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On the Dynamics of a Rational Difference Equation, Part 1

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Abstract

We investigate the global stability character, the periodic nature, and the boundedness of solutions of a rational difference equation with nonnegative parameters and with nonnegative initial conditions. We also pose several open problems and conjectures which we are unable to resolve at this time.

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1. Introduction and Preliminaries

Consider the rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, \ldots \tag{1.1}$$

with nonnegative parameters and with arbitrary nonnegative initial conditions such that the denominator is always positive. Our goal here and in Part 2 of this paper is to investigate the global stability character, the periodic nature, and the boundedness of solutions of Eq. (1.1). We also pose several open problems and conjectures which we are unable to resolve at this time.

Eq. (1.1), which contains some interesting and some challenging special cases of second-order rational difference equations, also arises from the rational system in the plane:

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \gamma_1 y_n}{y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \quad n = 0, 1, \ldots \tag{1.2}$$

when we reduce it to a single equation. See [3].

If we allow one or more of the parameters in Eq. (1.1) to be zero, then Eq. (1.1) contains

$$(2^3 - 1) \times (2^3 - 1) = 49$$

special cases of equations with positive parameters. One can see that 19 of these special cases are trivial, linear, Riccati, or reducible to linear or Riccati equations. The remaining 30 special cases are investigated here and in Part 2 of the paper and they are listed, in normalized form, in Appendix A. The results which we obtained for the 30 cases, together with some challenging open problems and conjectures, appear in Parts 1 and 2 of the paper. Part 1 investigates Equations #1 through #12 and Part 2 investigates Equations #13 through #30. A brief summary of our results and some conjectures are given in Appendix A.

The following well-known result, which is needed for the local asymptotic stability of the equilibrium points of Eq. (1.1), gives necessary and sufficient conditions for the two roots of a quadratic equation to have modulus less than one. See [12].

Theorem 1.1. Assume that p and q are real numbers. Then a necessary and sufficient condition for both roots of the equation

$$\lambda^2 + p\lambda + q = 0$$

to lie inside the unit disk is

$$|p| < 1 + q < 2.$$

We also list four global attractivity results which are needed in our investigation. These results have straight forward extensions to difference equations of any order but we only need them here for second-order difference equations.

In the next two theorems we make use of the following notation associated with a function $f(z_1, z_2)$ which is monotonic in both arguments.

For each pair of numbers (m, M) and for each $i \in \{1, 2\}$, define

$$M_i(m, M) = \begin{cases} M, & \text{if f is increasing in } z_i \\ m, & \text{if f is decreasing in } z_i \end{cases}$$

and

$$m_i(m, M) = M_i(M, m).$$

Theorem 1.2. (Kulenovic-Ladas-Sizer [13] or [14]) Let $[a, b]$ be a closed and bounded interval of real numbers and let $f \in C([a, b]^2, [a, b])$ satisfy the following conditions:

1. $f(z_1, z_2)$ is monotonic in each of its arguments.
2. If (m, M) is a solution of the system

$$\left.\begin{aligned} M &= f(M_1(m, M), M_2(m, M)) \\ m &= f(m_1(m, M), m_2(m, M)) \end{aligned}\right\}, \tag{1.3}$$

then $M = m$.

Then the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, \ldots \tag{1.4}$$

has a unique equilibrium point $\bar{x}$ in $[a, b]$ and every solution of Eq. (1.4), with initial conditions in $[a, b]$, converges to $\bar{x}$.

Theorem 1.3. (Camouzis-Ladas [5] or [6]) Assume that $f \in C([0, \infty)^2, [0, \infty))$ and $f(z_1, z_2)$ is either strictly increasing in z_1 and z_2, or strictly decreasing in z_1 and z_2, or strictly increasing in z_1 and strictly decreasing in z_2. Furthermore, assume that for every

$$m \in (0, \infty) \quad \text{and} \quad M > m,$$

either

$$[f(M_1(m, M), M_2(m, M)) - M][f(m_1(m, M), m_2(m, M)) - m] > 0$$

or

$$f(M_1(m, M), M_2(m, M)) = M \quad \text{and} \quad f(m_1(m, M), m_2(m, M)) = m.$$

Then every solution of Eq. (1.4) which is bounded from above and from below by positive constants converges to a finite limit.

Theorem 1.4. (El-Metwally, Grove, Ladas, and Voulov [9] or [10]) Let I be an interval of real numbers and let $f \in C(I^{k+1}, I)$. Assume that the following three conditions are satisfied:

1. f is increasing in each of its arguments.

2. $f(z_1, \ldots, z_{k+1})$ is strictly increasing in each of the arguments $z_{i_1}, \ldots, z_{i_l}$ where $1 \leq i_1 < i_2 < \ldots < i_l \leq k+1$, and the arguments $i_1, i_2, \ldots, i_l$ are relatively prime.

3. Every point c in I is an equilibrium point of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots. \tag{1.5}$$

Then every solution of Eq. (1.5) has a finite limit.

The following powerful result holds when the function $f(z_1, z_2)$ in Eq. (1.4) is decreasing in z_1 and increasing in z_2. Note that this case of monotonicity is not included in the hypotheses of Theorem 1.3.

Theorem 1.5. (Camouzis-Ladas [4] or [6]) Let I be a set of real numbers and let

$$f : I \times I \to I$$

be a function $f(z_1, z_2)$ which decreases in z_1 and increases in z_2. Then for every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (1.4) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:

(i) They are both monotonically increasing.

(ii) They are both monotonically decreasing.

(iii) Eventually, one of them is monotonically increasing and the other is monotonically decreasing.

In the remaining part of this section we present three results which were motivated by our investigation of the character of solutions of the following special cases of Eq. (1.1):

#3, #4, #6, #10, #17, #19, #21.

See the corresponding sections in Parts 1 and 2 of the paper. See also [18, 19] and the references cited there in.

Theorem 1.6. Let I be a set of real numbers and let

$$f : I \times I \to I$$

be a function $f(z_1, z_2)$ which increases in both variables. Then for every solution $\{x_n\}_{n=-1}^{\infty}$ of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots, \tag{1.6}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:

(i) Eventually they are both monotonically increasing.

(ii) Eventually they are both monotonically decreasing.

(iii) One of them is monotonically increasing and the other is monotonically decreasing.

