

## Symmetric Functions and Difference Equations with Asymptotically Period-two Solutions

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### Abstract

This paper introduces easily verified conditions which guarantee that all solutions to the equation  $y_n = f(y_{n-k}, y_{n-m})$ , with  $k, m \geq 1$  and  $\gcd(k, m) = 1$  are asymptotically periodic with period two. A recent result of Sun and Xi is employed. Several examples are included.

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## 1 Introduction

In this paper, we consider recursive equations of the form

$$y_n = f(y_{n-k}, y_{n-m}), \quad (1.1)$$

for  $n \geq 0$ , where  $k, m \geq 1$ ,  $\gcd(k, m) = 1$ ,  $s = \max\{k, m\}$  and  $y_{-s}, y_{-s+1}, \dots, y_{-1} \in (0, \infty)$ .

Recently, Sun and Xi [6] and Stević [4] proved the following interesting result regarding criteria for asymptotically two-periodic behavior of solutions to (1.1).

**Theorem 1.1.** *Suppose that  $\{y_i\}$  satisfies (1.1), and in addition*

(i)  $f \in C((0, \infty)^2, (a, \infty))$  with

$$a = \inf_{(u,v) \in (0,\infty)^2} f(u,v) \geq 0, \quad (1.2)$$

(ii)  $f(u, v)$  is increasing in  $u$  and decreasing in  $v$ ,

(iii) there exists a decreasing function  $g \in C((a, \infty), (a, \infty))$  such that

$$g(g(x)) = x, \text{ for } x > a, \quad (1.3)$$

$$x = f(x, g(x)), \text{ for } x > a, \quad (1.4)$$

and

$$\lim_{x \rightarrow a^+} g(x) = \infty \text{ and } \lim_{x \rightarrow \infty} g(x) = a. \quad (1.5)$$

Then, if  $k$  is even and  $m$  is odd, every positive solution to (1.1) converges to a (not necessarily prime) two-periodic solution. Otherwise, every solution converges to the unique equilibrium of (1.1).

For earlier related results and motivation, see [1, 2, 5].

*Remark 1.2.* Note that the self-inverse property in (1.3) gives that the range of  $g$  must comprise all of  $(a, \infty)$ . Hence under the assumption that  $g$  is decreasing, (1.5) is immediately satisfied and may be removed from the statement of Theorem 1.1.

## 2 Main Result and Examples

In this short note, we will give easily verified conditions for existence of a function  $g$  satisfying the requirements of Theorem 1.1. In particular, we will prove the following.

**Theorem 2.1.** *Suppose that  $\{y_i\}$  satisfies (1.1) with  $f(u, v) = h(u, v)/v$  for some function  $h$  where*

(i)  $h \in C((0, \infty)^2, (0, \infty))$  is such that  $h(u, v)$  is symmetric in  $u$  and  $v$  and increasing in  $u$ ,

(ii) the function  $f$  is decreasing in  $v$ ,

(iii) with  $a$  as in (1.2), for all  $v > a$ , there exist  $C_v$  and  $D_v$  (possibly infinite) such that

$$\lim_{u \rightarrow a^+} f(u, v)/u = C_v > 1 \text{ and } \lim_{u \rightarrow \infty} f(u, v)/u = D_v < 1. \quad (2.1)$$

Then, there exists a continuous function  $g$  satisfying (1.3), (1.4) and (1.5). Hence by Theorem 1.1, if  $k$  is even and  $m$  is odd, every positive solution to (1.1) converges to a (not necessarily prime) two-periodic solution, and otherwise, every positive solution converges to the unique equilibrium of (1.1).

Now, note that under assumptions (1) and (2) in Theorem 2.1, we have that, for fixed  $v > a$ ,

$$\frac{f(u, v)}{u} = \frac{h(u, v)}{uv} = \frac{h(v, u)}{u} \frac{1}{v} = \frac{f(v, u)}{v} \quad (2.2)$$

is decreasing in  $u$  and hence the (possibly infinite) limits in (2.1) exist. A key point here is that there is no need to have a closed form for  $g$  to verify the hypotheses of Theorem 1.1. In fact, finding such a closed form may not be very practical in practice (see for instance Example 2.6 below).

Before turning to a proof of Theorem 2.1, we give several examples of difference equations satisfying the requirements of the theorem.

