb{B} \cap \mathbb{T}_{\kappa^2},\ \sigma(t) \in \mathbb{A} \cap \mathbb{T}_{\kappa}, \end{cases}$$

$$\tilde{\beta} = \lim_{\substack{h_2 \to \eta(t) \\ h_4 \to \eta(\rho(t))}} \frac{h_4}{2h_2}, \quad M = \sup_{a<t<b} \frac{|f''(t)|}{2}, \quad 0 < s_2,\ s_4 \leq 1.$$

Proof. The proof is similar to that for Theorem 3.1. First, if $t \in \mathbb{B} \cap \mathbb{T}_{\kappa^2}$ and $\rho(t) \in \mathbb{B} \cap \mathbb{T}_{\kappa}$ then

$$\begin{aligned}
& \lim_{\substack{h_2 \to \eta(t) \\ h_4 \to \eta(\rho(t))}} f^{BB}(t) \\
&= \lim_{\substack{t-h_2 \to \rho(t) \\ t-h_2-h_4 \to \rho^2(t)}} \frac{h_2 f(t-h_2-h_4) - (h_2+h_4) f(t-h_2) + h_4 f(t)}{h_2^2 h_4} \\
&= \frac{\eta^{\rho}(t) f(t) - (\eta^{\rho}(t) + \eta(t)) f^{\rho}(t) + \eta(t) f^{\rho^2}(t)}{\eta^2(t) \eta^{\rho}(t)} \qquad (3.12)
\end{aligned}$$

which is consistent with results in [9, 14–16]. On the other hand, if $t \in \mathbb{A} \cap \mathbb{T}_{\kappa^2}$ then $\rho(t) = t$ and subsequently $\rho^2(t) = t$. It follows readily that

$$\begin{aligned}
& \lim_{\substack{h_2 \to \eta(t) \\ h_4 \to \eta(\rho(t))}} f^{BB}(t) \\
&= \lim_{h_2,\, h_4 \to 0^+} \frac{[f(t) - f(t-h_2)]/h_2 - [f(t-h_2) - f(t-h_2-h_4)]/h_4}{h_2} \\
&= \lim_{h_2 \to 0^+} \frac{[f(t) - f(t-h_2)]/h_2 - f^{\nabla}(t-h_2)}{h_2} = f^{\nabla\nabla}(t). \qquad (3.13)
\end{aligned}$$

Now, if $t \in \mathbb{B} \cap \mathbb{T}_{\kappa^2}$ and $\rho(t) \in \mathbb{A} \cap \mathbb{T}_{\kappa}$. According to [15],

$$\begin{aligned}
& \lim_{\substack{h_1 \to \eta(t) \\ h_3 \to \eta(\rho(t))}} f^{BB}(t) \\
&= \lim_{\substack{h_1 \to \eta(t) \\ h_3 \to 0^+}} \frac{[f(t) - f(t-h_1)]/h_1 - [f(t-h_1) - f(t-h_1-h_3)]/h_3}{h_1} \\
&= \frac{f^{\nabla}(t) - f'((\rho(t))^-)}{\eta(t)} = f^{\nabla\nabla}(t). \qquad (3.14)
\end{aligned}$$

Similar to (3.8), we arrive at

$$f^{BB}(t) = \frac{h_2 + h_4}{2h_2} f''(t) - \left(\frac{h_4^2}{6h_2} + \frac{h_2}{3} + \frac{h_4}{2} \right) f'''(t) + \cdots . \qquad (3.15)$$

To derive the error estimate, we first consider the case of $t \in \mathbb{B} \cap \mathbb{T}_{\kappa^2}$, $\rho(t) \in \mathbb{B} \cap \mathbb{T}_{\kappa}$. Recall (3.12). We have

$$f^{\nabla\nabla}(t) = \frac{\eta^\rho + \eta}{2\eta} f''(t) - \left(\frac{(\eta^\rho)^2}{6\eta} + \frac{\eta}{3} + \frac{\eta^\rho}{2} \right) f'''(t) + \cdots . \tag{3.16}$$

