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Oscillation Criteria for Sublinear Half-Linear Delay Dynamic Equations

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Abstract

This paper is concerned with oscillation of the second-order half-linear delay dynamic equation

$$(r(t)(x^{\Delta})^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0,$$

on a time scale $\mathbb{T}$ where $0 < \gamma \leq 1$ is the quotient of odd positive integers, $p : \mathbb{T} \to [0, \infty)$, and $\tau : \mathbb{T} \to \mathbb{T}$ are positive rd-continuous functions, $r(t)$ is a positive and (delta) differentiable function, $\tau(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \infty$. We establish some new sufficient conditions which ensure that every solution oscillates or converges to zero. Our results in the special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ involve and improve some oscillation results for second-order differential and difference equations; and when $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = q^{\mathbb{N}_0}$ and $\mathbb{T} = \mathbb{N}^2$ our oscillation results are essentially new. Some examples illustrating the importance of our results are also included.

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1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD Thesis in 1988 in order to unify continuous and discrete analysis, see [12]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [5]). This theory of these so-called "dynamic equations" not only unifies the corresponding theories for the differential equations and difference equations cases, but it also extends these classical cases to cases "in between". That is, we are able to treat the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0 \text{ for } q > 1\}$ (which has important applications in quantum theory (see [13])) and can be applied to different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ the set of the harmonic numbers. The books on the subject of time scales by Bohner and Peterson [5, 6] summarize and organize much of time scale calculus. In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales, and we refer the reader to the papers [1, 4, 8, 10, 11, 14] and the references cited therein. In this paper, we are concerned with the oscillatory behavior of the second-order half-linear delay dynamic equation

$$(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $0 < \gamma \leq 1$ is a quotient of odd positive integers, p is a positive rd-continuous function on $\mathbb{T}$, $r(t)$ is a positive and (delta) differentiable function and the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \leq t$ for $t \in \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. By a solution of (1.1) we mean a nontrivial real-valued function $x \in C_r^1[T_x, \infty)$, $T_x \geq t_0$ which has the property that $r(t)\left(x^{\Delta}(t)\right)^{\gamma} \in C_r^1[T_x, \infty)$ and satisfies equation (1.1) on $[T_x, \infty)$, where C_r is the space of rd-continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Recently there has also been a spate of papers on second-order nonlinear dynamic equations on time scales. For a few examples of work since then, Agarwal et al. [1] considered the second-order delay dynamic equations on time scales

$$x^{\Delta\Delta}(t) + p(t)x(\tau(t)) = 0, \tag{1.2}$$

and established some sufficient conditions for oscillation of (1.2). Erbe et al. [9] considered the pair of second-order dynamic equations

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(t) = 0, \tag{1.3}$$

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(\sigma(t)) = 0,$$

and established some necessary and sufficient conditions for nonoscillation of Hille–Kneser type. Saker [14] examines oscillation for half-linear dynamic equations on time scales (1.3), where $\gamma > 1$ is an odd positive integer and Agarwal et al. [3] studies oscillation for the same equation (1.3), where $\gamma > 1$ is the quotient of odd positive integers. Hassan [11] improved Agarwal's and Saker's results for the equation (1.3), when $\gamma > 0$ is the quotient of odd positive integers. Erbe et al. [7] considered the half-linear delay dynamic equations on time scales

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0,$$

where $\gamma > 1$ is the quotient of odd positive integers.

We herein utilize a Riccati transformation technique to establish oscillation criteria for (1.1), where $0 < \gamma \leq 1$ is the quotient of odd positive integers, which complete, improve and generalize the results that have been established by Agarwal et al. [3], Saker [14], Erbe et al. [7] and others. Also, interesting examples that illustrate the importance of our results are included in Section 4.

2 Main Results

Throughout the paper we assume that

$$r^{\Delta}(t) \geq 0, \quad \text{and} \quad \int_{t_0}^{\infty} \tau^{\gamma}(t)p(t)\Delta t = \infty \tag{2.1}$$

is satisfied. Before stating our main results, we begin with the following lemma which will play an important role in the proof of our main results.

Lemma 2.1. *[7] Assume that* (2.1) *and*

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \infty \tag{2.2}$$

hold and (1.1) *has a positive solution* x *on* $[t_0, \infty)_{\mathbb{T}}$*. Then there exists a* $T \in [t_0, \infty)_{\mathbb{T}}$*, sufficiently large, so that*

(i) $x^{\Delta}(t) > 0$*,* $x^{\Delta\Delta}(t) < 0$*,* $x(t) > tx^{\Delta}(t)$*, for* $t \in [T, \infty)_{\mathbb{T}}$*;*

(ii) $\dfrac{x(t)}{t}$ *is strictly decreasing on* $[T, \infty)_{\mathbb{T}}$*.*

Motivated by [7, Theorem 2.9], we can prove the following result which is a new oscillation result for equation (1.1).