Proof. Assume that (iii) is not true for a solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (1.6). Then for some $N \geq 0$, either

$$x_{2N+1} \geq x_{2N-1} \quad \text{and} \quad x_{2N+2} \geq x_{2N} \tag{1.7}$$

or

$$x_{2N+1} \leq x_{2N-1} \quad \text{and} \quad x_{2N+2} \leq x_{2N}. \tag{1.8}$$

Assume that (1.7) holds. The case where (1.8) holds is similar and will be omitted. Then

$$x_{2N+3} = f(x_{2N+2}, x_{2N+1}) \geq f(x_{2N}, x_{2N-1}) = x_{2N+1}$$

and

$$x_{2N+4} = f(x_{2N+3}, x_{2N+2}) \geq f(x_{2N+1}, x_{2N}) = x_{2N+2}$$

and the proof is complete. ■

Theorem 1.7. Assume that the function $f \in C([a, \infty)^2, [a, \infty))$ increases in both variables and that the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{1.9}$$

has no equilibrium point in (a, ∞). Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (1.9) with $x_{-1}, x_0 \in (a, \infty)$. Then

$$\lim_{n\to\infty} x_n = \begin{cases} \infty & \text{if} \quad f(x,x) > x, \quad \text{for all} \quad x > a. \\ a & \text{if} \quad f(x,x) < x, \quad \text{for all} \quad x > a. \end{cases}$$

Proof.
Case 1:

$$f(x,x) > x, \quad \text{for all} \quad x > a.$$

Choose a number z_0 such that

$$a < z_0 \leq \min\{x_{-1}, x_0\}$$

and let $\{z_n\}_{n=0}^{\infty}$ be the unique solution of the first-order difference equation

$$z_{n+1} = f(z_n, z_n), \quad n = 0, 1, \ldots \tag{1.10}$$

with initial condition z_0. The key idea behind the proof in this case is that every nontrivial solution of Eq. (1.10) converges to infinity and that $\{x_n\}_{n=-1}^{\infty}$ is bounded from below by $\{z_n\}$. Indeed,

$$x_1 = f(x_0, x_{-1}) \geq f(z_0, z_0) = z_1 > z_0$$

$$x_2 = f(x_1, x_0) \geq f(z_0, z_0) = z_1 > z_0$$

$$x_3 = f(x_2, x_1) \geq f(z_1, z_1) = z_2 > z_1 > z_0,$$

$$\dots$$

from which the result follows because

$$\lim_{n\to\infty} z_n = \infty.$$

Case 2:

$$f(x, x) < x, \quad \text{for all} \quad x > a.$$

Choose a number z_0 such that

$$z_0 \geq \max\{x_{-1}, x_0\}$$

and define $\{z_n\}_{n=0}^{\infty}$ as in Case 1. The key idea behind the proof now is that every nontrivial solution of Eq. (1.10) converges to a and $\{x_n\}_{n=-1}^{\infty}$ is bounded from above by $\{z_n\}$. Indeed,

$$x_1 = f(x_0, x_{-1}) \leq f(z_0, z_0) = z_1 < z_0$$

$$x_2 = f(x_1, x_0) \leq f(z_0, z_0) = z_1 < z_0$$

$$x_3 = f(x_2, x_1) \leq f(z_1, z_1) = z_2 < z_1 < z_0,$$

$$\dots$$

from which it follows that

$$\overline{\lim_{n\to\infty}} x_n \leq \lim_{n\to\infty} z_n = a.$$

The proof is complete. ■

Theorem 1.8. Assume that $f \in C([0, \infty)^2, [0, \infty))$ increases in both variables and that the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots \tag{1.11}$$

has two consecutive equilibrium points $\bar{x}_1$ and $\bar{x}_2$, with $\bar{x}_1 < \bar{x}_2$. Also assume that either

$$f(x, x) > x, \quad \text{for} \quad \bar{x}_1 < x < \bar{x}_2 \tag{1.12}$$

or

$$f(x, x) < x, \quad \text{for} \quad \bar{x}_1 < x < \bar{x}_2. \tag{1.13}$$

Then every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (1.11) with initial conditions

$$x_{-1}, x_0 \in (\bar{x}_1, \bar{x}_2)$$

converges to one of the two equilibrium points, and more precisely the following is true:

$$\lim_{n\to\infty} x_n = \begin{cases} \bar{x}_1\,, & \text{if} \quad (1.13) \quad \text{holds,} \\ \\ \bar{x}_2\,, & \text{if} \quad (1.12) \quad \text{holds.} \end{cases}$$

Proof. The proof is similar to the proof of Theorem 1.7 and will be omitted. ■

Remark 1.9. Some special cases of System (1.2) involve competitive systems in the plane. For such systems there is a vast literature dealing with various features of the solutions. See for example [7, 11, 15–19] and the references cited there in. In this paper our methods and techniques are those that we have developed in our treatment of rational difference equations to understand the global character of solutions. See [1–6, 8–10, 12–14], together with the global stability results that we presented in this section to understand the dynamics of Eq. (1.1).

2. Equation #1:

$$x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1}}, \qquad n = 0, 1, \ldots. \tag{2.1}$$

For this equation we conjecture that every solution has a finite limit but we can only confirm it when

$$\alpha \leq 2.$$

Eq. (2.1) has a unique equilibrium $\bar{x}$, and $\bar{x}$ is the unique positive root of the cubic equation:

$$\bar{x}^3 + \bar{x} - \alpha = 0.$$

The characteristic equation of the linearized equation of Eq. (2.1) about $\bar{x}$ is

$$\lambda^2 + \frac{\alpha - \bar{x}}{\alpha}\lambda + \frac{\alpha - \bar{x}}{\alpha} = 0.$$

From this it follows by Theorem 1.1 that $\bar{x}$ is locally asymptotically stable for all values of the parameter α.

Clearly every solution of Eq. (2.1) is bounded and more precisely,

$$\frac{\alpha}{1+\alpha^2} \leq x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1}} \leq \alpha, \quad \text{for all} \quad n \geq 1.$$

The following result is now a consequence of Theorem 1.2 and the fact that if (m, M) is a solution of system (1.3), namely,

$$M = \frac{\alpha}{1+m^2} \quad \text{and} \quad m = \frac{\alpha}{1+M^2},$$

then

$$m = M$$

when

$$\alpha \leq 2.$$

Theorem 2.1. Assume that

$$0 < \alpha \leq 2.$$

Then the positive equilibrium of Eq. (2.1) is globally asymptotically stable.

Conjecture 2.2. Show that every positive solution of Eq. (2.1) has a finite limit.

See also Section 7 in Part 2.

