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Bounded Solutions of Dynamic Equations on Time Scales

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Abstract

In this paper we discuss the asymptotic behavior of solutions of a dynamic equation

$$u^{\Delta}(t) = f(t, u(t)),$$

where $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ and $\mathbb{T}$ is a time scale. We formulate a principle which gives the guarantee that the graph of at least one solution of above mentioned equation stays in the prescribed domain. This principle uses the idea of the retraction method and is a suitable tool for investigating the asymptotic behavior of solutions of dynamic equations. This is illustrated by an example.

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1. Introduction

1.1. The Time Scale Calculus

Stefan Hilger initiated in [8, 9] the calculus of time scales in order to create a theory that unifies discrete and continuous analysis. We give some necessary definitions and theorems of the time scales calculus utilized in the sequel. This theoretical background is taken from [2].

Definition 1.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.

Definition 1.2. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma\colon \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} \colon s > t\},$$

while the *backward jump operator* $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} \colon s < t\}.$$

Definition 1.3. A point t is called *right-scattered* if $\sigma(t) > t$ while if $\rho(t) < t$ we say that t is *left-scattered*. The points that are right-scattered and left-scattered at the same time are called *isolated*.

Definition 1.4. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then a point t is called *right-dense* and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. The points that are right-dense and left-dense at the same time are called *dense*.

Definition 1.5. The function $\mu\colon \mathbb{T} \to [0, \infty)$ defined by $\mu(t) := \sigma(t) - t$ is called the *graininess function*.

We will need a set $\mathbb{T}^\kappa$ derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Thus

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

Definition 1.6. Assume $u\colon \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. We say that u is *delta differentiable* on $\mathbb{T}^\kappa$ if the limit

$$u^\Delta(t) = \lim_{s \to t,\, s \in \mathbb{T}} \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s}$$

exists for all $t \in \mathbb{T}^\kappa$. We call $u^\Delta(t)$ the *delta derivative* of u at t.

Definition 1.7. A function $u\colon \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits are finite at the left-dense points in $\mathbb{T}$.

Definition 1.8. Let $\mathbb{T}$ be a time scale. A function $f\colon \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is called

(i) *rd-continuous*, if g defined by $g(t) = f(t, u(t))$ is rd-continuous for any continuous function $u\colon \mathbb{T} \to \mathbb{R}$;

(ii) *bounded* on a set $S \subset \mathbb{T} \times \mathbb{R}$, if there exists a constant $M > 0$ such that

$$|f(t,u)| \le M \quad \text{for all} \quad (t,u) \in S;$$

(iii) *Lipschitz continuous* on a set $S \subset \mathbb{T} \times \mathbb{R}$, if there exists a constant $L > 0$ such that

$$|f(t,u_1) - f(t,u_2)| \le L|u_1 - u_2| \quad \text{for all} \quad (t,u_1), (t,u_2) \in S.$$

Theorem 1.9. (Local Existence and Uniqueness) Let $\mathbb{T}$ be a time scale, $t_* \in \mathbb{T}$, $u_0 \in \mathbb{R}$, $a > 0$ is with $\inf \mathbb{T} \le t_* - a$, $\sup \mathbb{T} \ge t_* + a$ and put

$$I_a = [t_* - a, t_* + a] \quad \textit{and} \quad U_m = \{u \in \mathbb{R}\colon |u - u_0| \le m\}$$

with $m > 0$. Suppose that $f\colon I_a \times U_m \to \mathbb{R}$ is rd-continuous, bounded (with bound $M > 0$), and Lipschitz continuous (with $L > 0$). Then the initial value problem

$$u^{\Delta} = f(t,u), \tag{1.1}$$

$$u(t_*) = u_0 \tag{1.2}$$

has exactly one solution on $[t_* - \alpha, t_* + \alpha]$, where

$$\alpha = \min\left\{a, \frac{m}{M}, \frac{1-\varepsilon}{L}\right\} \quad \text{for some } \varepsilon \in (0,1).$$

If t_* is right-scattered and $\alpha < \mu(t_*)$, then a unique solution exists on the interval $[t_* - \alpha, \sigma(t_*)]$.