A subtraction of (3.16) from (3.9) yields

$$\begin{aligned} f^{BB}(t) - f^{\nabla\nabla}(t) &= \left(\frac{h_2 + h_4}{2h_2} - \frac{\eta + \eta^\rho}{2\eta} \right) f''(t) \\ &\quad - \left(\frac{h_4^2}{6h_2} + \frac{h_2}{3} + \frac{h_4}{2} - \frac{(\eta^\rho)^2}{6\eta} - \frac{\eta}{3} - \frac{\eta^\rho}{2} \right) f'''(t) + \cdots . \end{aligned} \tag{3.17}$$

Note that $0 < \eta \leq h_2,\ 0 < \eta^\rho \leq h_4$. Define positive parameters $s_2 = \eta / h_2 \leq 1,\ s_4 = \eta^\rho / h_4 \leq 1$. Substitute them into (3.17), we acquire that

$$\begin{aligned} f^{BB}(t) - f^{\nabla\nabla}(t) &= \left(\frac{s_2 - s_4}{2s_2} \right) \frac{h_4}{h_2} f''(t) \\ &\quad - \left[h_4 \left(\frac{s_2 - s_4^2}{6s_2} \right) \frac{h_4}{h_2} + \frac{1 - s_2}{3} h_2 + \frac{1 - s_4}{2} h_4 \right] f'''(t) + \cdots . \end{aligned}$$

Thus the estimate is affirmative. If $t \in \mathbb{A} \cap \mathbb{T}_{\kappa^2}$ then $\rho(t) = \rho^2(t) = t$. Based on (3.13), (3.16) we obtain

$$f^{\nabla\nabla}(t) = \left(\frac{1}{2} + \tilde{\beta} \right) f''(t^-).$$

Hence,

$$\begin{aligned} f^{BB}(t) - f^{\nabla\nabla}(t) &= \frac{h_2 + h_4}{2h_2} f''(t^-) - \left(\frac{h_4^2}{6h_2} + \frac{h_2}{3} + \frac{h_4}{2} \right) f'''(t^-) + \cdots \\ &\quad - \left(\frac{1}{2} + \tilde{\beta} \right) f''(t^-) \\ &= \left(\frac{h_4}{2h_2} - \tilde{\beta} \right) f''(t) - \frac{1}{6} \left(\frac{h_4^2}{h_2} + 2h_2 + 3h_4 \right) f'''(t^-) + \cdots \end{aligned}$$

which leads to our estimate. Our last case is for $t \in \mathbb{B} \cap \mathbb{T}_{\kappa^2},\ \rho(t) \in \mathbb{A} \cap \mathbb{T}_{\kappa}$. Due to (3.14) and Theorem 2.2,

$$\begin{aligned} f^{\nabla\nabla}(t) &= \frac{f^{\nabla}(t) - f'((\rho(t))^-)}{\eta(t)} = \frac{\frac{f(t) - f(\rho(t))}{\eta(t)} - \left(f'(t) - \eta f''(t) + \frac{\eta^2}{2} f'''(t) + \cdots \right)}{\eta(t)} \\ &= \frac{1}{2} f''(t) - \frac{\eta(t)}{3} f'''(t) + \cdots . \end{aligned}$$

Recall (3.15). We deduce that

$$f^{FF}(t) - f^{\nabla\nabla}(t) = \frac{h_4}{2h_2} f''(t) - \left(\frac{h_4^2}{6h_2} + \frac{h_2}{3} + \frac{h_4}{2} - \frac{\eta(t)}{3} \right) f'''(t) + \cdots$$

which again indicates our error estimate.

A combination of the above discussions completes our proof. □

Remark 3.3. While Theorems 3.1 and 3.2 provide solid evidences that the second order dynamic differences (3.1), (3.2) converge to their corresponding dynamic derivatives, respectively, the error estimates (3.4), (3.11) offer applicable (probably not the best) local error estimates in the literature. The estimates provide safeguards in some sense in computations of the solution of dynamic equations whenever necessary.

