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New Retarded Discrete Inequalities with Applications

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Abstract

Some new nonlinear discrete inequalities of Gronwall type for retarded functions are established. These inequalities can be used as basic tools in the study of certain classes of functional difference equations as well as discrete delay equations.

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1 Introduction

The celebrated Gronwall inequality [11] states that if u and f are nonnegative continuous functions on the interval $[a,b]$ satisfying

$$u(t) \le c + \int_a^t f(s)u(s)ds, \quad t \in [a,b],$$

for some constant $c \ge 0$, then

$$u(t) \le c \exp\left(\int_a^t f(s)ds\right), \quad t \in [a,b]. \tag{1.1}$$

Since the inequality (1.1) provides an explicit bound to the unknown function u and hence furnishes a handy tool in the study of solutions of differential equations. Because of its fundamental importance, several generalizations and analogous results of the inequality (1.1) have been established over years [1–6, 9, 14, 16, 19]. Such generalizations are, in general, referred to as Gronwall type inequalities [1,2,6,8,13,19]. These inequalities provide necessary tools in the study of the theory of differential equations, integral equations, and inequalities of various types. Also in 1969, Sugiyama [20] proved the following most precise and complete discrete analogue of the well known Gronwall integral inequality [11]: Let $u(t)$ and $b(t)$ be nonnegative functions defined on $\mathbb{N}_0$ and $c \ge 0$ be a constant, where $\mathbb{N}_0$ denotes the set of nonnegative integers. If

$$u(t) \le c + \sum_{s=0}^{t-1} b(s)u(s)$$

for $t \in \mathbb{N}_0$, then

$$u(t) \le c \prod_{s=0}^{t-1} [1 + b(s)] \le c \exp\left(\sum_{s=0}^{t-1} b(s)\right)$$

for $t \in \mathbb{N}_0$. This result has evoked considerable interest in the literature and many generalizations and extensions of this inequality have been established, which find applications in the study of various classes of finite difference equations and sum-difference equations.

In addition, many authors [1, 2, 7, 10, 12, 15, 17, 19] have established several other very useful Gronwall-like discrete inequalities. Among these inequalities, the following one (Theorem 1.1 below) due to Pachpatte [15] needs specific mention. It is useful in the study of boundedness of certain difference equations.

Theorem 1.1 (Pachpatte [15]). *Let $u(t), a(t), b(t)$ and $p(t)$ be nonnegative functions defined on $\mathbb{N}_0$ and*

$$u(t) \le a(t) + p(t) \sum_{s=0}^{t-1} b(s)u(s)$$

for $t \in \mathbb{N}_0$*, then*

$$u(t) \le a(t) + p(t) \sum_{s=0}^{t-1} a(s)b(s) \prod_{\sigma=s+1}^{t-1} [1 + b(\sigma)p(\sigma)]$$

for $t \in \mathbb{N}_0$.

Pachpatte's inequality prompted researchers to devote considerable time for its generalizations and consequent applications [1, 2, 7, 10, 19]. For instance, Dragomir established the following generalization (Theorem 1.2 below) of Pachpatte's inequality in the process of establishing a connection between extended and used considerably in various contexts [10].

Theorem 1.2 (Dragomir [10]). *Let* $u(t), a(t), b(t)$ *be nonnegative functions defined for* $t \in \mathbb{N}_0$. *Let the function* $L : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies the condition*

$$0 \le L(t, x) - L(t, y) \le k(t, y)(x - y)$$

for $t \in \mathbb{N}_0$ *and* $x \ge y \ge 0$*, where* $k(t, r)$ *is a nonnegative function defined for* $t \in \mathbb{N}_0$ *and* $r \in \mathbb{R}_+$. *If*

$$u(t) \le a(t) + b(t) \sum_{s=0}^{t-1} L(s, u(s))$$

for $t \in \mathbb{N}_0$*, then*

$$u(t) \le a(t) + b(t) \sum_{s=0}^{t-1} L(s, a(s)) \prod_{\sigma=s+1}^{t-1} [1 + k(\sigma, a(\sigma))b(\sigma)]$$

for $t \in \mathbb{N}_0$.

More recently, Pachpatte established following useful inequality (Theorem 1.3 below) inspired by the discrete version of Bihari's inequality [4], which is handy in the study of the global existence of solutions to certain finite difference equations and sum-difference equations.

Theorem 1.3 (Pachpatte [18]). *Let* $u(t), a(t), b(t)$ *be nonnegative functions defined for* $t \in \mathbb{N}_0$ *and* c *be a nonnegative constant. Let* $g(u)$ *be a continuous nondecreasing function defined on* $\mathbb{R}_+$ *with* $g(u) > 0$ *for* $u > 0$. *If*

$$u^2(t) \le c^2 + 2 \sum_{s=0}^{t-1} [a(s)u(s)g(u(s)) + b(s)u(s)]$$

for $t \in \mathbb{N}_0$*, then for* $0 \le t \le t_1, t, t_1 \in \mathbb{N}_0$,

$$u(t) \le \Omega^{-1}\left[\Omega(p(t)) + \sum_{s=0}^{t-1} a(s)\right],$$

where

$$p(t) = c + \sum_{s=0}^{t-1} b(s)$$

for $t \in \mathbb{N}_0$,

$$\Omega(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \quad r > 0,$$

$r_0 > 0$ *is arbitrary,* Ω^{-1} *is a inverse function of* Ω *and* $t_1 \in \mathbb{N}_0$ *be chosen so that*

$$\Omega(p(t)) + \sum_{s=0}^{t-1} a(s) \in \mathrm{Dom}(\Omega^{-1}).$$

for all $t \in \mathbb{N}_0$ *such that* $0 \le t \le t_1$.

The main aim of the present paper is to establish some nonlinear retarded inequalities which extend the foregoing theorems. We also illustrate the use/application of these inequalities.