Remark 1.10. We recall that the function u is called the *solution* of (1.1), (1.2) if it satisfies (1.2) and the equation (1.1) holds on $[t_* - \alpha, t_* + \alpha]$ and if t_* is right-scattered and $\alpha < \mu(t_*)$ on the interval $[t_* - \alpha, \sigma(t_*)]$. It is easy to show that, if Theorem 1.9 holds, then the solution of the initial value problem (1.1), (1.2) depends continuously on the initial data.

1.2. Problem Under Consideration

In the sequel, we will assume that the time scale has the following properties: $\mathbb{T} \subset [t_0, \infty)$, $t_0 \in \mathbb{R}$ and $t_0 \in \mathbb{T}$. We note that Theorem 1.9 with $t_* = t_0$ can be modified for a right-hand neighborhood of the point t_* if necessary. Let $b,\ c\colon \mathbb{T} \to \mathbb{R}$ be delta differentiable functions such that $b(t) < c(t)$ for each $t \in \mathbb{T}$. We define a set $\Omega \subset \mathbb{T} \times \mathbb{R}$ as

$$\Omega := \{(t,u)\colon t \in \mathbb{T}, u \in \omega(t)\} \quad \text{with} \quad \omega(t) := \{u\colon b(t) < u < c(t)\}.$$

Then the closure $\overline{\Omega}$ equals

$$\overline{\Omega} := \{(t,u)\colon t \in \mathbb{T}, u \in \overline{\omega}(t)\} \quad \text{with} \quad \overline{\omega}(t) = \{u\colon b(t) \le u \le c(t)\}$$

and the boundary $\partial\Omega$ is

$$\partial\Omega := \{(t,u)\colon t \in \mathbb{T}, u \in \partial\omega(t)\} \text{ with } \partial\omega(t) := \{u\colon u = b(t) \text{ or } u = c(t)\}.$$

Let us consider the dynamic equation

$$u^{\Delta} = f(t,u) \tag{1.3}$$

where $f\colon \mathbb{T} \times \mathbb{R} \to \mathbb{R}$. Throughout we will assume that, for every fixed $t \in \mathbb{T}$, f is continuous with respect to the second coordinate. Moreover, for every fixed nonisolated point $t \in \mathbb{T}$, let $S(t) \subset \mathbb{T} \times \mathbb{R}$ be a closed set such that $[t-a, t+a] \cap \overline{\Omega} \subset S(t)$ for an $a > 0$, $\inf \mathbb{T} \le t - a$ and $\sup \mathbb{T} \ge t + a$ such that f is rd-continuous, bounded and Lipschitz continuous on $S(t)$; if $t = t_0$ is nonisolated, we define the set $S(t_0)$ in a similar way. This condition says that by Theorem 1.9 every initial value problem (1.3), (1.4) with

$$u(t) = u^* \in \overline{\omega}(t) \tag{1.4}$$

has exactly one solution on an interval $[t-\alpha, t+\alpha]$, $\alpha > 0$. It is also easy to show that the solution of the initial value problem (1.3), (1.4) with $t = t_0 \in \mathbb{T}$ has a unique solution, which depends continuously on the initial value $u^* \in \overline{\omega}(t_0)$.

Our aim is to establish the sufficient conditions to the right-hand side of the equation (1.3) in order to guarantee the existence of at least one solution of (1.3) defined on $\mathbb{T}$ such that $(t, u(t)) \in \Omega$ for each $t \in \mathbb{T}$. The main result generalizes some previous results of the authors concerning the asymptotic behavior of solutions of discrete equations (see, e.g., [5, 7]).

1.3. Points of Strict Egress

We define the auxiliary functions

$$B(t,u) := -u + b(t), \quad C(t,u) := \ u - c(t)$$

on $\mathbb{T} \times \mathbb{R}$ and divide the boundary $\partial\Omega$ into two disjoint subsets

$$\Omega_B := \{(t,u) : t \in \mathbb{T}, B(t,u) = 0\}$$

and

$$\Omega_C := \{(t,u) : t \in \mathbb{T}, C(t,u) = 0\}.$$

Definition 1.11. A point $M = (t, b(t)) \in \Omega_B$ is called a point of strict egress for the set Ω with respect to the equation (1.3) if

$$f(t, b(t)) < b^{\Delta}(t). \tag{1.5}$$

A point $M = (t, c(t)) \in \Omega_C$ is called a point of strict egress for the set Ω with respect to the equation (1.3) if

$$f(t, c(t)) > c^{\Delta}(t). \tag{1.6}$$

Remark 1.12. The geometrical meaning of the point of strict egress is evident. If a point $(\widetilde{t}, b(\widetilde{t})) \in \Omega_B$ is a point of strict egress for the set Ω with respect to (1.3) and u is a (unique) solution of (1.3) satisfying $u(\widetilde{t}) = b(\widetilde{t})$, then, due to (1.5),

$$\big(u(\widetilde{t}) - b(\widetilde{t})\big)^{\Delta} = f(\widetilde{t}, b(\widetilde{t})) - b^{\Delta}(\widetilde{t}) < 0.$$