Remark 3.4. From the above investigations we may conclude readily that neither of the second order dynamic differences, nor the second order dynamic derivatives, should be considered as natural approximations of the conventional derivative function $f''(t)$, should it exists in the domain considered. The arbitrary nonuniform mesh steps $h_\ell,\ \ell = 1,2,3,4$, often complicate the approximation desires [9, 18].

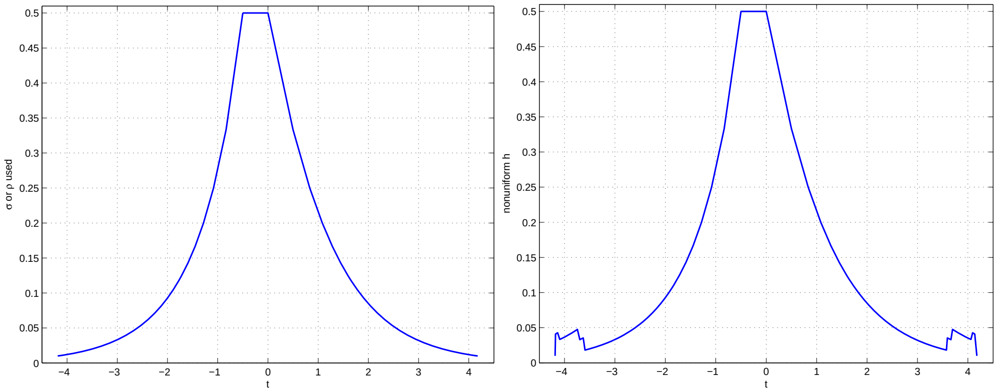
Remark 3.5. Investigations of the second order central dynamic difference,

$$\begin{aligned} f^{CC}(t) \;=\;& [(h_2+h_4)f(t+h_1+h_3)-(h_1+h_2+h_3+h_4)f(t) \\ & +(h_1+h_3)f(t-h_2-h_4)]/[(h_1+h_2)(h_1+h_3)(h_2+h_4)], \\ & \text{if } t-h_2-h_4,\ t-h_2,\ t,\ t+h_1, t+h_1+h_3 \in \mathbb{T}, \end{aligned}$$

as well as crossed second order dynamic difference formulae can be very promising but lengthy. We prefer to leave its discussions, together with those for crossed dynamic difference approximations, to our forthcoming papers.

4 Numerical Experiments

Figure 4.1: LEFT: Plot of the jump functions on $\mathbb{T}$. RIGHT: Plot of the nonuniform step size functions over $T \subset \mathbb{T}$.



For a given positive integer n, we consider the time scale $\mathbb{T} := \{t_n = 0;\ t_{i-1} = t_i - 1/(i-n+2),\ i = n, n-1, \ldots, 2; t_{j+1} = t_j + 1/(j-n+3),\ j = n, n+1, \ldots, 2n-2\}$.

The left end of $\mathbb{T}$, a, can be viewed as right dense while the right end of $\mathbb{T}$, b, can be viewed as left dense for large n from computational point-of-view. Set $n = 100$. The layout of the corresponding jump functions $\sigma(t)$, or $\rho(t)$, is plotted in the first frame of Figure 4.1. The second frame of Figure 4.1 shows a discrete set $T \subset \mathbb{T}$ on which the dynamic differences approximations are constructed. We let the nonuniform steps used in T be bounded below by 0.018 and above by 0.033. Irregular computational steps are used to replace the small jumps near the left and right ends of $\mathbb{T}$ for studying properties of the approximations. We only show results related to the dynamic differences $f^F(t)$ and $f^{FF}(t)$.

Figure 4.2: LEFT: The dynamic derivative $f^{\Delta}(t)$ and dynamic difference $f^F(t)$ on set T. A low frequency is used. The difference between the functions are hard to see. RIGHT: An enlarged image of the functions for $3.3 \leq t \leq 4.2$. The functions become distinguishable.