3. Equation #2:

$$x_{n+1} = \frac{\alpha}{(1 + x_n)x_{n-1}}, \quad n = 0, 1, \ldots. \tag{3.1}$$

This equation has some similarities with **Lyness's** Equation,

$$x_{n+1} = \frac{\alpha + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{3.2}$$

which is gifted with the **invariant**:

$$(\alpha + x_{n-1} + x_n)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_n}\right) = \text{constant}, \quad \forall\, n \geq 0.$$

See [12]. Indeed, as for Eq. (3.2), Eq. (3.1) possesses an invariant, namely,

$$x_{n-1} + x_n + x_{n-1}x_n + \alpha\left(\frac{1}{x_{n-1}} + \frac{1}{x_n}\right) = \text{constant}, \quad \forall\, n \geq 0. \tag{3.3}$$

By using (3.3) it follows that every positive solution of Eq. (3.1) is bounded from above and from below by positive constants.

Eq. (3.1) has a unique positive equilibrium $\bar{x}$, and $\bar{x}$ is the unique positive root of the cubic equation

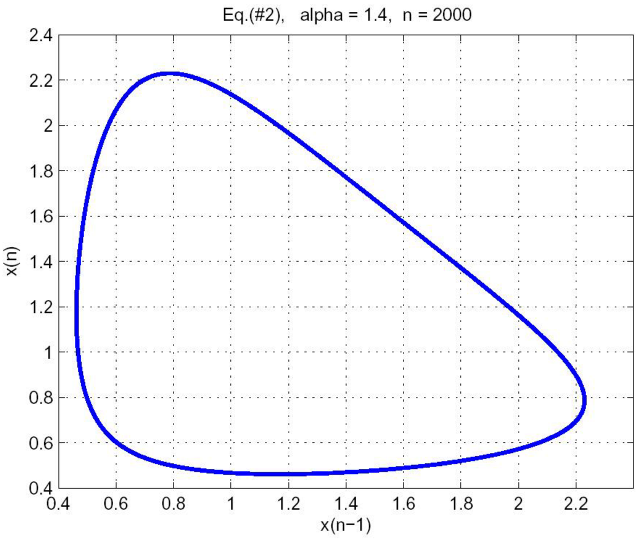
$$\bar{x}^3 + \bar{x}^2 - \alpha = 0.$$

The characteristic equation of the linearized equation of Eq. (3.1) about the equilibrium $\bar{x}$ is

$$\lambda^2 + \frac{\bar{x}}{1 + \bar{x}}\lambda + 1 = 0$$

which has two complex conjugate roots on $|\lambda| = 1$.

Conjecture 3.1. The (unique) positive equilibrium of Eq. (3.1) is stable (but not asymptotically stable).



Conjecture 3.2. No positive, non-equilibrium solution of Eq. (3.1) has a limit.

Open Problem 3.3.

(a) Determine all periodic solutions of Eq. (3.1).

(b) Is there a value of α for which every solution of Eq. (3.1) is periodic with the same period?

Open Problem 3.4. Assume that α is a real number. Determine the set G of real initial values x_{-1}, x_0 for which the equation

$$x_{n+1} = \frac{\alpha}{(1 + x_n)x_{n-1}}$$

is well defined for all $n \geq 0$, and investigate the character of solutions of Eq. (3.1) with $x_{-1}, x_0 \in G$.

Open Problem 3.5.

(a) Assume that $\{\alpha_n\}$ is a periodic sequence of positive real numbers. Are the positive solutions of the equation

$$x_{n+1} = \frac{\alpha_n}{(1 + x_n)x_{n-1}}, \qquad n = 0, 1, \ldots \tag{3.4}$$

bounded?

(b) Determine the periods of all periodic sequences $\{\alpha_n\}$ of positive real numbers for which Eq. (3.4) has an invariant.

4. Equation #3:

$$x_{n+1} = \frac{\beta x_n x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, \ldots. \tag{4.1}$$

The main result for this equation is the following:

Theorem 4.1. Every solution of Eq. (4.1) has a finite limit.

A solution of Eq. (4.1) with

$$x_{-1} x_0 = 0$$

is identically zero, for $n \geq 0$, and so in the sequel we will only consider positive solutions of Eq. (4.1).

Clearly

$$x_{n+1} < \beta, \quad \text{for all} \quad n \geq 0,$$

and so every solution of Eq. (4.1) is bounded.

It will also be useful to note that Eq. (4.1) has no prime period-two solutions.

Here

$$f(x, y) = \frac{\beta xy}{1 + xy}$$

is increasing in both variables and so by Theorem 1.6 the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of every solution of Eq. (4.1) should both converge to one and the same equilibrium point of Eq. (4.1).

In the remaining part of this section, we provide some additional details about the character of solutions of Eq. (4.1).

Zero is always an equilibrium point of Eq. (4.1) and it is locally asymptotically stable for all values of the parameter β.

Note that when

$$\beta < 2, \tag{4.2}$$

$\bar{x} = 0$ is the only equilibrium point of Eq. (4.1) and

$$f(x, x) = \frac{\beta x^2}{1 + x^2} < x, \quad \text{for all} \quad x > 0.$$

Hence by Theorem 1.7, with $a = 0$,

$$\lim_{n \to \infty} x_n = 0.$$

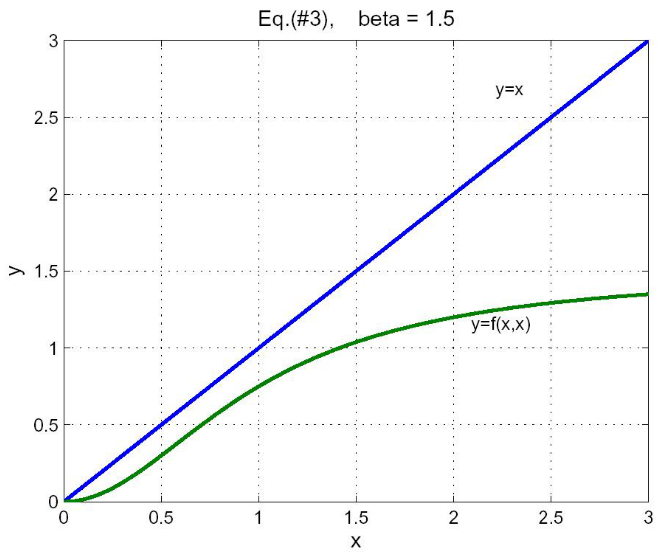
Therefore, when (4.2) holds, the zero equilibrium of Eq. (4.1) is globally asymptotically stable.