From Definition 1.6 of delta derivative we get $u(t) - b(t) < 0$ (or $(t, u(t)) \notin \overline{\Omega}$) for $t \in (\widetilde{t}, \widetilde{t} + \delta) \cap \mathbb{T}$ with a small positive δ if $\widetilde{t}$ is a right-dense point and for $t = \sigma(\widetilde{t})$ if $\widetilde{t}$ is right-scattered. By analogy, if $(\widetilde{t}, c(\widetilde{t})) \in \Omega_C$ is a point of strict egress for the set Ω with respect to (1.3) and u is a (unique) solution of (1.3) satisfying $u(\widetilde{t}) = c(\widetilde{t})$, then, due to (1.6), $u(t) - c(t) > 0$ (or $(t, u(t)) \notin \overline{\Omega}$) for $t \in (\widetilde{t}, \widetilde{t} + \delta) \cap \mathbb{T}$ with a small positive δ if $\widetilde{t}$ is a right-dense point and for $t = \sigma(\widetilde{t})$ if $\widetilde{t}$ is right-scattered.

Definition 1.13. [10] If $A \subset B$ are subsets of a topological space and $\pi : B \to A$ is a continuous mapping from B onto A such that $\pi(p) = p$ for every $p \in A$, then π is said to be a *retraction* of B onto A. When a retraction of B onto A exists, A is called a *retract* of B.

2. Existence Theorem

The proof of the following theorem is based on the retract method, which is well known for ordinary differential equations and goes back to Ważewski [11].

The idea of the proof is simple: we suppose that the statement of the theorem is not valid. Then it is possible to prove that there exists a retraction of a segment $B := [\alpha, \beta]$ with $\alpha < \beta$ onto two-point set $A := \{\alpha, \beta\}$. But it is well known that the boundary of a nonempty (closed) interval cannot be its retract (see [3]). So, in our case, such a retractive mapping cannot exist because it is incompatible with continuity.

For discrete systems and equations on discrete time scales with unique graininess function an extension of the retract method is given, e.g., in [1, 4–7].

Below we will assume that the function f, except for the indicated conditions, satisfies all the assumptions given in Part 1.2.

Theorem 2.1. Let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$. Let $b,\ c : \mathbb{T} \to \mathbb{R}$ be delta differentiable functions on $\mathbb{T}$ such that $b(t) < c(t)$ for each $t \in \mathbb{T}$. If, moreover, every point $M \in \Omega_B \cup \Omega_C$ is the point of strict egress for the set Ω with respect to the equation (1.3), then there exists a value $u^* \in (b(t_0), c(t_0))$ such that the initial problem

$$u(t_0) = u^*, \tag{2.1}$$

defines a solution $u = u^*(t)$ of the equation (1.3) satisfying

$$b(t) < u^*(t) < c(t) \tag{2.2}$$

for every $t \in \mathbb{T}$.

Proof. Without any special comment, throughout the proof, we use the property that the initial value problem in question has a unique solution and the property of continuous dependence of solutions on their initial data. Suppose now that a value u^* satisfying the inequality

$$b(t_0) < u^* < c(t_0)$$

and generating the solution

$$u = u^*(t), \;\; u(t_0) = u^*$$

which satisfies (2.2) for any $t \in \mathbb{T}$ does not exist. This means that, for any u_0 satisfying the inequality

$$b(t_0) < u_0 < c(t_0), \tag{2.3}$$

there exists a $t^0 \in \mathbb{T}$, $t^0 > t_0$ such that, for a corresponding solution $u = u^0(t)$ of the initial problem $u^0(t_0) = u_0$, we have

$$(t^0, u^0(t^0)) \notin \Omega \tag{2.4}$$

and, for all $t \in [t_0, t^0) \cap \mathbb{T}$,

$$(t, u^0(t)) \in \Omega. \tag{2.5}$$

The properties (2.4), (2.5) are a consequence of Theorem 1.9, applied to the initial problem (1.3), (2.1) with the property (2.3).