Theorem 2.2. *Assume that* (2.1) *and* (2.2) *hold. Furthermore, assume that there exists a positive* Δ*-differentiable function* $\delta(t)$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\delta(s)p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}-\frac{r(s)((\delta^{\Delta}(s))_+)^{\gamma+1}}{\delta^{\gamma}(s)(\gamma+1)^{\gamma+1}}\right]\Delta s=\infty, \tag{2.3}$$

where $d_+(t) := \max\{d(t),0\}$ *is the positive part of any function* $d(t)$*. Then every solution of equation* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$*.*

Proof. Assume (1.1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemma 2.1 on $[t_1,\infty)_{\mathbb{T}}$ with $x(\tau(t)) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. Let $\delta(t)$ be a positive Δ differentiable function and consider the generalized Riccati substitution

$$w(t)=\delta(t)r(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}.$$

Then by Lemma 2.1, we see that the function $w(t)$ is positive on $[t_1,\infty)_{\mathbb{T}}$. By the product rule and then the quotient rule (suppressing arguments)

$$\begin{aligned}
w^{\Delta} &= \delta^{\Delta}\left(\frac{r(x^{\Delta})^{\gamma}}{x^{\gamma}}\right)^{\sigma}+\delta\left(\frac{r(x^{\Delta})^{\gamma}}{x^{\gamma}}\right)^{\Delta}\\
&= \frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma}+\delta\frac{x^{\gamma}(r(x^{\Delta})^{\gamma})^{\Delta}-r(x^{\Delta})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}x^{\gamma\sigma}}\\
&= \frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma}+\frac{\delta x^{\gamma}(-px^{\tau\gamma})}{x^{\gamma}(x^{\sigma})^{\gamma}}-\frac{\delta r(x^{\Delta})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}\\
&= \frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma}-p\delta\left(\frac{x^{\tau}}{x^{\sigma}}\right)^{\gamma}-\delta\frac{r(x^{\Delta})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}.
\end{aligned}$$

Using the fact that $\dfrac{x(t)}{t}$ and $r(t)(x^{\Delta}(t))^{\gamma}$ are decreasing (from Lemma 2.1) we get

$$\frac{x^{\tau}(t)}{x^{\sigma}(t)}\geq\frac{\tau(t)}{\sigma(t)} \quad \text{and} \quad r(t)(x^{\Delta}(t))^{\gamma}\geq r^{\sigma}(t)(x^{\Delta}(t))^{\gamma\sigma}.$$

By these last two inequalities we obtain

$$w^{\Delta}\leq\frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma}-\delta p\left(\frac{\tau}{\sigma}\right)^{\gamma}-\delta\frac{r^{\sigma}(x^{\Delta\sigma})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}. \tag{2.4}$$

By the Pötzsche chain rule (see [5, Theorem 1.90]), and the fact that $x^{\Delta}(t) > 0$, we obtain

$$\begin{aligned}
(x^\gamma)^\Delta(t) &= \gamma \int_0^1 \left[x(t) + h\mu(t)x^\Delta(t)\right]^{\gamma-1} dh\, x^\Delta(t) \\
&= \gamma \int_0^1 \left[(1-h)\,x(t) + hx^\sigma(t)\right]^{\gamma-1} dh\, x^\Delta(t) \\
&\geq \gamma \int_0^1 (x^\sigma(t))^{\gamma-1}\, dh\, x^\Delta(t) \\
&= \gamma(x^\sigma(t))^{\gamma-1} x^\Delta(t). \qquad (2.5)
\end{aligned}$$

Using (2.4) and (2.5), we have that

$$w^\Delta \leq \frac{\delta^\Delta}{\delta^\sigma} w^\sigma - \delta p \left(\frac{\tau}{\sigma}\right)^\gamma - \gamma\delta \frac{r^\sigma (x^{\Delta\sigma})^\gamma x^\Delta}{x^\gamma x^\sigma}.$$

Since

$$x^\Delta(t) \geq \frac{(r^\sigma(t))^{\frac{1}{\gamma}} (x^\Delta(t))^\sigma}{r^{\frac{1}{\gamma}}(t)}, \quad \text{and} \quad x^\sigma(t) \geq x(t),$$

we get that

$$w^\Delta \leq \frac{\delta^\Delta}{\delta^\sigma} w^\sigma - \delta p \left(\frac{\tau}{\sigma}\right)^\gamma - \gamma \frac{\delta r^{\sigma(1+\frac{1}{\gamma})}}{r^{\frac{1}{\gamma}}} \left(\frac{x^{\Delta\sigma}}{x^\sigma}\right)^{\gamma+1}.$$

Using the definition of w we finally obtain

$$w^\Delta \leq \frac{(\delta^\Delta)_+}{\delta^\sigma} w^\sigma - \delta p \left(\frac{\tau}{\sigma}\right)^\gamma - \gamma \frac{\delta}{(\delta^\sigma)^\lambda r^{\frac{1}{\gamma}}} (w^\sigma)^\lambda, \qquad (2.6)$$

where $\lambda := \dfrac{\gamma+1}{\gamma}$. Define $A \geq 0$ and $B \geq 0$ by

$$A^\lambda := \frac{\gamma\delta}{(\delta^\sigma)^\lambda r^{\frac{1}{\gamma}}} (w^\sigma)^\lambda, \quad B^{\lambda-1} := \frac{r^{\frac{1}{\gamma+1}}}{\lambda(\gamma\delta)^{\frac{1}{\lambda}}} (\delta^\Delta)_+.$$