When

$$\beta = 2,$$

Eq. (4.1), in addition to the zero equilibrium, has the unique positive equilibrium

$$\bar{x} = 1.$$



This is a non-hyperbolic equilibrium point with characteristic roots (for the associated characteristic equation of the linearized equation about $\bar{x}$):

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -\frac{1}{2}.$$

By Theorems 1.6–1.8, the following statements are true for every non-equilibrium solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (4.1):

(i) If for some $N \geq 0$, $x_{N-1}, x_N \in [0, 1]$, then

$$x_n \in [0, 1], \quad \text{for all} \quad n \geq N,$$

and

$$\lim_{n\to\infty} x_n = 0.$$

(ii) If for some $N \geq 0$, $x_{N-1}, x_N \in [1, \infty)$, then

$$x_n \in [1, \infty), \quad \text{for all} \quad n \geq N,$$

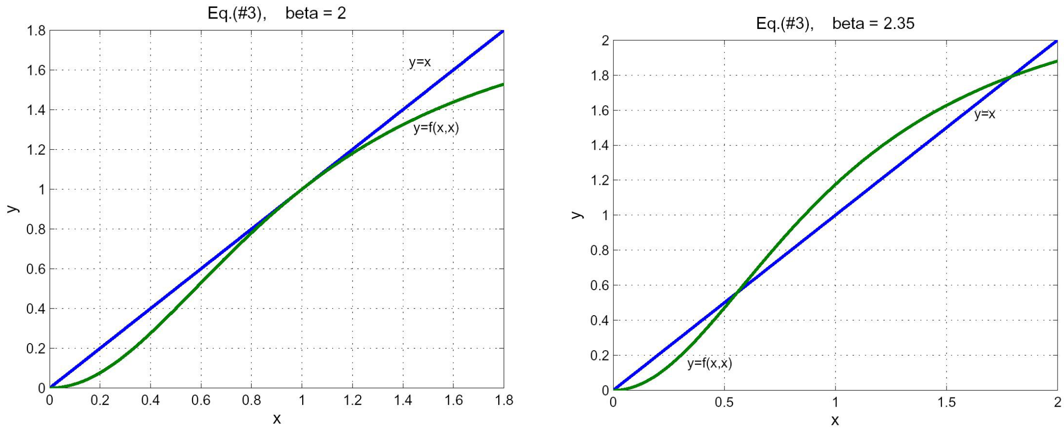
and

$$\lim_{n\to\infty} x_n = 1.$$

(iii) If the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ lie, one in the interval $[0, 1]$ and the other in $[1, \infty)$, then they must both converge monotonically to 1.

When

$$\beta > 2, \tag{4.3}$$



Eq. (4.1), in addition to the zero equilibrium, has the two positive equilibrium points:

$$\bar{x}_1 = \frac{\beta - \sqrt{\beta^2 - 4}}{2} \quad \text{and} \quad \bar{x}_2 = \frac{\beta + \sqrt{\beta^2 - 4}}{2}. \tag{4.4}$$

The characteristic equation of the linearized equation about a positive equilibrium $\{\bar{x}_i\}_{i=1,2}$ of Eq. (4.1) is

$$\lambda^2 - \frac{1}{\beta \bar{x}_i}\lambda - \frac{1}{\beta \bar{x}_i} = 0$$

and so by Theorem 1.1, $\bar{x}_1$ is unstable (saddle point) and $\bar{x}_2$ is locally asymptotically stable.

When (4.3) holds, the following is true for Eq. (4.1):

$$f(x,x) < x, \quad \text{for} \quad x \in (0, \bar{x}_1) \cup (\bar{x}_2, \infty)$$

and

$$f(x,x) > x, \quad \text{for} \quad x \in (\bar{x}_1, \bar{x}_2).$$

Therefore by Theorems 1.6–1.8, the following statements are true for every non-equilibrium solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (4.1):

(i) If for some $N \geq 0$, $x_{N-1}, x_N \in [0, \bar{x}_1]$, then

$$x_n \in [0, \bar{x}_1], \quad \text{for all} \quad n \geq N,$$

and

$$\lim_{n\to\infty} x_n = 0.$$

(ii) If for some $N \geq 0$, $x_{N-1}, x_N \in [\bar{x}_1, \bar{x}_2]$, then

$$x_n \in [\bar{x}_1, \bar{x}_2], \quad \text{for all} \quad n \geq N,$$

and

$$\lim_{n\to\infty} x_n = \bar{x}_2.$$

(iii) If for some $N \geq 0$, $x_{N-1}, x_N \in [\bar{x}_2, \infty)$, then

$$x_n \in [\bar{x}_2, \infty), \quad \text{for all} \quad n \geq N,$$

and

$$\lim_{n\to\infty} x_n = \bar{x}_2.$$

(iv) If the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ lie eventually, one in $[0, \bar{x}_1]$ and the other in $[\bar{x}_1, \bar{x}_2]$, then they both converge monotonically to $\bar{x}_1$, and if they lie eventually, one of them in $[\bar{x}_1, \bar{x}_2]$ and the other in $[\bar{x}_2, \infty)$, then they both converge monotonically to $\bar{x}_2$.

It should be mentioned that Eq. (4.1) cannot have a solution with the property that the subsequences of even $\{x_{2n}\}$ and odd $\{x_{2n+1}\}$ terms lie eventually, one of them in $[0, \bar{x}_1]$ and the other in $[\bar{x}_2, \infty)$. This is because, by Theorem 1.6 the two subsequences are eventually monotonic and also because every solution of Eq. (4.1) is bounded and Eq. (4.1) has no period-two solutions.

An interesting feature of Eq. (4.1), and also for Eqs. (5.1) and (19.1) (see Part 2), is that the **local stability of the zero equilibrium does not imply its global stability**.

5. Equation #4:

$$x_{n+1} = \frac{\beta x_n x_{n-1}}{1 + x_{n-1}}, \quad n = 0, 1, \ldots. \tag{5.1}$$

Here

$$f(x, y) = \frac{\beta x y}{1 + y}$$

is increasing in both variables and we can employ Theorems 1.6–1.8 to investigate the character of solutions of the equation.