2 Main Results

For discrete inequalities it is customary that the functions occurring in them are defined on countable sets, which can, without loss of generality, be assumed to be subsets of the set $\mathbb{Z}$ of integers. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+ = [0, \infty)$ be the given subset of $\mathbb{R}$. Let $\alpha, \beta \in \mathbb{R}, \alpha \le \beta$, $\mathbb{Z}_\alpha = \{n \in \mathbb{Z} : n \ge \alpha\}$, $\mathbb{Z}_{[\alpha,\beta]} = \{n \in \mathbb{Z} : \alpha \le n \le \beta\}$. Let $\sum_{j=\alpha}^{\beta} x(j)$ and $\prod_{j=\alpha}^{\beta} x(j)$ be the sum and the product of $x(j), j \in \mathbb{Z}_{[\alpha,\beta]}$, respectively and assume that $\sum_{j=\alpha}^{\alpha-1} x(j) = 0$, $\prod_{j=\alpha}^{\alpha-1} x(j) = 1$. Denote by $C^i(M, N)$ the class of all i–times continuously differentiable functions defined on the set M to the set N for $i = 1, 2, \cdots$ and $C^0(M, N) = C(M, N)$. Further, denote by $\mathcal{F}(M, N)$ the collection of functions defined on the set M to the set N. As usual, let u be a real-valued function on $\mathbb{Z}_{[\alpha,\beta]}$ and the difference operator $\triangle$ on u be defined as

$$\triangle u(n) = u(n+1) - u(n), \quad n \in \mathbb{Z}_{[\alpha,\beta-1]}.$$

Also assume that all the sums and products involved throughout the discussion exist on the respective domains of their definitions.

Theorem 2.1. *Let* $b, f_i, g_i \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_+)$, $i = 1, \ldots, n$ *with* b *nondecreasing and let* $\sigma \in \mathcal{F}(\mathbb{Z}_0, \mathbb{Z})$ *be nondecreasing with* $\sigma(t) \le t$ *and* $-\infty < a = \inf\{\sigma(s) : s \in \mathbb{Z}_0\}$. *Suppose that* $q \ge 0$ *is a constant,* $\varphi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *is an increasing function with*

$\varphi(\infty)=\infty$ *on* $\mathbb{R}_+$*, and* $\psi(u)$ *is a nondecreasing continuous function for* $u\in\mathbb{R}_+$ *with* $\psi(u)>0$ *for* $u>0$*. If* $u\in\mathcal{F}(\mathbb{Z}_a,\mathbb{R}_+)$ *and*

$$\varphi(u(n))\leq b(n)+\sum_{i=1}^{l}\sum_{s=0}^{n-1}u^q(\sigma(s))[f_i(s)\psi(u(\sigma(s)))+g_i(s)]$$

for $t\in I$*, then*

$$u(n)\leq\varphi^{-1}\bigg\{G^{-1}\bigg[\Omega^{-1}\bigg(\Omega\bigg(G(b(n))+\sum_{s=0}^{n-1}\sum_{i=1}^{l}g_i(s)\bigg)+\sum_{s=0}^{n-1}\sum_{i=1}^{l}f_i(s)\bigg)\bigg]\bigg\}\tag{2.1}$$

for $n\in\mathbb{Z}_{[0,\alpha]}$*, where*

$$G(r)=\int_{r_0}^{r}\frac{ds}{[\varphi^{-1}(s)]^q},\quad r\geq r_0>0,$$
$$\Omega(r)=\int_{r_0}^{r}\frac{ds}{\psi[\varphi^{-1}(G^{-1}(s))]},\quad r\geq r_0>0,$$

G^{-1},Ω^{-1} *denote the inverse functions of* G,Ω*, respectively and* $\alpha\geq 0$ *is so chosen that*

$$\Omega\bigg(G(b(n))+\sum_{s=0}^{n-1}\sum_{i=1}^{l}g_i(s)\bigg)+\sum_{s=0}^{n-1}\sum_{i=1}^{l}f_i(s)\in\mathrm{Dom}(\Omega^{-1}).$$

Proof. Let $\epsilon>0$ and $N\in\mathbb{Z}_0$. Define a function $z:\mathbb{Z}_{[0,N]}\to\mathbb{R}_0$ by

$$z(n)=\epsilon+b(N)+\sum_{i=1}^{l}\sum_{s=0}^{n-1}u^q(\sigma(s))[f_i(s)\psi(u(\sigma(s)))+g_i(s)].$$

Clearly, $z(n)$ is nondecreasing, $u(n)\leq\varphi^{-1}(z(n))$ for $n\in\mathbb{Z}_{[0,N]}$ and $z(0)=\epsilon+b(N)$. We get

$$\begin{aligned}\triangle z(n)&=\sum_{i=1}^{l}u^q(\sigma(n))[f_i(n)\psi(u(\sigma(n)))+g_i(n)]\\&\leq[\varphi^{-1}(z(n))]^q\sum_{i=1}^{l}[f_i(n)\psi(\varphi^{-1}(z(n)))+g_i(n)].\end{aligned}$$

Using the monotonicity of φ^{-1} and z, we deduce

$$[\varphi^{-1}(z(n))]^q\geq[\varphi^{-1}(z(0))]^q=[\varphi^{-1}(\epsilon+b(N))]^q>0.$$

That is

$$\frac{\triangle z(t)}{[\varphi^{-1}(z(n))]^q}\leq\sum_{i=1}^{l}[f_i(n)\psi(\varphi^{-1}(z(n)))+g_i(n)].\tag{2.2}$$

On the other hand, by the mean value theorem, we have

$$\triangle G(z(n)) = G(z(n+1)) - G(z(n)) = G'(\xi)\triangle z(n) \tag{2.3}$$

for some $\xi \in [z(n), z(n+1)]$. From (2.2), (2.3) and using the function G, we obtain

$$\triangle G(z(n)) \leq \sum_{i=1}^{l}[f_i(n)\psi(\varphi^{-1}(z(n))) + g_i(n)]. \tag{2.4}$$

Setting $n = s$ in the inequality (2.4), summing up, we get

$$\sum_{s=0}^{n-1} \triangle G(z(s)) \leq \sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s)\psi(\varphi^{-1}(z(s))) + g_i(s)]. \tag{2.5}$$