Then, using the inequality

$$\lambda A B^{\lambda-1} - A^\lambda \leq (\lambda-1)B^\lambda,$$

we get that

$$\begin{aligned}
\frac{(\delta^\Delta)_+}{\delta^\sigma} w^\sigma - \gamma \frac{\delta}{(\delta^\sigma)^\lambda r^{\frac{1}{\gamma}}} (w^\sigma)^\lambda &= \lambda A B^{\lambda-1} - A^\lambda \\
&\leq (\lambda-1)B^\lambda \\
&\leq \frac{r((\delta^\Delta)_+)^{\gamma+1}}{\delta^\gamma(\gamma+1)^{\gamma+1}}.
\end{aligned}$$

By this last inequality and (2.6) we get

$$w^{\Delta} \leq \frac{r((\delta^{\Delta})_{+})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}.$$

Integrating both sides from t_1 to t we get

$$-w(t_1) \leq w(t) - w(t_1) \leq \int_{t_1}^{t} \left[\frac{r((\delta^{\Delta})_{+})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}\right] \Delta s,$$

which leads to a contradiction, since the right-hand side tends to $-\infty$ by (2.3). □

We introduce the notation

$$p_* := \liminf_{t\to\infty} \frac{t^{\gamma}}{r(t)} \int_{\sigma(t)}^{\infty} P(s)\Delta s, \quad q_* := \liminf_{t\to\infty} \frac{1}{t} \int_{T}^{t} \frac{s^{\gamma+1}}{r(s)} P(s)\Delta s,$$

$$r_* := \liminf_{t\to\infty} \frac{t^{\gamma} w^{\sigma}(t)}{r\,(t)}, \quad R := \limsup_{t\to\infty} \frac{t^{\gamma} w^{\sigma}(t)}{r(t)},$$

where $P(t) = \left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma} p(t)$ and assume that $l := \liminf_{t\to\infty} \frac{t}{\sigma(t)}$. Note that $0 \leq l \leq 1$. In order for the definition of p_* to make sense we assume that

$$\int_{t_0}^{\infty} P(s)\Delta s = \int_{t_0}^{\infty} p(s) \left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma} \Delta s < \infty. \tag{2.7}$$

Theorem 2.3. *Assume that* (2.1), (2.2) *and* (2.7) *hold. Furthermore, assume that* $l > 0$ *and*

$$p_* > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}, \tag{2.8}$$

or

$$p_* + q_* > \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.9}$$

Then every solution of equation (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume (1.1) has a nonoscillatory solution on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T \in (t_0, \infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemma 2.1 on $[T, \infty)_{\mathbb{T}}$ with $x(\tau(t)) > 0$ on $[T, \infty)_{\mathbb{T}}$. Again we define $w(t)$ as in Theorem 2.2 with $\delta(t) = 1$. We get from (2.6) that

$$-w^{\Delta}(t) \geq P(t) + \frac{\gamma}{r^{\frac{1}{\gamma}}(t)} (w^{\sigma}(t))^{\frac{\gamma+1}{\gamma}}, \quad \text{for} \quad t \in [T, \infty)_{\mathbb{T}}. \tag{2.10}$$

First, we assume (2.8) holds. It follows from Lemma 2.1 that

$$w(t) = r(t) \left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma} < \left(\int_{t_0}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\right)^{-\gamma}, \quad \text{for} \quad t \in [T, \infty)_{\mathbb{T}},$$

which implies (using (2.2)) that $\lim_{t\to\infty} w(t) = 0$. Integrating (2.10) from $\sigma(t)$ to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w^\sigma(t) \ge \int_{\sigma(t)}^\infty P(s)\Delta s + \gamma \int_{\sigma(t)}^\infty \frac{(w^\sigma(s))^{\frac{1}{\gamma}} w^\sigma(s)}{r^{\frac{1}{\gamma}}(s)}\Delta s. \tag{2.11}$$

It follows from (2.11) that

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \ge \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^\infty P(s)\Delta s + \gamma \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^\infty \frac{(w^\sigma(s))^{\frac{1}{\gamma}} w^\sigma(s)}{r^{\frac{1}{\gamma}}(s)}\Delta s. \tag{2.12}$$

Let $\epsilon > 0$. Then by the definition of p_* and r_* we can pick $t_1 \in [T,\infty)_{\mathbb{T}}$, sufficiently large, so that

$$\frac{t^\gamma}{r(t)} \int_{\sigma(t)}^\infty P(s)\Delta s \ge p_* - \epsilon, \quad \text{and} \quad \frac{t^\gamma w^\sigma(t)}{r(t)} \ge r_* - \epsilon, \tag{2.13}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. From (2.12) and (2.13) and using the fact $r^\Delta(t) \ge 0$, we get that

$$\begin{aligned} \frac{t^\gamma w^\sigma(t)}{r(t)} &\ge (p_* - \epsilon) + \gamma \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^\infty \frac{s\,(w^\sigma(s))^{\frac{1}{\gamma}}\, s^\gamma w^\sigma(s)}{s^{\gamma+1} r^{\frac{1}{\gamma}}(s)}\Delta s \\ &\ge (p_* - \epsilon) + (r_* - \epsilon)^{1+\frac{1}{\gamma}} \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^\infty \frac{\gamma r(s)}{s^{\gamma+1}}\Delta s \\ &\ge (p_* - \epsilon) + (r_* - \epsilon)^{1+\frac{1}{\gamma}}\, t^\gamma \int_{\sigma(t)}^\infty \frac{\gamma}{s^{\gamma+1}}\Delta s. \end{aligned} \tag{2.14}$$