Zero is always an equilibrium point of Eq. (5.1) and when

$$\beta > 1, \tag{5.2}$$

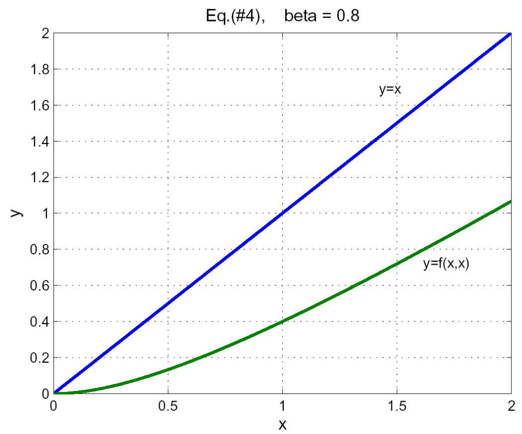
Eq. (5.1) also has the unique positive equilibrium

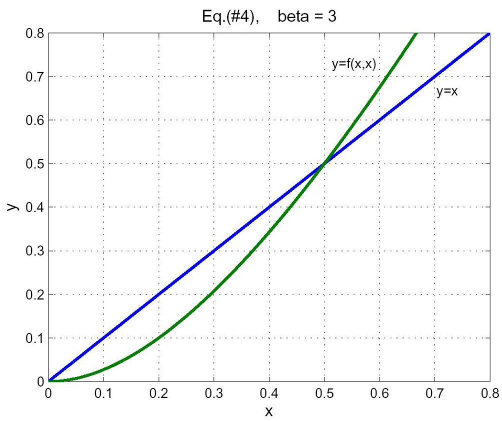
$$\bar{x} = \frac{1}{\beta - 1}.$$

The characteristic equation of the linearized equation of Eq. (5.1) about the zero equilibrium is

$$\lambda^2 = 0$$

and so $\bar{x} = 0$ is locally asymptotically stable for all values of the parameter β.





When (5.2) holds, the characteristic equation of the linearized equation of Eq. (5.1) about the positive equilibrium $\bar{x} = \dfrac{1}{\beta - 1}$ is

$$\lambda^2 - \lambda + \frac{1}{\beta} - 1 = 0.$$

From this it follows by Theorem 1.1 that $\bar{x}$ is unstable (saddle point).

It is important to note that the following identities for a solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (5.1)

$$x_{2n+1} = \beta\left(\frac{x_{2n-1}}{1 + x_{2n-1}}\right)x_{2n}, \quad \text{for all} \quad n \geq 0,$$

$$x_{2n+2} = \beta\left(\frac{x_{2n}}{1 + x_{2n}}\right)x_{2n+1}, \quad \text{for all} \quad n \geq 0,$$

imply that if one of the subsequences $\{x_{2n}\}$ or $\{x_{2n+1}\}$ has a finite limit, then so does the other.

For the long-term behavior of solutions of Eq. (5.1) we have the following result:

Theorem 5.1.

(a) Assume that

$$\beta \leq 1.$$

Then every solution of Eq. (5.1) converges to zero.

(b) Assume that

$$\beta > 1.$$

Then the following statements are true for every non-equilibrium solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (5.1):

(i) If for some $N \geq 0$, $x_{N-1}, x_N \in \left[0, \dfrac{1}{\beta - 1}\right]$, then

$$x_n \in \left[0, \frac{1}{\beta - 1}\right], \quad \text{for all} \quad n \geq N,$$

and

$$\lim_{n \to \infty} x_n = 0.$$

(ii) If for some $N \geq 0$, $x_{N-1}, x_N \in \left[\dfrac{1}{\beta - 1}, \infty\right)$, then

$$x_n \in \left[\frac{1}{\beta - 1}, \infty\right), \quad \text{for all} \quad n \geq N,$$

and

$$\lim_{n \to \infty} x_n = \infty.$$

(iii) If either

$$x_{2n} < \frac{1}{\beta - 1} < x_{2n+1}, \quad \text{for all} \;\; n \geq 0,$$

or

$$x_{2n} > \frac{1}{\beta - 1} > x_{2n+1}, \quad \text{for all} \;\; n \geq 0,$$

then the solution is bounded and

$$\lim_{n \to \infty} x_n = \frac{1}{\beta - 1}.$$

Proof. The proof follows by the preceding discussion and by employing Theorems 1.6–1.8. ■

6. Equation #5:

$$x_{n+1} = \frac{\gamma x_{n-1}}{1 + x_n x_{n-1}}, \quad n = 0, 1, \ldots. \tag{6.1}$$

When one of the initial conditions of a solution of Eq. (6.1) is zero, Eq. (6.1) reduces to the linear equation

$$x_{n+1} = \gamma x_{n-1}$$

with one initial condition equal to zero. If the other initial condition of a solution is ϕ, then the solution of the equation is

$$\ldots, 0, \; \phi, \; 0, \; \gamma\phi, \; 0, \; \gamma^2\phi, \; \ldots.$$

Hence the solution converges to zero when

$$\gamma < 1.$$

When

$$\gamma = 1,$$

the solution is the period-two sequence

$$\ldots,\ 0,\ \phi,\ 0,\ \phi,\ 0,\ \phi,\ \ldots$$

and when

$$\gamma > 1 \quad \text{and} \quad \phi > 0,$$

the solution is **unbounded**.

The key results for positive solutions of Eq. (6.1) are contained in the following lemma:

Lemma 6.1.

(a) Every positive solution of Eq. (6.1) is bounded.

(b) Eq. (6.1) has positive prime period-two solutions if and only if

$$\gamma > 1.$$

Proof. (a) When

$$\gamma \leq 1,$$

we have

$$x_{n+1} = \frac{\gamma x_{n-1}}{1 + x_n x_{n-1}} \leq x_{n-1}$$

and so the solutions of Eq. (6.1) are bounded. Now assume that

$$\gamma > 1$$

and let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (6.1). Choose a positive number m such that

$$x_{-1}, x_0 \in \left(m, \frac{\gamma - 1}{m}\right).$$

Then by using the monotonic character of the function:

$$f(x, y) = \frac{\gamma y}{1 + xy}$$

we find that

$$m = \frac{\gamma m}{1 + \frac{\gamma - 1}{m} m} < x_1 = \frac{\gamma x_{-1}}{1 + x_0 x_{-1}} < \frac{\gamma \frac{\gamma - 1}{m}}{1 + m \frac{\gamma - 1}{m}} = \frac{\gamma - 1}{m}$$

and so by induction

$$x_n \in \left(m, \frac{\gamma - 1}{m}\right), \quad \text{for all} \quad n \geq -1.$$

(b) The proof is straightforward and will be omitted. ■

One can easily see that when

$$\gamma > 1, \tag{6.2}$$

all positive prime period-two solutions of Eq. (6.1) are given by

$$\ldots, \ \phi, \ \psi, \ \ldots$$

with

$$\phi\psi = \gamma - 1 \quad \text{and} \quad \phi \neq \psi.$$

Also note that when (6.2) holds, the characteristic roots of the linearized equation of Eq. (6.1) about the positive equilibrium $\bar{x} = \sqrt{\gamma - 1}$ are

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = \frac{1}{\gamma} \in (0, 1).$$

Finally note that when (6.2) holds, then for every positive solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (6.1)

$$x_{n+1}x_n = \frac{\gamma x_n x_{n-1}}{1 + x_n x_{n-1}}, \quad n \geq 0,$$

which implies that

$$\lim_{n\to\infty} (x_n x_{n-1}) = \gamma - 1 > 0.$$

The main results for Eq. (6.1) are summarized in the following theorem.