From the inequality (2.5), we observe that

$$\begin{aligned} G(z(n)) &\leq G(\epsilon + b(N)) + \sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s)\psi(\varphi^{-1}(z(s))) + g_i(s)] \\ &\leq G(\epsilon + b(N)) + \sum_{s=0}^{N-1}\sum_{i=1}^{l} g_i(s) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} f_i(s)\psi(\varphi^{-1}(z(s))) \end{aligned} \tag{2.6}$$

for all $n \in \mathbb{N}_{[0,N]}$. Now define a function $v(n)$ by the right-hand side of (2.6). Clearly, $v(n)$ is nondecreasing, $z(n) \leq G^{-1}(v(n))$ for $n \in \mathbb{N}_{[0,N]}$ and

$$v(0) = G(\epsilon + b(N)) + \sum_{s=0}^{N-1}\sum_{i=1}^{l} g_i(s).$$

Therefore, for any $t \in \mathbb{N}_{[0,N-1]}$, we get

$$\begin{aligned} \triangle v(t) &= \sum_{i=1}^{l} f_i(t)\psi(\varphi^{-1}(z(t))) \\ &\leq \psi(\varphi^{-1}(G^{-1}(v(t)))) \sum_{i=1}^{l} f_i(t). \end{aligned}$$

Using the monotonicity of $\psi, \varphi^{-1}, G^{-1}$ and v, we deduce

$$\frac{\triangle v(t)}{\psi(\varphi^{-1}(G^{-1}(v(t))))} \leq \sum_{i=1}^{l} f_i(t). \tag{2.7}$$

On the other hand, by the mean value theorem, we have

$$\triangle \Omega(v(t)) = \Omega(v(t+1)) - \Omega(v(t)) = \Omega'(\xi)\triangle v(t) \tag{2.8}$$

for some $\xi \in [v(t), v(t+1)]$. From (2.7), (2.8) and using the function Ω, we obtain

$$\triangle\Omega(v(t)) \le \sum_{i=1}^{l} f_i(t). \tag{2.9}$$

Setting $t = s$ in the inequality (2.9), summing it from 0 to $n-1$, we obtain

$$\Omega(v(n)) \le \Omega\Bigg(G(\epsilon + b(N)) + \sum_{s=0}^{N-1}\sum_{i=1}^{l} g_i(s)\Bigg) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} f_i(s). \tag{2.10}$$

From the inequalities (2.6) and (2.10), we conclude that

$$z(n) \le G^{-1}\Bigg[\Omega^{-1}\Bigg(\Omega\Bigg(G(\epsilon + b(N)) + \sum_{s=0}^{N-1}\sum_{i=1}^{l} g_i(s)\Bigg) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} f_i(s)\Bigg)\Bigg]$$

for $n \in \mathbb{N}_{[0,N]}$. Now, letting $\varepsilon \to 0$, a combination of $u(n) \le \varphi^{-1}(z(n))$ and the last inequality produces the required inequality in (2.1) for $N = n$, since $N \in \mathbb{N}_{[0,\alpha]}$ was arbitrary. This completes the proof. $\square$

For the special case $\varphi(u) = u^p$, $G(r) = r^{(p-q)/p}$ ($p > q \ge 0$ is a constant), Theorem 2.1 gives the following retarded integral inequality for nonlinear functions.

Corollary 2.2. *Let $b, f_i, g_i \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_+), i = 1, \ldots, n$ with b nondecreasing and let $\sigma \in \mathcal{F}(\mathbb{Z}_0, \mathbb{Z})$ be nondecreasing with $\sigma(t) \le t$ and $-\infty < a = \inf\{\sigma(s) : s \in \mathbb{Z}_0\}$. Suppose that p and q are constants with $p > q \ge 0$, and $\psi(u)$ is a nondecreasing continuous function for $u \in \mathbb{R}_+$ with $\psi(u) > 0$ for $u > 0$. If $u \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_+)$ and*

$$u^p(n) \le b(n) + \sum_{i=1}^{l}\sum_{s=0}^{n-1} u^q(\sigma(s))[f_i(s)\psi(u(\sigma(s))) + g_i(s)]$$

for $t \in I$, then

$$u(n) \le \Bigg\{\Omega_1^{-1}\Bigg[\Omega_1\Bigg([b(n)]^{\frac{p-q}{p}} + \frac{p-q}{p}\sum_{s=0}^{n-1}\sum_{i=1}^{l} g_i(s)\Bigg) + \frac{p-q}{p}\sum_{s=0}^{n-1}\sum_{i=1}^{l} f_i(s)\Bigg]\Bigg\}^{\frac{1}{p-q}}$$

for $n \in \mathbb{Z}_{[0,\alpha_1]}$, where

$$\Omega_1(r) = \int_{r_0}^{r} \frac{ds}{\psi[s^{\frac{1}{p-q}}]}, \quad r \ge r_0 > 0,$$

Ω_1^{-1} denotes the inverse function of Ω_1 and $\alpha_1 \ge 0$ is so chosen that

$$\Omega_1\Bigg([b(n)]^{\frac{p-q}{p}} + \frac{p-q}{p}\sum_{s=0}^{n-1}\sum_{i=1}^{l} g_i(s)\Bigg) + \frac{p-q}{p}\sum_{s=0}^{n-1}\sum_{i=1}^{l} f_i(s) \in \mathrm{Dom}(\Omega_1^{-1}).$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 2.1 with suitable modification. We omit the details here. □

Remark 2.3. When $p = 2$, $q = 1$, $b(n) = c^2$, $\sigma(s) = s$ and $i = 1$, from Corollary 2.2, we derive Theorem 1.3.

Theorem 2.1 can easily be applied to generate other useful nonlinear integral inequalities in more general situations. For example, we have the following result.

Theorem 2.4. *Let $b, f_i, g_i \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_+), i = 1, \dots, n$ with b nondecreasing and let $\sigma \in \mathcal{F}(\mathbb{Z}_0, \mathbb{Z})$ be nondecreasing with $\sigma(t) \le t$ and $-\infty < a = \inf\{\sigma(s) : s \in \mathbb{Z}_0\}$. Suppose that $q \ge 0$ is constant, $\varphi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing function with $\varphi(\infty) = \infty$ on $\mathbb{R}_+$, and $\psi_j(u), j = 1, 2$ are nondecreasing continuous functions for $u \in \mathbb{R}_+$ with $\psi_j(u) > 0$ for $u > 0$. If $u \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_1)$ and*

$$\varphi(u(n)) \le b(n) + \sum_{i=1}^{l} \sum_{s=0}^{n-1} u^q(\sigma(s))[f_i(s)\psi_1(u(\sigma(s))) + g_i(s)\psi_2(\log u(\sigma(s)))] \quad (2.11)$$