Using the Pötzsche chain rule [5, Theorem 1.90], we get

$$\begin{aligned} \left(\frac{-1}{s^\gamma}\right)^\Delta &= \gamma \int_0^1 \frac{1}{[s + h\mu(s)]^{\gamma+1}} dh \\ &\le \int_0^1 \left(\frac{\gamma}{s^{\gamma+1}}\right) dh \\ &= \frac{\gamma}{s^{\gamma+1}}. \end{aligned} \tag{2.15}$$

Then from (2.14) and (2.15), we have

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \ge (p_* - \epsilon) + (r_* - \epsilon)^{1+\frac{1}{\gamma}} \left(\frac{t}{\sigma(t)}\right)^\gamma.$$

Taking the $\liminf$ of both sides as $t \to \infty$ we get that

$$r_* \ge p_* - \epsilon + (r_* - \epsilon)^{1+\frac{1}{\gamma}}\, l^\gamma.$$

Since $\epsilon > 0$ is arbitrary, we get

$$p_* \le r_* - r_*^{1+\frac{1}{\gamma}} l^\gamma. \tag{2.16}$$

Using the inequality

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma},$$

with $B = 1$ and $A = l^\gamma$ we get that

$$p_* \le \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

which contradicts (2.8). Next, we assume (2.9) holds. Multiplying both sides of (2.10) by $\frac{t^{\gamma+1}}{r(t)}$, and integrating from T to t ($t \ge T$) we get

$$\int_T^t \frac{s^{\gamma+1}}{r(s)} w^\Delta(s)\Delta s \le -\int_T^t \frac{s^{\gamma+1}}{r(s)} P(s)\Delta s - \gamma \int_T^t \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s.$$

Using integration by parts, we obtain

$$\begin{aligned} \frac{t^{\gamma+1}w(t)}{r(t)} \le \; & \frac{T^{\gamma+1}w(T)}{r(T)} + \int_T^t \left(\frac{s^{\gamma+1}}{r(s)}\right)^\Delta w^\sigma(s)\Delta s - \int_T^t \frac{s^{\gamma+1}}{r(s)} P(s)\Delta s \\ & -\gamma \int_T^t \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s. \end{aligned}$$

By the quotient rule and applying the Pötzsche chain rule,

$$\begin{aligned} \left(\frac{s^{\gamma+1}}{r(s)}\right)^\Delta &= \frac{(s^{\gamma+1})^\Delta}{r^\sigma(s)} - \frac{s^{\gamma+1} r^\Delta(s)}{r(s) r^\sigma(s)} \\ &\le \frac{(\gamma+1)\sigma^\gamma(s)}{r^\sigma(s)} \\ &\le \frac{(\gamma+1)\sigma^\gamma(s)}{r(s)}. \end{aligned} \tag{2.17}$$

Hence

$$\begin{aligned} \frac{t^{\gamma+1}w(t)}{r(t)} \le \; & \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)} P(s)\Delta s + \int_T^t (\gamma+1)\left(\frac{\sigma^\gamma(s) w^\sigma(s)}{r(s)}\right)\Delta s \\ & - \gamma \int_T^t \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s. \end{aligned}$$

Let $0 < \epsilon \leq l$ be given. Then using the definition of l, we can assume, without loss of generality, that T is sufficiently large so that

$$\frac{s}{\sigma(s)} > l - \epsilon, \quad s \geq T.$$

It follows that

$$\sigma(s) \leq Ks, \qquad s \geq T \quad \text{where} \quad K := \frac{1}{l-\epsilon}.$$

We then get that

$$\begin{aligned} \frac{t^{\gamma+1}w(t)}{r(t)} \quad \leq \quad & \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)} P(s)\Delta s \\ & + \int_T^t \{(\gamma+1)K^\gamma \frac{s^\gamma w^\sigma(s)}{r(s)} - \gamma \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}}\}\Delta s. \end{aligned}$$

Let

$$u(s) := \frac{s^\gamma w^\sigma(s)}{r(s)}.$$

Then

$$u^\lambda(s) = \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^\lambda,$$

where $\lambda = \dfrac{\gamma+1}{\gamma}$. It follows that

$$\begin{aligned} \frac{t^{\gamma+1}w(t)}{r(t)} \quad \leq \quad & \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)} P(s)\Delta s \\ & + \int_T^t \{(\gamma+1)K^\gamma u(s) - \gamma u^\lambda(s)\}\Delta s. \end{aligned}$$

Again, using the inequality

$$Bu - Au^\lambda \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma},$$

where A, B are constants, we get

$$\begin{aligned} \frac{t^{\gamma+1}w(t)}{r(t)} \quad \leq \quad & \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)} P(s)\Delta s \\ + \quad & \int_T^t \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{[(\gamma+1)K^\gamma]^{\gamma+1}}{\gamma^\gamma}\Delta s \\ \leq \quad & \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)} P(s)\Delta s + K^{\gamma(\gamma+1)}(t-T). \end{aligned}$$