Theorem 6.2.

(a) Assume that

$$\gamma < 1.$$

Then the zero equilibrium of Eq. (6.1) is globally asymptotically stable.

(b) Assume that

$$\gamma = 1.$$

Then every solution of Eq. (6.1) converges to a (not necessarily prime) period-two solution of the form

$$\ldots, \ 0, \ \phi, \ \ldots$$

with

$$\phi \geq 0.$$

(c) Assume that

$$\gamma > 1.$$

Then every positive solution of Eq. (6.1) converges to a (not necessarily prime) period-two solution of the form

$$\ldots, \ \phi, \ \psi, \ \ldots$$

with

$$\phi\psi = \gamma - 1.$$

On the other hand when

$$\gamma > 1 \quad \text{and} \quad x_{-1}x_0 = 0 \quad \text{with} \quad x_{-1} + x_0 > 0,$$

the solutions of Eq. (6.1) are **unbounded**.

Corollary 6.3. Eq. (6.1) has a period-two trichotomy which can be described as follows:

(i) Every solution of Eq. (6.1) converges to zero when

$$\gamma < 1.$$

(ii) Every solution of Eq. (6.1) converges to a (not necessarily prime) period-two solution when

$$\gamma = 1.$$

(iii) Eq. (6.1) has unbounded solutions when

$$\gamma > 1.$$

7. Equation #6:

$$x_{n+1} = \alpha + x_n x_{n-1}, \quad n = 0, 1, \ldots . \tag{7.1}$$

Here

$$f(x, y) = \alpha + xy$$

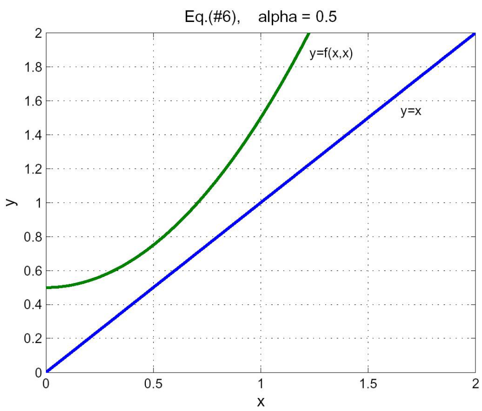
is increasing in both variables and we can employ Theorems 1.6–1.8 to investigate the character of solutions of the equation.

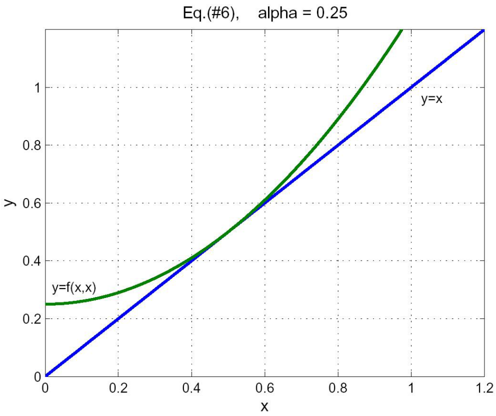
Eq. (7.1) has no equilibrium points when

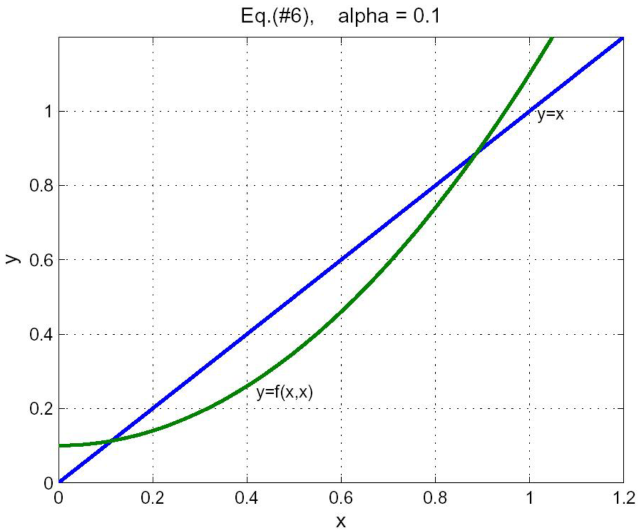
$$\alpha > \frac{1}{4}.$$

When

$$\alpha = \frac{1}{4},$$







Eq. (7.1) has the unique equilibrium point $\bar{x} = \dfrac{1}{2}$ which is unstable (saddle point).

Finally when

$$\alpha < \frac{1}{4},$$

Eq. (7.1) has the two positive equilibrium points:

$$\bar{x}_1 = \frac{1 - \sqrt{1 - 4\alpha}}{2} \quad \text{and} \quad \bar{x}_2 = \frac{1 + \sqrt{1 - 4\alpha}}{2}$$

of which $\bar{x}_1$ is locally asymptotically stable and $\bar{x}_2$ is unstable (saddle point).

It is important to note that the following identities for a solution of Eq. (7.1)

$$x_{2n+1} = \alpha + x_{2n}x_{2n-1}, \quad \text{for all} \quad n \geq 0,$$

$$x_{2n+2} = \alpha + x_{2n+1}x_{2n}, \quad \text{for all} \quad n \geq 0,$$

imply that if one of the subsequences $\{x_{2n}\}$ or $\{x_{2n+1}\}$ has a finite limit, then so does the other. Please note that Eq. (7.1) has no prime period-two solutions.

For the long-term behavior of solutions of Eq. (7.1) we have the following result:

Theorem 7.1.

(a) Assume that

$$\alpha > \frac{1}{4}.$$

Then every solution of Eq. (7.1) converges to ∞.