for $n \in \mathbb{Z}_0$, then

(*i*) *for the case $\psi_1(u) \ge \psi_2(\log(u))$,*

$$u(n) \le \varphi^{-1}\left\{G^{-1}\left[\Omega_2^{-1}\left(\Omega_2[G(b(n))] + \sum_{s=0}^{n-1} \sum_{i=1}^{l} [f_i(s) + g_i(s)]\right)\right]\right\} \quad (2.12)$$

for $n \in \mathbb{Z}_{[0,\alpha_2]}$, and

(*ii*) *for the case $\psi_1(u) < \psi_2(\log(u))$,*

$$u(n) \le \varphi^{-1}\left\{G^{-1}\left[\Omega_3^{-1}\left(\Omega_3[G(b(n))] + \sum_{s=0}^{n-1} \sum_{i=1}^{l} [f_i(s) + g_i(s)]\right)\right]\right\} \quad (2.13)$$

for $n \in \mathbb{Z}_{[0,\alpha_3]}$,

where

$$\Omega_m(r) = \int_{r_0}^{r} \frac{ds}{\psi_{m-1}(\varphi^{-1}(G^{-1}(s)))}, \quad r \ge r_0 > 0,$$

$G^{-1}, \Omega_m^{-1}, m = 2, 3$, denote the inverse functions of $G, \Omega_m, m = 2, 3$, the function $G(t)$ is as defined in Theorem 2.1 for $t > 0$, and $\alpha_m \ge 0, m = 2, 3$, are chosen so that

$$\Omega_m[G(b(n))] + \sum_{s=0}^{n-1} \sum_{i=1}^{l} [f_i(s) + g_i(s)] \in \mathrm{Dom}(\Omega_m^{-1}).$$

Proof. Let $\epsilon > 0$ and $N \in \mathbb{Z}_0$. Define a function $z : \mathbb{Z}_{[0,N]} \to \mathbb{R}_0$ by

$$z(n) = \epsilon + b(N) + \sum_{i=1}^{l} \sum_{s=0}^{n-1} u^q(\sigma(s))[f_i(s)\psi_1(u(\sigma(s))) + g_i(s)\psi_2(\log u(\sigma(s)))]. \quad (2.14)$$

Clearly, $z(n)$ is nondecreasing, $u(n) \leq \varphi^{-1}(z(n))$ for $n \in \mathbb{Z}_{[0,N]}$ and $z(0) = \epsilon + b(N)$. We get

$$\begin{aligned} \triangle z(n) &= \sum_{i=1}^{l} u^q(\sigma(n))[f_i(n)\psi_1(u(\sigma(n))) + g_i(n)\psi_2(\log u(\sigma(n)))] \\ &\leq [\varphi^{-1}(z(n))]^q \sum_{i=1}^{l} [f_i(s)\psi_1(\varphi^{-1}(z(n))) + g_i(s)\psi_2(\log \varphi^{-1}(z(n)))]. \end{aligned}$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\frac{\triangle z(t)}{[\varphi^{-1}(z(n))]^q} \leq \sum_{i=1}^{l} [f_i(s)\psi_1(\varphi^{-1}(z(n))) + g_i(s)\psi_2(\log \varphi^{-1}(z(n)))]. \quad (2.15)$$

On the other hand, by the mean value theorem, we have

$$\triangle G(z(n)) = G(z(n+1)) - G(z(n)) = G'(\xi)\triangle z(n) \quad (2.16)$$

for some $\xi \in [z(n), z(n+1)]$. From (2.15), (2.16) and using the function G, we obtain

$$\triangle G(z(n)) \leq \sum_{i=1}^{l} [f_i(s)\psi_1(\varphi^{-1}(z(n))) + g_i(s)\psi_2(\log \varphi^{-1}(z(n)))]. \quad (2.17)$$

Setting $n = s$ in the inequality (2.17), summing up, we get

$$G(z(n)) \leq G(\epsilon + b(N)) + \sum_{s=0}^{n-1} \sum_{i=1}^{l} [f_i(s)\psi_1(\varphi^{-1}(z(s))) + g_i(s)\psi_2(\log \varphi^{-1}(z(s)))] \quad (2.18)$$

for all $n \in \mathbb{N}_{[0,N]}$.

When $\psi_1(u) \geq \psi_2(\log(u))$, from the inequality (2.18), we find

$$G(z(n)) \leq G(\epsilon + b(N)) + \sum_{s=0}^{n-1} \sum_{i=1}^{l} [f_i(s) + g_i(s)]\psi_1(\varphi^{-1}(z(s))). \quad (2.19)$$

Now, define a function $v(n)$ by the right-hand side of (2.19). Clearly, $v(n)$ is nondecreasing, $z(n) \leq G^{-1}(v(n))$ for $n \in \mathbb{N}_{[0,N]}$ and $v(0) = G(\epsilon + b(N))$. Therefore, for any $t \in \mathbb{N}_{[0,N-1]}$, we get

$$\begin{aligned} \triangle v(t) &= \sum_{i=1}^{l} [f_i(t) + g_i(t)] \psi_1(\varphi^{-1}(z(t))) \\ &\leq \psi_1(\varphi^{-1}(G^{-1}(v(t)))) \sum_{i=1}^{l} [f_i(t) + g_i(t)]. \end{aligned}$$

Using the monotonicity of $\psi_1, \varphi^{-1}, G^{-1}$ and v, we deduce

$$\frac{\triangle v(t)}{\psi_1(\varphi^{-1}(G^{-1}(v(t))))} \leq \sum_{i=1}^{l} [f_i(t) + g_i(t)]. \tag{2.20}$$

On the other hand, by the mean value theorem, we have

$$\triangle \Omega_2(v(t)) = \Omega_2(v(t+1)) - \Omega_2(v(t)) = \Omega_2'(\xi) \triangle v(t) \tag{2.21}$$

for some $\xi \in [v(t), v(t+1)]$. From (2.20), (2.21) and using the function Ω_2, we obtain

$$\triangle \Omega_2(v(t)) \leq \sum_{i=1}^{l} [f_i(t) + g_i(t)]. \tag{2.22}$$

Setting $t = s$ in the inequality (2.22), summing it from 0 to $n-1$, we obtain

$$\Omega_2(v(n)) \leq \Omega_2\Big(G(\epsilon + b(N))\Big) + \sum_{s=0}^{n-1} \sum_{i=1}^{l} [f_i(s) + g_i(s)]. \tag{2.23}$$

From the inequalities (2.19) and (2.23), we conclude that

$$z(n) \leq G^{-1}\left[\Omega_2^{-1}\left(\Omega_2[G(\epsilon + b(N))] + \sum_{s=0}^{n-1} \sum_{i=1}^{l} [f_i(s) + g_i(s)]\right)\right] \tag{2.24}$$

for $n \in \mathbb{N}_{[0,N]}$.