It follows from this that

$$\frac{t^\gamma w(t)}{r(t)} \le \frac{\frac{T^{\gamma+1}w(T)}{r(T)}}{t} - \frac{1}{t}\int_T^t \frac{s^{\gamma+1}}{r(s)}P(s)\Delta s + K^{\gamma(\gamma+1)}\left(1-\frac{T}{t}\right).$$

Since $w^\sigma(t) \le w(t)$ we get

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \le \frac{\frac{T^{\gamma+1}w(T)}{r(T)}}{t} - \frac{1}{t}\int_T^t \frac{s^{\gamma+1}}{r(s)}P(s)\Delta s + K^{\gamma(\gamma+1)}\left(1-\frac{T}{t}\right).$$

Taking the $\limsup$ of both sides as $t \to \infty$ we obtain

$$R \le -q_* + K^{\gamma(\gamma+1)} = -q_* + \frac{1}{(l-\epsilon)^{\gamma(\gamma+1)}}.$$

Since $\epsilon > 0$ is arbitrary, we get that

$$R \le -q_* + \frac{1}{l^{\gamma(\gamma+1)}}.$$

Using this and the inequality (2.16) we get

$$p_* \le r_* - l^\gamma r_*^{1+\frac{1}{\gamma}} \le r_* \le R \le -q_* + \frac{1}{l^{\gamma(\gamma+1)}}.$$

Therefore

$$p_* + q_* \le \frac{1}{l^{\gamma(\gamma+1)}},$$

which contradicts (2.9). □

Remark 2.4. We give an example which shows that the inequality (2.8) and hence the inequality (2.9) cannot be weakened. To see this let $\mathbb{T} = [1,\infty)$, $r(t) = 1$, and

$$p(t) := \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\frac{1}{t^{\gamma+1}}, \quad t \ge 1.$$

We have that

$$p_* = \liminf_{t\to\infty} t^\gamma \int_t^\infty p(s)ds = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}},$$

and the second-order half-linear differential equation

$$\left((x'(t))^\gamma\right)' + p(t)x^\gamma(t) = 0,$$

has a nonoscillatory solution $x(t) = t^{\frac{\gamma}{\gamma+1}}$. This shows that the constant $\frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}$ is sharp for the oscillation for all solutions of this equation. Note in the case when $\gamma = 1$ this constant is $\frac{1}{4}$.

Theorem 2.5. *Assume that* (2.1) *and* (2.2) *hold and*

$$\limsup_{t\to\infty} \frac{t^\gamma}{r(t)} \int_t^\infty p(s) \left(\frac{\tau(s)}{s}\right)^\gamma \Delta s > 1. \tag{2.18}$$

Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume x is an eventually positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Using Lemma 2.1 there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x^\Delta(t) > 0, \quad x^{\Delta\Delta}(t) < 0, \quad \frac{x(t)}{t} > x^\Delta(t),$$

on $[t_1, \infty)_{\mathbb{T}}$ and $\frac{x(t)}{t}$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Then integrating both sides of the dynamic equation (1.1) from t to T, $T \geq t \geq t_1$, we obtain

$$\int_t^T p(s) x^\gamma(\tau(s)) \Delta s = r(t)(x^\Delta(t))^\gamma - r(T)(x^\Delta(T))^\gamma.$$

Since $x^\Delta(t) > 0$, we get that

$$\frac{1}{r(t)} \int_t^T p(s) x^\gamma(\tau(s)) \Delta s \leq (x^\Delta(t))^\gamma.$$

Since $\frac{x(t)}{t}$ is strictly decreasing and using $x^\Delta(t) < \frac{x(t)}{t}$ we obtain

$$\frac{1}{r(t)} \int_t^T p(s) \left(\frac{\tau(s)}{s}\right)^\gamma x^\gamma(s) \Delta s \leq \frac{x^\gamma(t)}{t^\gamma}.$$

Since $x(t)$ is increasing we get

$$\frac{t^\gamma}{r(t)} \int_t^T p(s) \left(\frac{\tau(s)}{s},\right)^\gamma \Delta s \leq 1$$

which implies that

$$\frac{t^\gamma}{r(t)} \int_t^\infty p(s) \left(\frac{\tau(s)}{s}\right)^\gamma \Delta s \leq 1,$$

which gives us the contradiction

$$\limsup_{t\to\infty} \frac{t^\gamma}{r(t)} \int_t^\infty p(s) \left(\frac{\tau(s)}{s}\right)^\gamma \Delta s \leq 1.$$

This concludes the proof. □

In the following, we assume that

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} < \infty \tag{2.19}$$

holds and establish some sufficient conditions which ensure that every solution $x(t)$ of (1.1) oscillates or converges to zero. The proof is similar to the proof of [14, Theorem 3.3] and hence is omitted.

Theorem 2.6. *Assume that* (2.1), (2.19) *and*

$$\int_{t_0}^{\infty} \left[\frac{1}{r(t)} \int_{t_0}^{t} p(s)\Delta s\right]^{\frac{1}{\gamma}} \Delta t = \infty \tag{2.20}$$

hold. If one of the conditions (2.3) *or* (2.18) *holds, then every solution of* (1.1) *oscillates or converges to zero on* $[t_0, \infty)_{\mathbb{T}}$.