**Example 2.2.** Consider the equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}}. \quad (2.3)$$

Here we have  $h(u, v) = u + v$ ,  $f(u, v) = 1 + u/v$ ,  $a = 1$ , and for  $v > 1$ ,

$$\lim_{u \rightarrow 1^+} \frac{f(u, v)}{u} = \lim_{u \rightarrow 1^+} \frac{1}{u} + \frac{1}{v} = 1 + \frac{1}{v} > 1 \quad (2.4)$$

and

$$\lim_{u \rightarrow \infty} \frac{f(u, v)}{u} = \lim_{u \rightarrow \infty} \frac{1}{u} + \frac{1}{v} = \frac{1}{v} < 1. \quad (2.5)$$

Hence Theorem 2.1 is applicable and all positive solutions are asymptotically two-periodic whenever  $k$  is even and  $m$  is odd, and otherwise all positive solutions converge to the unique equilibrium. See [1–3] and the references therein for further discussion of Eq. (2.3).

**Example 2.3.** Consider the equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}} + \sqrt{\frac{x_{n-k}}{x_{n-m}}}. \quad (2.6)$$

Here we have  $h(u, v) = u + v + \sqrt{uv}$ ,  $f(u, v) = 1 + u/v + \sqrt{u/v}$ ,  $a = 1$ , and for  $v > 1$ ,

$$\lim_{u \rightarrow 1^+} \frac{f(u, v)}{u} = \lim_{u \rightarrow 1^+} \frac{1}{u} + \frac{1}{v} + \frac{1}{\sqrt{uv}} = 1 + \frac{1}{v} + \frac{1}{\sqrt{v}} > 1 \quad (2.7)$$

and

$$\lim_{u \rightarrow \infty} \frac{f(u, v)}{u} = \lim_{u \rightarrow \infty} \frac{1}{u} + \frac{1}{v} + \frac{1}{\sqrt{uv}} = \frac{1}{v} < 1. \quad (2.8)$$

By Theorem 2.1, we have the required asymptotic two-periodic behavior.

**Example 2.4.** Consider the equation

$$x_n = \frac{1}{x_{n-m}} \left( \frac{x_{n-k}}{x_{n-k} + 1} + \frac{x_{n-m}}{x_{n-m} + 1} \right). \quad (2.9)$$

Here we have  $h(u, v) = u/(u + 1) + v/(v + 1)$ ,  $f(u, v) = (u/(u + 1) + v/(v + 1))/v$ ,  $a = 0$ , and for  $v > 0$ ,

$$\lim_{u \rightarrow 0^+} \frac{f(u, v)}{u} = \lim_{u \rightarrow 0^+} \frac{1}{v(u + 1)} + \frac{1}{u(v + 1)} = \infty \quad (2.10)$$

and

$$\lim_{u \rightarrow \infty} \frac{f(u, v)}{u} = \lim_{u \rightarrow \infty} \frac{1}{v(u + 1)} + \frac{1}{u(v + 1)} = 0 < 1. \quad (2.11)$$

By Theorem 2.1, we have the required asymptotic two-periodic behavior.

**Example 2.5.** Consider the equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}} + \frac{\log(x_{n-k}x_{n-m})}{x_{n-m}}. \quad (2.12)$$

Here we have  $h(u, v) = u + v + \log(uv)$ ,  $f(u, v) = 1 + u/v + \log(uv)/v$ ,  $a = 1$ , and for  $v > 1$ ,

$$\lim_{u \rightarrow 1^+} \frac{f(u, v)}{u} = \lim_{u \rightarrow 1^+} \frac{1}{u} + \frac{1}{v} + \frac{\log(uv)}{uv} = 1 + \frac{1}{v} + \frac{\log(v)}{v} > 1 \quad (2.13)$$

and

$$\lim_{u \rightarrow \infty} \frac{f(u, v)}{u} = \lim_{u \rightarrow \infty} \frac{1}{u} + \frac{1}{v} + \frac{\log(uv)}{uv} = \frac{1}{v} < 1. \quad (2.14)$$

By Theorem 2.1, we have the required asymptotic two-periodic behavior.