When $\psi_1(u) < \psi_1(\log(u))$, from the inequality (2.18), we find

$$G(z(n)) \leq G(\epsilon + b(N)) + \sum_{s=0}^{n-1} \sum_{i=1}^{l} [f_i(s) + g_i(s)] \psi_2(\varphi^{-1}(z(s)))$$

Now, by a suitable application of the process of obtaining (2.24), we conclude that

$$z(n) \leq G^{-1}\left[\Omega_3^{-1}\left(\Omega_3[G(\epsilon + b(N))] + \sum_{s=0}^{n-1} \sum_{i=1}^{l} [f_i(s) + g_i(s)]\right)\right] \tag{2.25}$$

for $n \in \mathbb{N}_{[0,N]}$.

Letting $\varepsilon \to 0$, a combination of $u(n) \leq \varphi^{-1}(z(n))$ and the inequalities (2.24), (2.25) produces the required inequalities in (2.12), (2.13), respectively, for $N = n$, since $N \in \mathbb{N}_{[0,\alpha_m]}$, $m = 2, 3$ was arbitrary. This completes the proof. $\square$

For the special case $\varphi(u) = u^p, G(r) = r^{(p-q)/p}$ ($p > q \geq 0$ is a constant), Theorem 2.4 gives the following retarded integral inequality for nonlinear functions.

Corollary 2.5. *Let $b, f_i, g_i \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_+), i = 1, \dots, n$ with b nondecreasing and let $\sigma \in \mathcal{F}(\mathbb{Z}_0, \mathbb{Z})$ be nondecreasing with $\sigma(t) \leq t$ and $-\infty < a = \inf\{\sigma(s) : s \in \mathbb{Z}_0\}$. Suppose that p, q are constants with $p > q \geq 0$, and $\psi_j(u), j = 1, 2$ are nondecreasing continuous functions for $u \in \mathbb{R}_+$ with $\psi_j(u) > 0$ for $u > 0$. If $u \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_1)$ and*

$$u^p(n) \leq b(n) + \sum_{i=1}^{l} \sum_{s=0}^{n-1} u^q(\sigma(s))[f_i(s)\psi_1(u(\sigma(s))) + g_i(s)\psi_2(\log u(\sigma(s)))]$$

for $n \in \mathbb{Z}_0$, then

(*i*) *for the case $\psi_1(u) \geq \psi_2(\log(u))$,*

$$u(n) \leq \left\{\Omega_4^{-1}\left[\Omega_4\left([b(n)]^{\frac{p-q}{p}}\right) + \frac{p-q}{p}\sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s) + g_i(s)]\right]\right\}^{\frac{1}{p-q}}$$

for $n \in \mathbb{Z}_{[0,\alpha_4]}$, and

(*ii*) *for the case $\psi_1(u) < \psi_2(\log(u))$,*

$$u(n) \leq \left\{\Omega_5^{-1}\left[\Omega_5\left([b(n)]^{\frac{p-q}{p}}\right) + \frac{p-q}{p}\sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s) + g_i(s)]\right]\right\}^{\frac{1}{p-q}}$$

for $n \in \mathbb{Z}_{[0,\alpha_5]}$,

where

$$\Omega_m(r) = \int_{r_0}^{r} \frac{ds}{\psi_{m-3}(s^{\frac{1}{p-q}})}, \quad r \geq r_0 > 0,$$

$G^{-1}, \Omega_m^{-1}, m = 4, 5$, denote the inverse functions of $G, \Omega_m, m = 4, 5$, the function $G(t)$ is as defined in Theorem 2.1 for $t > 0$, and $\alpha_m \geq 0, m = 4, 5$, are chosen so that

$$\Omega_m\left([b(n)]^{\frac{p-q}{p}}\right) + \frac{p-q}{p}\sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s) + g_i(s)] \in \mathrm{Dom}(\Omega_m^{-1}).$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 2.4 with suitable modification. We omit the details here. □

Theorem 2.1 can easily be applied to generate another useful nonlinear integral inequalities in more general situations. For example, we have the following result.

Theorem 2.6. *Let $b, f_i, g_i \in \mathcal{F}(\mathbb{Z}_0, \mathbb{R}_+), i = 1, \ldots, n$ with b nondecreasing and let $\sigma \in \mathcal{F}(\mathbb{Z}_0, \mathbb{Z})$ be nondecreasing with $\sigma(t) \le t$ and $-\infty < a = \inf\{\sigma(s) : s \in \mathbb{Z}_0\}$. Suppose that $q \ge 0$ is constant, $\varphi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is an increasing function with $\varphi(\infty) = \infty$ on $\mathbb{R}_+$, and $L, M \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ satisfy*

$$0 \le L(t,v) - L(t,w) \le M(t,w)(v-w) \tag{2.26}$$

for $t, v, w \in \mathbb{R}_+$ with $v \ge w \ge 0$. If $u \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ and

$$\varphi(u(n)) \le b(n) + \sum_{i=1}^{l}\sum_{s=0}^{n-1} u^q(\sigma(s))[f_i(s)L(s, u(\sigma(s))) + g_i(s)u(\sigma(s))] \tag{2.27}$$

for $n \in \mathbb{Z}_0$, then

$$u(n) \le \varphi^{-1}\left\{G^{-1}\left[H^{-1}\left(H[B(n)] + \sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s)M(s) + g_i(s)]\right)\right]\right\} \tag{2.28}$$

for $n \in \mathbb{Z}_{[0,\beta]}$, where

$$\begin{aligned} H(r) &= \int_{r_0}^{r} \frac{ds}{\varphi^{-1}(G^{-1}(s)))}, \quad r \ge r_0 > 0, \\ B(n) &= G(b(n)) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} f_i(s)L(s), \end{aligned}$$