3 Applications

In this section, we apply the oscillation criteria to various time scales. For example if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, and (1.1) becomes the sublinear half-linear delay differential equation

$$\left(r(t)\left(x'(t)\right)^{\gamma}\right)' + p(t)x^{\gamma}(\tau(t)) = 0. \tag{3.1}$$

Then we have from Theorems 2.2–2.6 the following oscillation criteria for equation (3.1).

Theorem 3.1. *Assume that*

$$r'(t) \geq 0, \quad \int_{t_0}^{\infty} \tau^{\gamma}(t)p(t)dt = \infty \tag{3.2}$$

and

$$\int_{t_0}^{\infty} \frac{dt}{r^{\frac{1}{\gamma}}(t)} = \infty \tag{3.3}$$

hold. Furthermore, assume that there exists a positive differentiable function $\delta(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\delta(s)p(s)\left(\frac{\tau(s)}{s}\right)^{\gamma} - \frac{r(s)((\delta')_{+}(s))^{\gamma+1}}{\delta^{\gamma}(s)(\gamma+1)^{\gamma+1}}\right] ds = \infty, \tag{3.4}$$

where $d_{+}(t) := \max\{d(t), 0\}$ *is the positive part of any function* $d(t)$. *Then every solution of equation* (3.1) *is oscillatory on* $[t_0, \infty)$.

Theorem 3.2. *Assume that* (3.2), (3.3) *and*

$$\int_{t_0}^{\infty} p(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} ds < \infty$$

hold. Furthermore, assume

$$p_* > \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \tag{3.5}$$

or

$$p_* + q_* > 1, \tag{3.6}$$

where

$$p_* = \liminf_{t\to\infty} \frac{t^{\gamma}}{r(t)} \int_t^{\infty} \left(\frac{\tau(s)}{s}\right)^{\gamma} p(s)ds,$$

and

$$q_* = \liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{s\tau^{\gamma}(s)}{r(s)} p(s)ds.$$

Then every solution of equation (3.1) *is oscillatory on* $[t_0, \infty)$.

Theorem 3.3. *Assume that* (3.2) *and* (3.3) *hold. Furthermore, assume*

$$\limsup_{t\to\infty} \frac{t^{\gamma}}{r(t)} \int_t^{\infty} p(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} ds > 1. \tag{3.7}$$

Then every solution of (3.1) *is oscillatory on* $[t_0, \infty)$.

Theorem 3.4. *Assume that* (3.2) *and*

$$\int_{t_0}^{\infty} \frac{dt}{r^{\frac{1}{\gamma}}(t)} < \infty$$

hold. Furthermore, assume that

$$\int_{t_0}^{\infty} \left[\frac{1}{r(t)} \int_{t_0}^{t} p(s)ds\right]^{\frac{1}{\gamma}} dt = \infty$$

hold. If one of the conditions (3.4) *or* (3.7) *holds, then every solution of* (3.1) *oscillates or converges to zero on* $[t_0, \infty)$.

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^{\Delta}(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, and (1.1) becomes the sublinear half-linear delay difference equation

$$\Delta(r(t)\,(\Delta x(t))^{\gamma}) + p(t)x^{\gamma}(\tau(t)) = 0. \tag{3.8}$$

Then we have from Theorems 2.2–2.6 the following oscillation criteria for equation (3.8).

Theorem 3.5. *Assume that*

$$\Delta r(t) \geq 0, \quad \sum_{t=t_0}^{\infty} \tau^{\gamma}(t)p(t) = \infty \tag{3.9}$$

and

$$\sum_{t=t_0}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(t)} = \infty \tag{3.10}$$

hold. Furthermore, assume that there exists a positive sequence $\delta(t)$ *such that*

$$\limsup_{t\to\infty} \sum_{t=t_0}^{t-1} \left[\delta(s)p(s) \left(\frac{\tau(s)}{s+1} \right)^{\gamma} - \frac{r(s)((\Delta\delta(s))_+)^{\gamma+1}}{\delta^{\gamma}(s)(\gamma+1)^{\gamma+1}} \right] = \infty, \tag{3.11}$$

where $d_+(t) := \max\{d(t), 0\}$ *is the positive part of any sequence* $d(t)$*. Then every solution of equation* (3.8) *is oscillatory on* $\mathbb{N}$*.*

Theorem 3.6. *Assume that* (3.9)*,* (3.10)*, and*

$$\sum_{s=t_0}^{\infty} \left(\frac{\tau(s)}{s+1} \right)^{\gamma} p(s) < \infty$$

hold. Furthermore, assume

$$p_* > \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \tag{3.12}$$

or

$$p_* + q_* > 1, \tag{3.13}$$

where

$$p_* = \liminf_{t\to\infty} \frac{t^{\gamma}}{r(t)} \sum_{s=t+1}^{\infty} \left(\frac{\tau(s)}{s+1} \right)^{\gamma} p(s),$$

and

$$q_* = \liminf_{t\to\infty} \frac{1}{t} \sum_{s=N}^{t} \frac{s^{\gamma+1}}{r(s)} \left(\frac{\tau(s)}{s+1} \right)^{\gamma} p(s),$$

where N *is sufficiently large. Then every solution of equation* (3.8) *is oscillatory on* $\mathbb{N}$*.*

Theorem 3.7. *Assume that* (3.9) *and* (3.10) *hold. Furthermore, assume that*

$$\limsup_{t\to\infty} \frac{t^{\gamma}}{r(t)} \sum_{s=t}^{\infty} p(s) \left(\frac{\tau(s)}{s} \right)^{\gamma} > 1 \tag{3.14}$$

Then every solution of (3.8) *is oscillatory on* $\mathbb{N}$*.*

Theorem 3.8. *Assume that* (3.9) *and*

$$\sum_{t=t_0}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(t)} < \infty$$

hold. Furthermore, assume

$$\sum_{t=t_0}^{\infty} \left[\frac{1}{r(t)} \sum_{s=t_0}^{t-1} p(s) \right]^{\frac{1}{\gamma}} = \infty.$$

If one of the conditions (3.11) *or* (3.14) *holds, then every solution of* (3.8) *oscillates or converges to zero on* $\mathbb{N}$.