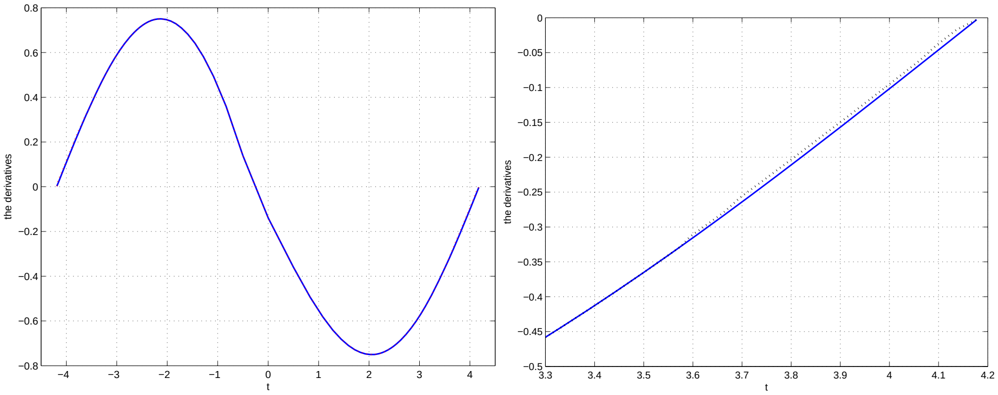
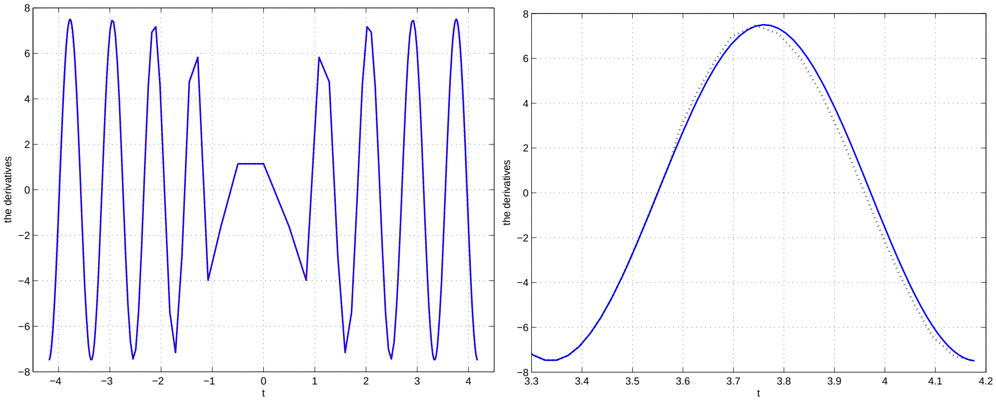


Figure 4.3: LEFT: The dynamic derivative $f^{\Delta}(t)$ and dynamic difference $f^F(t)$ on set T. RIGHT: A local image of the functions when $3.3 \leq t \leq 4.2$.