G^{-1}, H^{-1} denote the inverse functions of G, H,, respectively, the function $G(r)$ is as defined in Theorem 2.1 for $r > 0$, and $\beta \ge 0$ is chosen so that

$$H[B(n)] + \sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s)M(s) + g_i(s)] \in \mathrm{Dom}(H^{-1}).$$

Proof. Let $\epsilon > 0$ and $N \in \mathbb{Z}_0$. Define a function $z : \mathbb{Z}_{[0,N]} \to \mathbb{R}_0$ by

$$z(n) = \epsilon + b(N) + \sum_{i=1}^{l}\sum_{s=0}^{n-1} u^q(\sigma(s))[f_i(s)L(s, u(\sigma(s))) + g_i(s)u(\sigma(s))]. \tag{2.29}$$

Clearly, $z(n)$ is nondecreasing, $u(n) \le \varphi^{-1}(z(n))$ for $n \in \mathbb{Z}_{[0,N]}$ and $z(0) = \epsilon + b(N)$. We get

$$\begin{aligned} \triangle z(n) &= \sum_{i=1}^{l} u^q(\sigma(n))[f_i(n)L(n, u(\sigma(n))) + g_i(n)u(\sigma(n))] \\ &\le [\varphi^{-1}(z(n))]^q \sum_{i=1}^{l}[f_i(n)L(n, u(\sigma(n))) + g_i(n)u(\sigma(n))]. \end{aligned}$$

Using the monotonicity of φ^{-1} and z, we deduce

$$\frac{\triangle z(t)}{[\varphi^{-1}(z(n))]^q} \leq \sum_{i=1}^{l}[f_i(n)L(n,u(\sigma(n))) + g_i(n)u(\sigma(n))]. \tag{2.30}$$

On the other hand, by the mean value theorem, we have

$$\triangle G(z(n)) = G(z(n+1)) - G(z(n)) = G'(\xi)\triangle z(n) \tag{2.31}$$

for some $\xi \in [z(n), z(n+1)]$. From (2.30), (2.31), and using the function G, we obtain

$$\triangle G(z(n)) \leq \sum_{i=1}^{l}[f_i(n)L(n,u(\sigma(n))) + g_i(n)u(\sigma(n))]. \tag{2.32}$$

Setting $n = s$ in the inequality (2.32) and summing up, we get

$$G(z(n)) \leq G(\epsilon + b(N)) + \sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s)L(s,u(\sigma(s))) + g_i(s)u(\sigma(s))] \tag{2.33}$$

for all $n \in \mathbb{N}_{[0,N]}$. From the inequalities (2.26), (2.33), we find

$$\begin{aligned} G(z(n)) \;\; \leq \;\; & G(\epsilon + b(N)) + \sum_{s=0}^{N-1}\sum_{i=1}^{l} f_i(s)L(s) \\ & + \sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s)M(s) + g_i(s)]\varphi^{-1}(z(s)). \end{aligned} \tag{2.34}$$

Now, define a function $v(n)$ by the right-hand side of (2.34). Clearly, $v(n)$ is nondecreasing, $z(n) \leq G^{-1}(v(n))$ for $n \in \mathbb{N}_{[0,N]}$ and

$$v(0) = G(\epsilon + b(N)) + \sum_{s=0}^{N-1}\sum_{i=1}^{l} f_i(s)L(s).$$

Therefore, for any $t \in \mathbb{N}_{[0,N-1]}$, we get

$$\begin{aligned} \triangle v(t) \;\; = \;\; & \sum_{i=1}^{l}[f_i(t)M(t) + g_i(t)]\varphi^{-1}(z(t)) \\ \leq \;\; & \varphi^{-1}(G^{-1}(v(t)))\sum_{i=1}^{l}[f_i(t)M(t) + g_i(t)]. \end{aligned}$$

Using the monotonicity of φ^{-1}, G^{-1} and v, we deduce

$$\frac{\triangle v(t)}{\varphi^{-1}(G^{-1}(v(t)))} \leq \sum_{i=1}^{l}[f_i(t)M(t) + g_i(t)]. \tag{2.35}$$

On the other hand, by the mean value theorem, we have

$$\triangle H(v(t)) = H(v(t+1)) - H(v(t)) = H'(\xi)\triangle v(t) \tag{2.36}$$

for some $\xi \in [v(t), v(t+1)]$. From (2.35), (2.36), and using the function H, we obtain

$$\triangle H(v(t)) \le \sum_{i=1}^{l}[f_i(t)M(t) + g_i(t)]. \tag{2.37}$$

Setting $t = s$ in the inequality (2.37), summing it from 0 to $n-1$, we obtain

$$H(v(n)) \le H(v(0)) + \sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s)M(s) + g_i(s)]. \tag{2.38}$$

From the inequalities (2.34) and (2.38), we conclude that

$$z(n) \le G^{-1}\left[H^{-1}\left(H(v(0)) + \sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s)M(s) + g_i(s)]\right)\right] \tag{2.39}$$

for $n \in \mathbb{N}_{[0,N]}$. Now, letting $\varepsilon \to 0$, a combination of $u(n) \le \varphi^{-1}(z(n))$ and the inequality (2.39) produces the required inequality in (2.28) for $N = n$, since $N \in \mathbb{N}_{[0,\beta]}$ was arbitrary. This completes the proof. $\square$

For the special case $\varphi(u) = u^p, G(r) = r^{(p-q)/p}$ ($p > q \ge 0$ is a constant), Theorem 2.6 gives the following retarded integral inequality for nonlinear functions.