Similarly, we can state oscillation criteria for many other time scales, e.g., $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \{t : t = q^k,\, k \in \mathbb{N}_0,\, q > 1\}$, $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$, or $\mathbb{T} = \{H_n : n \in \mathbb{N}\}$ where H_n is the so-called n-th harmonic number defined by $H_0 = 0$, $H_n = \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}_0$.

4 Examples

In this section we give some examples to illustrate our main results.

Example 4.1. Consider the half-linear delay dynamic equation

$$\left(t^\gamma \left(x^\Delta(t)\right)^\gamma\right)^\Delta + \frac{\alpha}{t} \left(\frac{\sigma(t)}{\tau(t)}\right)^\gamma x^\gamma(\tau(t)) = 0, \tag{4.1}$$

for $t \in [t_0, \infty)_\mathbb{T}$, where α is a positive constant and $0 < \gamma \leq 1$ is the quotient of odd positive integers and $\tau(t) \leq t$. Here $p(t) = \frac{\alpha}{t} \left(\frac{\sigma(t)}{\tau(t)}\right)^\gamma$ and $r(t) = t^\gamma$. It is clear that

$$\int_{t_0}^{\infty} \tau^\gamma(t) p(t) \Delta t = \alpha \int_{t_0}^{\infty} \frac{\sigma^\gamma(t)}{t} \Delta t \geq \alpha \int_{t_0}^{\infty} \frac{\Delta t}{t^{1-\gamma}} = \infty,$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \int_{t_0}^{\infty} \frac{\Delta t}{t} = \infty, \quad \text{for} \quad 0 < \gamma \leq 1,$$

by [6, Example 5.60]. (i.e., (2.1) and (2.2) hold). To apply Theorem 2.2, with $\delta(t) = t$, it remains to prove that condition (2.3) holds. To see this note that

$$\begin{aligned} &\limsup_{t\to\infty} \int_{t_0}^{t} \left[sp(s) \left(\frac{\tau(s)}{\sigma(s)}\right)^\gamma - \frac{r(s)}{(\gamma+1)^{\gamma+1} s^\gamma} \right] \Delta s \\ &\quad = \left(\alpha - \frac{1}{(\gamma+1)^{\gamma+1}}\right) \limsup_{t\to\infty} \int_{t_0}^{t} \Delta s = \infty, \end{aligned}$$

if $\alpha > \dfrac{1}{(\gamma+1)^{\gamma+1}}$. We conclude, by Theorem 2.2, that if

$$\alpha > \frac{1}{(\gamma+1)^{\gamma+1}},$$

then every solution of (4.1) is oscillatory.

Example 4.2. Consider the half-linear delay dynamic equation

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \tag{4.2}$$

where $p(t) := \dfrac{\beta}{t^{\gamma+1}}\left(\dfrac{\sigma(t)}{\tau(t)}\right)^{\gamma}$ with $\tau(t) \leq t$, $r(t) = 1$, where β is a positive constant and $0 < \gamma \leq 1$ is the quotient of odd positive integers. It is clear that conditions (2.1) and (2.2) are satisfied since

$$\int_{t_0}^{\infty} \tau^{\gamma}(t)p(t)\Delta t = \beta\int_{t_0}^{\infty}\left(\frac{\sigma(t)}{t}\right)^{\gamma}\cdot\frac{1}{t}\,\Delta t \geq \beta\int_{t_0}^{\infty}\frac{\Delta t}{t} = \infty,$$

and

$$\int_{t_0}^{\infty}\frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \int_{t_0}^{\infty}\Delta t = \infty,$$

by [6, Example 5.60]. For equation (4.2), we have

$$\begin{aligned} p_* &= \liminf_{t\to\infty} t^{\gamma}\int_{\sigma(t)}^{\infty} p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}\Delta s \\ &= \beta\ \liminf_{t\to\infty} t^{\gamma}\int_{\sigma(t)}^{\infty}\frac{\Delta s}{s^{\gamma+1}}. \end{aligned}$$

But, by the Pötzsche chain rule

$$\left(-\frac{1}{t^{\gamma}}\right)^{\Delta} = \gamma\int_0^1\frac{1}{(t+h\mu(t))^{\gamma+1}}dh \leq \gamma\int_0^1\frac{1}{t^{\gamma+1}}dh = \frac{\gamma}{t^{\gamma+1}},$$

so we get that

$$p_* \geq \frac{\beta}{\gamma}\liminf_{t\to\infty}\left(\frac{t}{\sigma(t)}\right)^{\gamma} = \frac{\beta}{\gamma}l^{\gamma}.$$

So if

$$\beta > \frac{\gamma^{\gamma+1}}{l^{\gamma(\gamma+1)}(\gamma+1)^{\gamma+1}},$$

then (2.7) and (2.8) hold and we have by Theorem 2.3 that (4.2) is oscillatory if $\beta > \dfrac{\gamma^{\gamma+1}}{l^{\gamma(\gamma+1)}(\gamma+1)^{\gamma+1}}$.