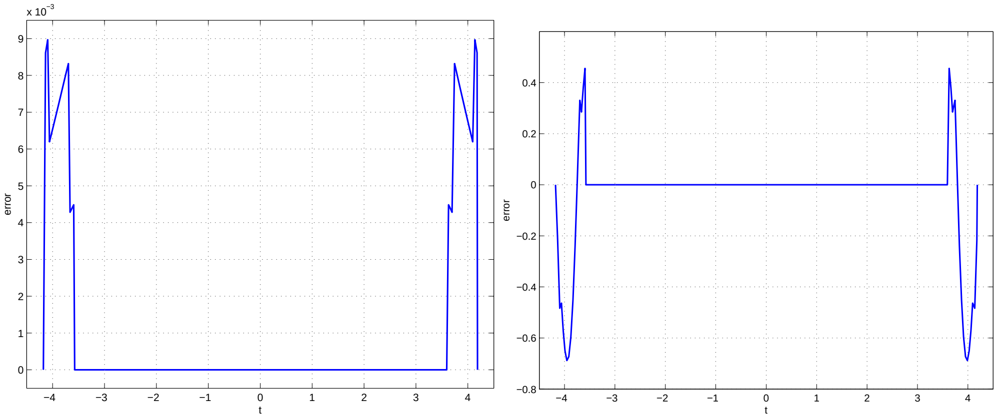


In the experiments, we consider the wave function

$$f(t) = \sin\left[\omega\left(t - \frac{b-a}{2}\right)\right], \tag{4.1}$$

where ω is the wave number involved [13, 18]. In the first case a low frequency with $\omega = 2\pi/(b-a)$ is used. The dynamic derivatives $f^{\Delta}(t)$ and $f^{F}(t)$ are plotted in Figure 4.2. The solid curve is for the dynamic derivative while the dotted curve is for the dynamic difference. The second frame in Figure 4.2 offers an enlarged image of the functions in the most turbulent area. The numerical error is significantly small compared with the magnitudes of the functions (Figure 4.4). In fact, most of the central part of T is overlapped with $\mathbb{T}$ based on the step size bounds. The maximal error appears in areas where steps used for $f^{F}(t)$ on T are significantly larger than the corresponding jump functions on $\mathbb{T}$. This can be clearly observed in Figure 4.4. Our second set of experiments are designed with a high frequency wave with $\omega = 20\pi/(b-a)$ in (4.1). The numerical results of $f^{\Delta}(t)$ and $f^{F}(t)$ are given in Figure 4.3, where the second frame is again for an enlarged picture. It is found that the approximation is excellent and acceptable, even with the relatively large irregular steps used near the two ends of T. The maximal relative error of $f^{F}(t)$ is less than $0.7/8 \approx 8.5\%$ in computations (Figure 4.4).

Figure 4.4: LEFT: Distributions of the computational error on T. A low frequency is used. RIGHT: Distributions of the computational error on T. A high frequency is used.



We now consider approximations of the dynamic derivative $f^{\Delta\Delta}(t)$ with function (4.1) on $\mathbb{T}$ and T. Figure 4.5 provides curves of the dynamic derivative (solid curve) and dynamic difference (dotted curve). It can be observed that the latter is overlapped with the former in most of the domains, except in areas near the two ends, where T is significantly different from $\mathbb{T}$. In the more detailed right frame, we may see that the maximal relative error of the numerical approximation is almost 25%. This suggests

Figure 4.5: LEFT: The dynamic derivative $f^{\Delta\Delta}(t)$ and dynamic difference $f^{FF}(t)$ on set T. A low frequency is used. The differences between the functions are hard to observe. RIGHT: A locally enlarged image of the functions as $3.3 \le t \le 4.2$. The functions become distinguishable.

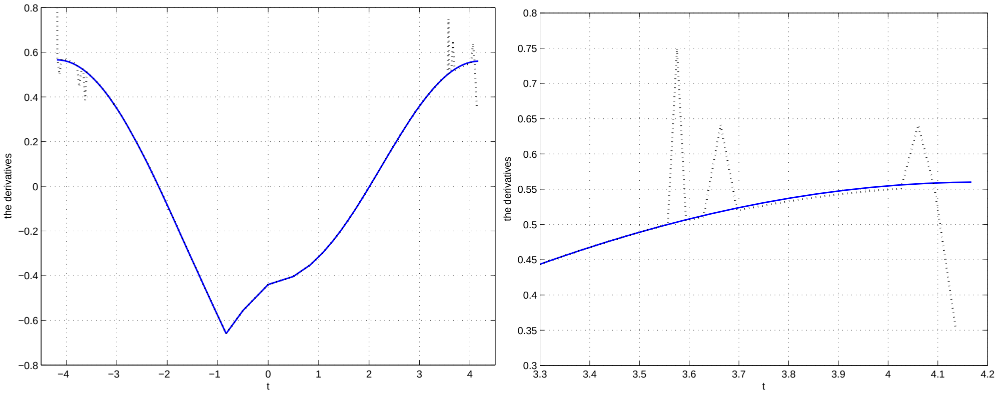
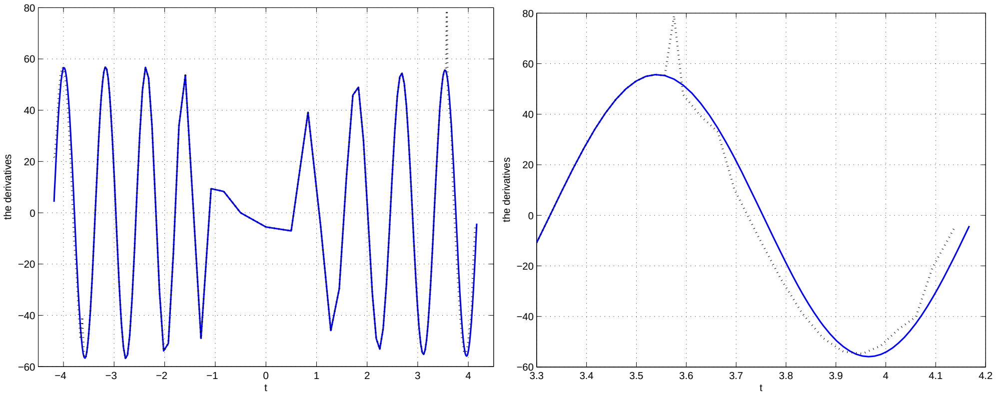