(b) Assume that

$$\alpha = \frac{1}{4}.$$

Then the following statements are true for every non-equilibrium solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (7.1):

(i) If for some $N \geq 0$, $x_{N-1}, x_N \in \left[0, \frac{1}{2}\right]$, then

$$x_n \in \left[0, \frac{1}{2}\right], \quad \text{for all } n \geq N,$$

and

$$\lim_{n\to\infty} x_n = \frac{1}{2}.$$

(ii) If for some $N \geq 0$, $x_{N-1}, x_N \in \left[\frac{1}{2}, \infty\right)$, then

$$x_n \in \left[\frac{1}{2}, \infty\right), \quad \text{for all } n \geq N,$$

and

$$\lim_{n\to\infty} x_n = \infty.$$

(iii) If either

$$x_{2n} < \frac{1}{2} < x_{2n+1}, \quad \text{for all } n \geq 0,$$

or

$$x_{2n+1} < \frac{1}{2} < x_{2n}, \quad \text{for all } n \geq 0,$$

then the solution is bounded and

$$\lim_{n\to\infty} x_n = \frac{1}{2}.$$

(c) Assume that

$$\alpha < \frac{1}{4}.$$

Then the following statements are true for every non-equilibrium solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (7.1):

(i) If for some $N \geq 0$, $x_{N-1}, x_N \in [0, \bar{x}_2]$, then

$$x_n \in [0, \bar{x}_2], \quad \text{for all } \; n \geq N,$$

and

$$\lim_{n\to\infty} x_n = \bar{x}_1.$$

(ii) If for some $N \geq 0$, $x_{N-1}, x_N \in [\bar{x}_2, \infty)$, then

$$x_n \geq \bar{x}_2, \quad \text{for all } \; n \geq N,$$

and

$$\lim_{n\to\infty} x_n = \infty.$$

(iii) If either

$$\bar{x}_1 < x_{2n} < \bar{x}_2 < x_{2n+1}, \quad \text{for all } \; n \geq 0,$$

or

$$\bar{x}_1 < x_{2n+1} < \bar{x}_2 < x_{2n}, \quad \text{for all } \; n \geq 0,$$

then the solution is bounded and

$$\lim_{n\to\infty} x_n = \bar{x}_2.$$

Proof. The proof follows by the preceding discussion and by employing Theorems 1.6–1.8. ■

8. Equation #7:

$$x_{n+1} = \beta + \frac{1}{x_n x_{n-1}}, \quad n = 0, 1, \ldots. \tag{8.1}$$

The change of variables:

$$x_n = \frac{1}{y_n\sqrt{\beta}}$$

transforms Eq. (8.1) to the difference equation (2.1). See Section 1.

Conjecture 8.1. Every positive solution of Eq. (8.1) has a finite limit.

Open Problem 8.2. Assume that β is a real number. Determine the set G of real initial values x_{-1}, x_0 for which the equation

$$x_{n+1} = \beta + \frac{1}{x_n x_{n-1}}$$

is well defined for all $n \geq 0$, and investigate the character of solutions of Eq. (8.1) with $x_{-1}, x_0 \in G$.

9. Equation #8:

$$x_{n+1} = \beta x_n + \frac{1}{x_{n-1}}, \quad n = 0, 1, \ldots. \tag{9.1}$$

The main result for this equation is the following:

Theorem 9.1.

(a) Eq. (9.1) has bounded solutions, if and only if

$$\beta < 1. \tag{9.2}$$

(b) When (9.2) holds, the equilibrium of Eq. (9.1) is globally asymptotically stable.

Proof. (a) Note that

$$x_{n+1} > \beta x_n$$

from which it follows that Eq. (9.1) has unbounded solutions for

$$\beta \geq 1.$$

On the other hand when (9.2) holds, we claim that every positive solution of Eq. (9.1) is bounded. Indeed if $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of Eq. (9.1) and if we choose positive numbers m and M such that

$$x_{-1}, x_0 \in [m, M] \quad \text{and} \quad mM = \frac{1}{1-\beta},$$

then

$$m = \frac{1}{(1-\beta)M} = \beta m + \frac{1}{M} \leq x_1 = \beta x_0 + \frac{1}{x_{-1}} \leq \beta M + \frac{1}{m} = \frac{1}{(1-\beta)m} = M$$

and inductively,

$$x_n \in [m, M], \quad \text{for all } n \geq -1$$

which establishes our claim.

(b) When (9.2) holds, Eq. (9.1) has the unique equilibrium

$$\bar{x} = \frac{1}{\sqrt{1-\beta}}.$$

The characteristic equation of the linearized equation of Eq. (9.1) about $\bar{x}$ is

$$\lambda^2 - \beta\lambda + (1-\beta) = 0.$$

From this it follows by Theorem 1.1 that $\bar{x}$ is locally asymptotically stable.

To complete the proof it remains to show that when (9.2) holds, every solution of Eq. (9.1) converges to the equilibrium $\bar{x}$. This follows now by applying Theorem 1.3. Indeed for every $m \in (0, \infty)$ and $M > m$,

$$\left(\beta M + \frac{1}{m} - M\right)\left(\beta m + \frac{1}{M} - m\right) = \left(\frac{(\beta-1)mM+1}{m}\right)\left(\frac{(\beta-1)mM+1}{M}\right)$$

and the hypotheses of Theorem 1.3 are satisfied. The proof is complete. ■

10. Equation #9:

$$x_{n+1} = \frac{\alpha + x_{n-1}}{x_n x_{n-1}}, \quad n = 0, 1, \ldots. \tag{10.1}$$

The main result for this equation is the following:

Theorem 10.1. The equilibrium of Eq. (10.1) is globally asymptotically stable.

Proof. Observe that

$$x_{n+1} = \frac{x_{n-2}(\alpha + x_{n-1})}{\alpha + x_{n-2}}, \quad n = 1, 2, \ldots \tag{10.2}$$

and that the function

$$f(x, y) = \frac{y(\alpha + x)}{\alpha + y}$$

is strictly increasing in both arguments and every point $\bar{x} \geq 0$ is an equilibrium point of Eq. (10.2). By Theorem 1.4 it follows that every solution of Eq. (10.2) converges to a finite limit.