Now we construct the desired retraction. Let us define auxiliary mappings P_1, P_2 and P_3. For $(t_0, u_0) \in [b(t_0), c(t_0)]$ we define

$$P_1\colon (t_0, u_0) \to (t^0, u^0(t^0))$$

where the value t^0 was defined above for $u_0 \in \omega(t_0)$ and if $u^0 = b(t_0)$ or $u^0 = c(t_0)$, we put $t^0 = t_0$,

$$P_1\colon (t_0, b(t_0)) \to (t^0, u^0(t^0)) = (t_0, b(t_0))$$

and

$$P_1\colon (t_0, c(t_0)) \to (t^0, u^0(t^0)) = (t_0, c(t_0));$$

$$P_2\colon (t^0, u^0(t^0)) \to \begin{cases} (t^0, c(t^0)) & \text{if} \quad u^0(t^0) \geq c(t^0), \\ (t^0, b(t^0)) & \text{if} \quad u^0(t^0) \leq b(t^0), \end{cases}$$

and, for $(t^0, \tilde{u}) \in \partial\Omega$,

$$P_3\colon (t^0, \tilde{u}) \to \begin{cases} (t_0, c(t_0)) & \text{if} \quad \tilde{u} = c(t^0), \\ (t_0, b(t_0)) & \text{if} \quad \tilde{u} = b(t^0). \end{cases}$$

We will show that the composite mapping

$$P\colon (t_0, u_0) \to \{(t_0, b(t_0)), (t_0, c(t_0))\} = \partial\omega(t_0),$$

where

$$P := P_3 \circ P_2 \circ P_1$$

and $(t_0, u_0) \in [b(t_0), c(t_0)]$ is continuous with respect to the second coordinate u_0 of the point (t_0, u_0).

The definition of the mapping P implies that only two resulting points are possible, namely, either $P(t_0, u_0) = (t_0, c(t_0))$ or $P(t_0, u_0) = (t_0, b(t_0))$.

We consider the first possibility, i.e., $P(t_0, u_0) = (t_0, c(t_0))$. Let $u_0 \in \omega(t_0)$. Then

$$P_1(t_0, u_0) = (t^0, u^0(t^0)),\ (t^0, u^0(t^0)) \notin \Omega \ \text{ and } \ u^0(t^0) \geq c(t^0).$$

Let $u^0(t^0) > c(t^0)$. Then $\rho(t^0) < t^0$ and the continuity of the mapping P_1 is obvious. Indeed, if

$$u^{0,\varepsilon}(t_0) = u_0 + \varepsilon \tag{2.6}$$

is an initial problem defining solution $u = u^{0,\varepsilon}(t)$ and ε is a sufficiently small number, then, due to the property of continuous dependence of solutions on their initial data, $t^{0,\varepsilon} = t^0$ and

$$P_1(t_0, u_0 + \varepsilon) = (t^{0,\varepsilon}, u^{\varepsilon}(t^{0,\varepsilon})) = (t^0, u^{\varepsilon}(t^0)).$$

Consequently $P(t_0, u_0 + \varepsilon) = (t_0, c(t_0))$.

Let $u^0(t^0) = c(t^0)$. By assumption of the theorem, every boundary point of $\partial\Omega$ is a point of strict egress for the set Ω with respect to the equation (1.3). Then, for the solution $u = u^{0,\varepsilon}(t)$ defined by (2.6), we have

$$P_1(t_0, u_0 + \varepsilon) = (t^{0,\varepsilon}, u^{\varepsilon}(t^{0,\varepsilon}))$$

either with $u^{\varepsilon}(t^{0,\varepsilon}) > c(t^{0,\varepsilon})$ or with $t^{0,\varepsilon} = t^0$, $u^{\varepsilon}(t^{0,\varepsilon}) = c(t_0)$. (We do not describe all the possibilities for occurrence of the first or of the second alternative.) In both cases we get $P(t_0, u_0 + \varepsilon) = (t_0, c(t_0))$ again.