**Example 2.6.** Consider the equation

$$x_n = \frac{x_{n-k}^\alpha + x_{n-k}^\beta + x_{n-m}^\alpha + x_{n-m}^\beta}{x_{n-m}}. \quad (2.15)$$

Here, for  $0 < \alpha, \beta < 1$ , we have  $h(u, v) = u^\alpha + u^\beta + v^\alpha + v^\beta$ ,  $f(u, v) = u^\alpha/v + u^\beta/v + v^{-(1-\alpha)} + v^{-(1-\beta)}$ ,  $a = 0$ , and for  $v > 0$ ,

$$\lim_{u \rightarrow 0^+} \frac{f(u, v)}{u} = \lim_{u \rightarrow 0^+} \frac{1}{vu^{1-\alpha}} + \frac{1}{vu^{1-\beta}} + \frac{1}{uv^{1-\alpha}} + \frac{1}{uv^{1-\beta}} = \infty \quad (2.16)$$

and

$$\lim_{u \rightarrow \infty} \frac{f(u, v)}{u} = \lim_{u \rightarrow \infty} \frac{1}{vu^{1-\alpha}} + \frac{1}{vu^{1-\beta}} + \frac{1}{uv^{1-\alpha}} + \frac{1}{uv^{1-\beta}} = 0 < 1. \quad (2.17)$$

By Theorem 2.1, we have the required asymptotic two-periodic behavior.

*Remark 2.7.* To see that the requirement in (3) cannot be removed, consider the equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}} + x_{n-k}. \quad (2.18)$$

Here, we have  $h(u, v) = u + v + uv$ ,  $f(u, v) = 1 + u/v + u$  and  $a = 1$ , yet the equation possesses no equilibrium. Indeed, for  $v > 1$ ,

$$\lim_{u \rightarrow 1^+} \frac{f(u, v)}{u} = \lim_{u \rightarrow 1^+} 1 + \frac{1}{u} + \frac{1}{v} = 2 + \frac{1}{v} > 1, \quad (2.19)$$

but

$$\lim_{u \rightarrow \infty} \frac{f(u, v)}{u} = \lim_{u \rightarrow \infty} 1 + \frac{1}{u} + \frac{1}{v} = 1 + \frac{1}{v} > 1. \quad (2.20)$$

Condition (3) is not satisfied and Theorem 2.1 is not applicable.

### 3 Proof of the Main Result

We now turn to a proof of Theorem 2.1.

*Proof of Theorem 2.1.* By Theorem 1.1 and Remark 1.2, we need only show that there exists a decreasing continuous function  $g$  which satisfies (1.3) and (1.4). To that end, note that by (2.2),  $f(u, v)/u$  is decreasing in  $u$  and hence for fixed  $v > a$ , by (3), there exists a unique  $u = g(v)$ , say, which satisfies

$$f(g(v), v)/g(v) = 1. \quad (3.1)$$

To see that the function  $g$  is decreasing, note that for  $x > a$  and  $\epsilon > 0$ , by (2) and the definition of  $g$ ,

$$\frac{f(g(x), x + \epsilon)}{g(x)} < \frac{f(g(x), x)}{g(x)} = 1 = \frac{f(g(x + \epsilon), x + \epsilon)}{g(x + \epsilon)}, \quad (3.2)$$

and hence since  $f(u, v)/u$  is decreasing in  $u$ ,  $g(x + \epsilon) < g(x)$ , as required.

Now, note that for  $x > a$ , by the definition of  $g$  and the assumptions on  $f$  and  $h$ , we have

$$f(x, g(x)) = \frac{h(x, g(x))}{g(x)} = \frac{h(g(x), x)}{g(x)} = \frac{f(g(x), x)x}{g(x)} = x, \quad (3.3)$$

and (1.4) is satisfied. Since  $u = g(g(x))$  is the unique value satisfying  $f(u, g(x)) = u$ , (3.3) gives that (1.3) also holds.

The continuity of  $g$  follows from its monotonicity and the fact that the range of  $g$  is  $(a, \infty)$  (see Remark 1.2), and the theorem is proven.  $\square$

*Remark 3.1.* Expanding on ideas in [1], Stević [4,5] showed that similar asymptotically two-periodic behavior can occur for multivariable functions  $f$  in (1.1), with varying delays. The interested reader may verify that the ideas introduced here are applicable in that case as well.

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