Corollary 2.7. *Let b, f_i, g_i and σ be as defined in Theorem 2.6. Suppose that $p > q \ge 0$ are constants and $L, M \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ satisfy*

$$0 \le L(t,v) - L(t,w) \le M(t,w)(v-w)$$

for $t, v, w \in \mathbb{R}_+$ with $v \ge w \ge 0$. If $u \in \mathcal{F}(\mathbb{Z}_a, \mathbb{R}_0)$ and

$$u^p(n) \le b(n) + \sum_{i=1}^{l}\sum_{s=0}^{n-1} u^q(\sigma(s))[f_i(s)L(s, u(\sigma(s))) + g_i(s)u(\sigma(s))]$$

for $n \in \mathbb{Z}_0$, then

$$u(n) \le \left[H_1^{-1}\left(H_1(B_1(n)) + \frac{p-q}{p}\sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s)M(s) + g_i(s)]\right)\right]^{\frac{1}{p-q}}$$

for $n \in \mathbb{Z}_{[0,\beta_1]}$, where

$$\begin{aligned} H_1(r) &= \int_{r_0}^{r} \frac{ds}{s^{\frac{1}{p-q}}}, \quad r \ge r_0 > 0, \\ B_1(n) &= G(b(n)) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} f_i(s)L(s), \end{aligned}$$

H_1^{-1} *denotes the inverse function of* H_1 *and* $\beta_1 \geq 0$ *is so chosen that*

$$H_1(B_1(n)) + \frac{p-q}{p}\sum_{s=0}^{n-1}\sum_{i=1}^{l}[f_i(s)M(s) + g_i(s)] \in \mathrm{Dom}(H_1^{-1}).$$

Proof. The proof follows by an argument similar to that in the proof of Theorem 2.6 with suitable modification. We omit the details here. □

Remark 2.8. When $p = 1$, $q = 0$, $\sigma(s) = s$ and $i = 1$, from Corollary 2.7, we derive an analogue of the result of Theorem 1.2.

3 Applications

In this section we will show that our results are useful in proving the global existence of solutions to certain differential equations with time delay. First consider the functional sum-difference equation

$$\begin{cases} \triangle\phi(x(n)) = h(n) + \displaystyle\sum_{i=1}^{l} F_i[n, x(\sigma(n)), w(x(\sigma(n)))], \\ \phi(x(0)) = x_0, \end{cases} \tag{3.1}$$

where x_0 is a constant, $\phi \in C(\mathbb{R}, \mathbb{R}_+)$ is increasing function with $\phi(|x|) \leq |\phi(x)|$, $h \in \mathcal{F}(\mathbb{N}_0, \mathbb{R})$ be nondecreasing, $x \in \mathcal{F}(\mathbb{N}_a, \mathbb{R})$, $\sigma \in \mathcal{F}(\mathbb{Z}_0, \mathbb{Z})$ be nondecreasing with $\sigma(t) \leq t$ and $-\infty < a = \inf\{\sigma(s) : s \in \mathbb{Z}_0\}$, $w \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing function, and $F \in C(\mathbb{N}_0 \times R^2, R)$. The following theorem deals with a bound on the solution of the problem (3.1).

Theorem 3.1. *Assume that* $F_i : \mathbb{N}_0 \times R^2 \to R, i = 1, \ldots, l$ *is a continuous function for which there exists continuous nonnegative functions* $f_i, g_i \in \mathcal{F}(\mathbb{N}_0, \mathbb{R}_+), i = 1, \ldots, l$ *such that*

$$|F_i[n, x(\sigma(n)), w(x(\sigma(n)))]| \leq |x(\sigma(n))|^q (f_i(n)\psi(|x(\sigma(n))|) + g_i(n)), \tag{3.2}$$

$$|x_0| + \sum_{s=0}^{n-1} |h(s)| \leq b(n), \tag{3.3}$$

where $q \geq 0$ *is a constant and* $b(n), \psi$ *are as in Theorem 2.1. If* $x(t)$ *is any solution of the problem* (3.1)*, then*

$$|x(n)| \leq \phi^{-1}\Big\{G^{-1}\Big[\Omega^{-1}\Big(\Omega\Big(G(b(n)) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} g_i(s)\Big) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} f_i(s)\Big)\Big]\Big\} \tag{3.4}$$

for $n, s \in \mathbb{N}_0$*, where* G, Ω *are as in Theorem 2.1.*

Proof. It is easy to see that the solution $x(n)$ of the problem (3.1) satisfies the equivalent equation

$$\phi(x(n)) = x_0 + \sum_{s=0}^{n-1} h(s) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} F_i[s, x(\sigma(s)), w(x(\sigma(s)))]. \tag{3.5}$$

From (3.5), we have

$$|\phi(x(n))| \le |x_0| + \sum_{s=0}^{n-1} |h(s)| + \sum_{s=0}^{n-1}\sum_{i=1}^{l} |F_i[s, x(\sigma(s)), w(x(\sigma(s)))]| \tag{3.6}$$

for $n, s \in \mathbb{N}_0$. Using the conditions (3.2), (3.3) on the right-hand side of (3.6) and rewriting we have

$$\phi(|x(n)|) \le b(n) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} |x(\sigma(s))|^q [f_i(s)\psi(|x(\sigma(s))|) + g_i(s)],$$

where $n, s \in \mathbb{N}_0$. Now an immediate application of the inequality established in Theorem 2.1 to the inequality (3.4) yields the result. □

Remark 3.2. Consider the functional difference equation with the initial condition

$$\begin{cases} \triangle x^p(n) = h(n) + \sum_{i=1}^{l} F_i[n, x(\sigma(n)), w(x(\sigma(n)))], \\ x^p(0) = x_1, \end{cases} \tag{3.7}$$

where $p > 0, x_1$ are constants. Assume that $F_i : \mathbb{N}_0 \times R^2 \to R$, $i = 1, \dots, l$ is a continuous function for which there exists continuous nonnegative functions $f_i, g_i \in \mathcal{F}(\mathbb{N}_0, \mathbb{R}_+)$, $i = 1, \dots, l$ such that the inequalities (3.2) and (3.3) hold, where $q \ge 0$ ($p > q$) is a constant and $b(n), \psi$ are as in Corollary 2.2. If $x(n)$ is any solution of the problem (equation) (3.7), then it satisfies the equivalent equation

$$x^p(n) = x_1 + \sum_{s=0}^{n-1} h(s) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} F_i[s, x(\sigma(s)), w(x(\sigma(s)))]. \tag{3.8}$$

From (3.8), we have

$$|x(n)|^p \le |x_1| + \sum_{s=0}^{n-1} |h(s)| + \sum_{s=0}^{n-1}\sum_{i=1}^{l} |F_i[s, x(\sigma(s)), w(x(\sigma(s)))]| \tag{3.9}$$

for $n, s \in \mathbb{N}_0$. Using the conditions (3.2), (3.3) on the right-hand side of (3.9) and rewriting we have

$$|x(n)|^p \le b(n) + \sum_{s=0}^{n-1}\sum_{i=1}^{l} |x(\sigma(s))|^q [f_i(s)\psi(|x(\sigma(s))|) + g_i(s)], \tag{3.10}$$

where $n, s \in \mathbb{N}_0$. Now an immediate application of the inequality established in Corollary 2.2 to the inequality (3.10) yields

$$|x(n)| \leq \left\{ \Omega_1^{-1} \left[\Omega_1 \left([b(n)]^{\frac{p-q}{p}} + \frac{p-q}{p} \sum_{s=0}^{n-1} \sum_{i=1}^{l} g_i(s) \right) + \frac{p-q}{p} \sum_{s=0}^{n-1} \sum_{i=1}^{l} f_i(s) \right] \right\}^{\frac{1}{p-q}}$$

for $n, s \in \mathbb{N}_0$, where Ω_1 is as in Corollary 2.2.