We proceed similarly if $(t_0, u_0) = (t_0, c(t_0))$. So, the mapping P is continuous in the considered case.

We proceed analogously in the case $P(t_0, u_0) = (t_0, b(t_0))$ too.

The continuity of P was proved for $b(t_0) \leq u_0 \leq c(t_0)$. Thus the desired retraction is realized by P because the mapping

$$[b(t_0), c(t_0)] \quad \xrightarrow{P} \quad \{b(t_0), c(t_0)\}$$

is continuous and

$$\{b(t_0)\} \ \xrightarrow{P} \ \{b(t_0)\}, \qquad \{c(t_0)\} \ \xrightarrow{P} \ \{c(t_0)\},$$

i.e., the points $\{b(t_0)\}$, $\{c(t_0)\}$ are stationary.

In this situation we proved that there exists a retraction of the set $B := [b(t_0), c(t_0)]$ onto the two-point set $A := \{b(t_0), c(t_0)\}$ (see Definition 1.13). In regard of the above mentioned fact this is impossible. Our supposition is false and there exists an initial problem (2.1) such that the corresponding solution $u = u^*(t)$ satisfies the inequalities (2.2) for every $t \in \mathbb{T}$. The theorem is proved. ■

3. Example

Let us consider the dynamic equation

$$u^{\Delta} = f(t,u) := \frac{2}{1+t^2} \cdot u + \frac{\cos^2(tu)}{1+t^4} \tag{3.1}$$

defined for each $t \in \mathbb{T} \subset [t_0, \infty)$, where $t_0 \in \mathbb{R}$, $t_0 > 0$ and $t_0 \in \mathbb{T}$. With the aid of Theorem 2.1, we will show that there exists an initial value

$$u^* \in \left(-t_0^{-1}, t_0^{-1}\right) \tag{3.2}$$

such that the initial problem $u(t_0) = u^*$ defines a solution $u = u^*(t)$ of the dynamic equation (3.1) satisfying

$$|u^*(t)| < t^{-1} \tag{3.3}$$

for every $t \in \mathbb{T}$.

We define delta differentiable functions $b, c\colon \mathbb{T} \to \mathbb{R}$ satisfying $b(t) < c(t)$ for each $t \in \mathbb{T}$ as

$$b(t) := -t^{-1}, \quad c(t) := t^{-1}.$$

We will verify that every point $M \in \Omega_B \cup \Omega_C$ where

$$\begin{aligned} \Omega_B &:= \left\{(t,u) : t \in \mathbb{T}, -u - t^{-1} = 0\right\}, \\ \Omega_C &:= \left\{(t,u) : t \in \mathbb{T}, \quad u - t^{-1} = 0\right\} \end{aligned}$$

is a point of strict egress for the set

$$\Omega := \left\{(t,u) : t \in \mathbb{T}, -t^{-1} < u < t^{-1}\right\}$$

with respect to the dynamic equation (3.1). Indeed, for $M = (t,u) \in \Omega_B$,

$$f(t, b(t)) - b^{\Delta}(t) = -\frac{2}{1+t^2} \cdot \frac{1}{t} - \frac{\cos^2(1)}{1+t^4} - \left(-\frac{1}{t}\right)^{\Delta} < 0,$$

where $\left(-t^{-1}\right)^{\Delta}$ is positive because the function $-t^{-1}$ is increasing for every $t \in \mathbb{T}$. Inequality (1.5) holds. Similarly, for $M = (t,u) \in \Omega_C$,

$$f(t, c(t)) - c^{\Delta}(t) = \frac{2}{1+t^2} \cdot \frac{1}{t} + \frac{\cos^2(1)}{1+t^4} - \left(\frac{1}{t}\right)^{\Delta} > 0,$$

where $\left(t^{-1}\right)^{\Delta}$ is negative because the function t^{-1} is decreasing for every $t \in \mathbb{T}$ and inequality (1.6) is valid. In view of Definition 1.11, every point $M \in \Omega_B \cup \Omega_C$ is a point of strict egress for the set Ω. Therefore, all the assumptions of Theorem 2.1 hold and there exists an initial value u^* with property (3.2) such that the initial problem $u(t_0) = u^*$ defines a solution $u = u^*(t)$ of (3.1) satisfying inequality (3.3) for every $t \in \mathbb{T}$.

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