Figure 4.6: LEFT: The dynamic derivative $f^{\Delta\Delta}(t)$ and dynamic difference $f^{FF}(t)$ on set T. A high frequency is used. The difference between the functions are hard to observe. RIGHT: An locally enlarged image of the functions as $3.3 \le t \le 4.2$. The functions become distinguishable.

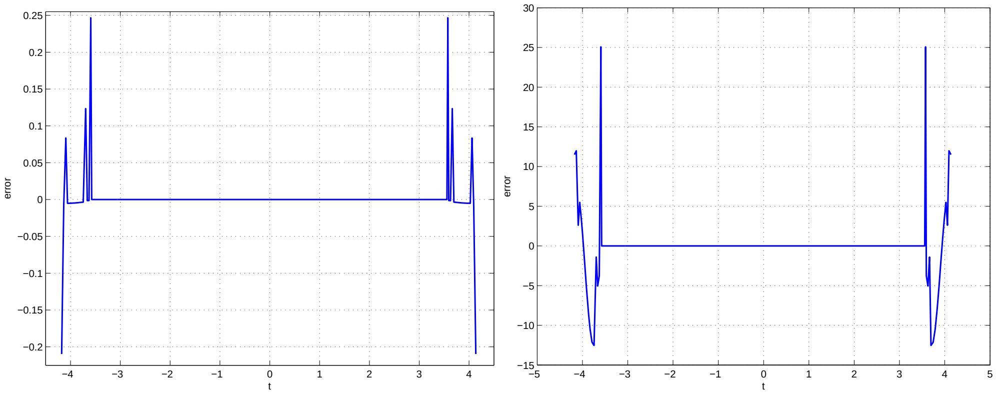


that approximations of the second order dynamic derivatives can be far more difficult than the first order dynamic derivatives [9, 15].

Corresponding curves via the high frequency wave function (4.1) are given in Figure 4.6. It is again found that the approximation works satisfactorily except in the areas near the two ends of $\mathbb{T}$, where irregular mesh points are used in T. Oscillations of the dynamic difference values indicate a relatively poor approximation on the mesh. The maximal relative error reaches about 25% in this scenario. The phenomenon observed supports the common practice for not using “nonsmooth” grids during adaptive computations [9, 13, 17].

Finally, in Figure 4.7, we present more precise numerical errors. Though the error curve associated with the high frequency function seems to be more violent, the maximal relative errors are still around 25% in both low and high frequency cases. The oscillations are more significant than that in the first order dynamic derivative approximations.

Figure 4.7: Numerical error on T. LEFT: A low frequency is used. RIGHT: A high frequency is used.



Experiments involving other dynamic derivatives, dynamic differences and testing functions are similar. All numerical experiments suggest that approximations of the second and higher order dynamic derivatives need to be extremely careful. The irregularity of the time scale structures may bring in tremendous amount of numerical uncertainties [9, 12, 16, 17]. Optimization procedures may need to be imposed to assure more effective approximations.

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