Note that in the case $\mathbb{T} = \mathbb{R}$, $\tau(t) = t$ and $\gamma = 1$, we get that $l = 1$ and we see that $\beta > \dfrac{1}{4}$ which is the sharp condition for the Euler–Cauchy differential equation to be oscillatory (see [1] for related results for the delay case). Also, note that the results by Agarwal et al. [2] and Thandapani et al. [15] cannot be applied to equation (4.2) in the cases of differential and difference equations.

Example 4.3. Consider the half-linear delay dynamic equation

$$\left(t^{\gamma}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{t^{\gamma-1}}{\tau^{\gamma}(t)} x^{\gamma}(\tau(t)) = 0, \tag{4.3}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $0 < \gamma \leq 1$ is the quotient of odd positive integers and $\tau(t) \leq t$. Here $p(t) = \dfrac{t^{\gamma-1}}{\tau^{\gamma}(t)}$ and $r(t) = t^{\gamma}$. It is clear that condition (2.1) is satisfied since

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^{1-\gamma}} = \infty, \quad \text{for} \quad 0 < \gamma \leq 1,$$

by [6, Example 5.60]. As in Example 4.1, it is clear that condition (2.2) holds. To apply Theorem 2.5, it remains to prove that condition (2.18) holds. To see this note that

$$\limsup_{t\to\infty} \frac{t^{\gamma}}{r(t)} \int_t^{\infty} p(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} \Delta s = \limsup_{t\to\infty} \int_t^{\infty} \frac{\Delta s}{s} = \infty.$$

Then, by Theorem 2.5 every solution of (4.3) is oscillatory.

Example 4.4. Consider the half-linear delay dynamic equation

$$\left(t^{\gamma+1}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \left(\frac{\sigma(s)}{\tau(s)}\right)^{\gamma} x^{\gamma}(\tau(t)) = 0, \tag{4.4}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $0 < \gamma \leq 1$ is the quotient of odd positive integers. In this case $p(t) = \left(\dfrac{\sigma(s)}{\tau(s)}\right)^{\gamma}$ and $r(t) = t^{\gamma+1}$. It is clear that (2.1) holds. Also

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^{\frac{\gamma+1}{\gamma}}} < \infty, \quad 0 < \gamma \leq 1,$$

for those time scales $[t_0, \infty)_{\mathbb{T}}$, where $\displaystyle\int_{t_0}^{\infty} \frac{1}{t^p} \Delta t < \infty$ when $p > 1$, and hence (2.19) holds for such time scales. The condition $\displaystyle\int_{t_0}^{\infty} \frac{1}{t^p} \Delta t < \infty$ when $p > 1$ holds for many time scales (see [6, Theorems 5.64 and 5.65], and see [6, Example 5.63] where this result does not hold). To see that (2.20) holds note that

$$\begin{aligned}\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t}p(s)\Delta s\right]^{\frac{1}{\gamma}}\Delta t &= \int_{t_0}^{\infty}\left[\frac{1}{t^{\gamma+1}}\int_{t_0}^{t}\left(\frac{\sigma(s)}{\tau(s)}\right)^{\gamma}\Delta s\right]^{\frac{1}{\gamma}}\Delta t\\ &\geq \int_{t_0}^{\infty}\left[\frac{1}{t^{\gamma+1}}\int_{t_0}^{t}\Delta s\right]^{\frac{1}{\gamma}}\Delta t\\ &= \int_{t_0}^{\infty}\left[\frac{t-t_0}{t^{\gamma+1}}\right]^{\frac{1}{\gamma}}\Delta t.\end{aligned}$$

We can find $0<k<1$ such that $t-t_0>kt$, for $t\geq t_k>t_0$. Therefore, we get

$$\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t}p(s)\Delta s\right]^{\frac{1}{\gamma}}\Delta t \geq k^{\frac{1}{\gamma}}\int_{t_k}^{\infty}\left[\frac{1}{t^{\gamma+1}}\int_{t_k}^{t}\Delta s\right]^{\frac{1}{\gamma}}\Delta t = k^{\frac{1}{\gamma}}\int_{t_k}^{\infty}\frac{1}{t}\Delta t=\infty.$$

To apply Theorem 2.6, it remains to prove that the condition (2.3) holds. To see this note that if $\delta(t)=1$, then

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\delta(s)p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}-\frac{r(s)\left(\left(\delta^{\Delta}(s)\right)_+\right)^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}}\right]\Delta s=\int_{t_0}^{\infty}\Delta t=\infty.$$

We conclude that if $[t_0,\infty)_{\mathbb{T}}$ is a time scale where $\int_{t_0}^{\infty}\frac{1}{t^p}\Delta t<\infty$ when $p>1$, then, by Theorem 2.6, every solution of (4.4) is oscillatory or converges to zero.

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