The following theorem provides a uniqueness on the solution of the problem (3.7).

Theorem 3.3. *Assume that $F : I \times R^3 \to R$ is a continuous function for which there exists continuous nonnegative functions $f_i(n), i = 1, \ldots, n$ for $n \in \mathbb{N}_0$ such that*

$$|F(n, x, w(x)) - F(t, \overline{x}, w(\overline{x}))| \leq f_i(n)|x^p - \overline{x}^p|, \tag{3.11}$$

where $p > 1$ is a constant, then the problem (3.7) *has at most one solution on $n \in \mathbb{N}_0$.*

Proof. Let $x(n)$ and $\overline{x}(n)$ be two solutions of the problem (3.7). We have

$$x^p(n) - \overline{x}^p(n) = \sum_{s=0}^{n-1} \sum_{i=1}^{l} \{F_i[n, x(\sigma(n)), w(x(\sigma(n)))] - F_i[n, \overline{x}(\sigma(n)), w(\overline{x}(\sigma(n)))]\}. \tag{3.12}$$

From (3.11) and (3.12), we find

$$|x^p(n) - \overline{x}^p(n)| \leq \sum_{s=0}^{n-1} \sum_{i=1}^{l} f_i(s)|x^p(\sigma(n)) - \overline{x}^p(\sigma(n))| \tag{3.13}$$

for $n, s \in \mathbb{N}_0$. Rewriting the right-hand and the left-hand sides of (3.13) we have

$$(|x^p(n) - \overline{x}^p(n)|^{\frac{1}{p}})^p \leq \sum_{s=0}^{n-1} \sum_{i=1}^{l} [|A(x, \overline{x}; \sigma(s))|^{\frac{1}{p}}]^{p-1} f_i(s) [|A(x, \overline{x}; \sigma(s))|^{\frac{1}{p}}], \tag{3.14}$$

where $A(x, \overline{x}; \sigma(s)) = x^p(\sigma(s)) - \overline{x}^p(\sigma(s))$ for $s \in \mathbb{N}_0$. When $\psi(u) = u, q = p - 1$, a suitable application of the inequality in Corollary 2.2 to the function $|x^p(n) - \overline{x}^p(n)|^{1/p}$ and the inequality (3.14) lead us to the inequality

$$|x^p(n) - \overline{x}^p(n)|^{1/p} \leq 0$$

for all $t \in \mathbb{N}_0$. Hence $x(n) = \overline{x}(n)$. $\square$

References

[1] R. P. Agarwal, *Difference Equations and Inequalities,* Marcel Dekker, New York, 1992.

[2] D. Bainov and P. Simeonov, *Integral Inequalities and Applications,* Kluwer Academic Publishers, Dordrecht, 1992.

[3] R. Bellman, The stability of solutions of linear differential equations, *Duke Math. J.,* 10 (1943), 643–647.

[4] I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, *Acta Math. Acad. Sci. Hungar.,* 7 (1956), 71–94.

[5] W. S. Cheung, Some new nonlinear inequalities and applications to boundary value problems, *Nonlinear Anal.,* 64 (2006), 2112–2128.

[6] W. S. Cheung and Q. H. Ma, On certain new Gronwall–Ou–Iang type integral inequalities in two variables and their applications, *J. Inequal. Appl.,* 4 (2005), 347–361.

[7] W. S. Cheung and S. Tseng, Some new discrete nonlinear delay inequalities and application to discrete delay equations, *J. Inequal. Pure and Appl. Math.,* 7 (4) Art. 122 (2006), [ONLINE http://jipam. vu. edu. au/].

[8] Y. J. Cho, S. S. Dragomir and Y.-H. Kim, On some integral inequalities with iterated integrals, *J. Korean Math. Soc.,* 43 no. 3 (2006), 563–578.

[9] C. M. Dafermos, The second law of thermodynamics and stability, *Arch. Rational Mech. Anal.,* 70 (1979), 167–179.

[10] S. S. Dragomir, On discrete generalization of Pachpatte's inequality and applications, *Bull. Math. Soc. Sci. Math. Roumanie* 36 (1992), 45–58.

[11] T. H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, *Ann. of Math.,* 20 (1919), 292–296.

[12] Y.-H. Kim, On some discrete inequalities in two independent variables, *Czechoslovak Math. J.* 55 (130) (2005), 113–124.

[13] O. Lipovan, A retarded integral inequality and its applications, *J. Math. Anal. Appl.,* 285 (2003), 336–443.

[14] L. Ou-Iang, The boundedness of solutions of linear differential equations $y'' + A(t)y = 0$, *Shuxue Jinzhan,* 3 (1957), 409–415.

[15] B. G. Pachpatte, On the discrete generalizations of Gronwall's inequality, *J. Indian Math. Soc.* 37 (1973), 147–156.

[16] B. G. Pachpatte, A note on Gronwall–Bellman inequality, *J. Math. Anal. Appl.* 44 (1973), 758–762.

[17] B. G. Pachpatte, Finite difference inequalities and their applications, *Proc. Nat. Acad. Sci. India,* 43(A), (1973).

[18] B. G. Pachpatte, On some new inequalities related to certain inequalities in the theory of differential equations, *J. Math. Anal. Appl.* 189 (1995), 128–144.

[19] B. G. Pachpatte, *Inequalities for Finite Difference Equations,* Marcel Dekker, New York, (2002).

[20] S. Sugiyama, On the stability of difference equations, *Bull. Sci. Engr. Research Lab., Waseda Univ.* 45, (1969).