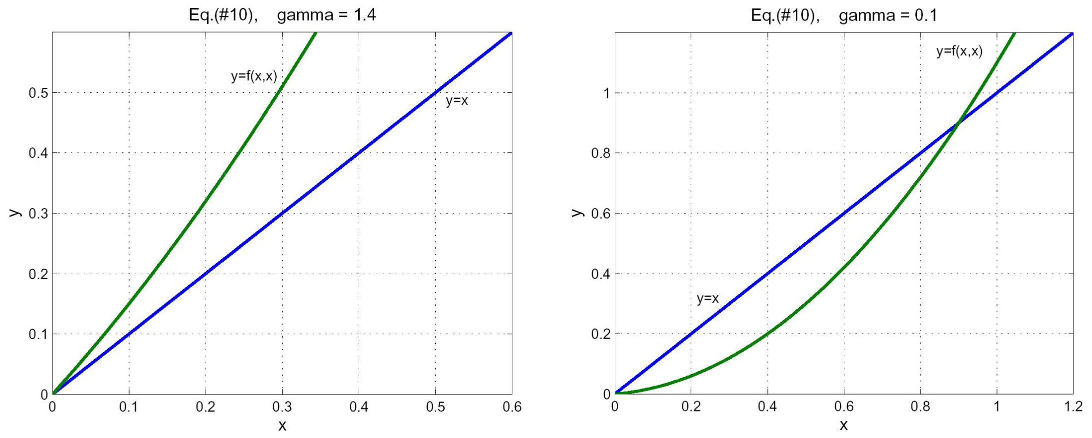
Also note that Eq. (10.1) has a unique positive equilibrium $\bar{x}$, and $\bar{x}$ is the unique solution of the cubic equation

$$\bar{x}^3 - \bar{x} - \alpha = 0.$$

The characteristic equation of the linearized equation of Eq. (10.1) about the equilibrium $\bar{x}$ is

$$\lambda^2 + \lambda + \frac{\alpha}{\bar{x}^3} = 0.$$

Hence by Theorem 1.1 the equilibrium $\bar{x}$ is locally asymptotically stable for all values of the parameter α. The proof is complete. ■



11. Equation #10:

$$x_{n+1} = \gamma x_{n-1} + x_n x_{n-1}, \qquad n = 0, 1, \ldots . \tag{11.1}$$

Here

$$f(x, y) = \gamma y + xy$$

is increasing in both variables and we can employ Theorems 1.6–1.8 to investigate the character of solutions of Eq. (11.1).

Zero is always an equilibrium solution of Eq. (11.1) and when

$$\gamma < 1,$$

Eq. (11.1) also has the unique positive equilibrium $\bar{x} = 1 - \gamma$.

The characteristic equation of the linearized equation of Eq. (11.1) about the zero equilibrium is

$$\lambda^2 - \gamma = 0.$$

From this it follows by Theorem 1.1 that $\bar{x} = 0$ is locally asymptotically stable when $\gamma < 1$.

The characteristic equation of the linearized equation of Eq. (11.1) about the positive equilibrium when $\gamma < 1$ is

$$\lambda^2 + (\gamma - 1)\lambda - 1 = 0.$$

From this it follows by Theorem 1.1 that $\bar{x}$ is unstable (saddle point).

It is important to note that, when $\gamma < 1$, the following identities for a solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (11.1)

$$x_{2n+1} = (\gamma + x_{2n})x_{2n-1}, \qquad \text{for all} \quad n \geq 0,$$

$$x_{2n+2} = (\gamma + x_{2n+1})x_{2n}, \qquad \text{for all} \quad n \geq 0,$$

imply that one of the subsequences $\{x_{2n}\}$ or $\{x_{2n+1}\}$ has a finite limit if and only if the other has too. Indeed this is clear if the limit of one of the subsequences is positive. On the other hand when one of the subsequences converges to zero, say

$$\{x_{2n}\} \to 0,$$

then eventually

$$x_{2n+1} = (\gamma + x_{2n})x_{2n-1} < x_{2n-1}$$

and so $\{x_{2n+1}\}$ also converges to a finite limit.

For the long-term behavior of solutions of Eq. (11.1) we have the following result:

Theorem 11.1.

(a) Assume that

$$\gamma \geq 1.$$

Then every solution of Eq. (11.1) converges to ∞.

(b) Assume that

$$\gamma < 1.$$

Then the following statements are true for every non-equilibrium solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (11.1):

(i) If for some $N \geq 0$, $x_{N-1}, x_N \in [0, 1-\gamma]$, then

$$x_n \in [0, 1-\gamma], \quad \text{for all } n \geq N,$$

and

$$\lim_{n\to\infty} x_n = 0.$$

(ii) If for some $N \geq 0$, $x_{N-1}, x_N \in [1-\gamma, \infty)$, then

$$x_n \in [1-\gamma, \infty), \quad \text{for all } n \geq N,$$

and

$$\lim_{n\to\infty} x_n = \infty.$$

(iii) If either

$$x_{2n} < 1-\gamma < x_{2n+1}, \quad \text{for all } n \geq 0,$$

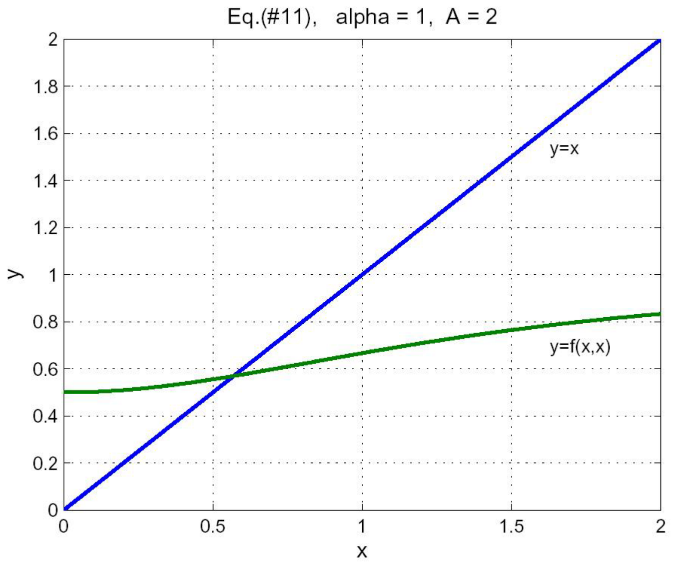
or

$$x_{2n+1} < 1-\gamma < x_{2n}, \quad \text{for all } n \geq 0,$$

then the solution is bounded and

$$\lim_{n\to\infty} x_n = 1-\gamma.$$

Proof. The proof follows by the preceding discussion and by employing Theorems 1.6–1.8. ■



12. Equation #11:

$$x_{n+1} = \frac{\alpha + x_n x_{n-1}}{A + x_n x_{n-1}}, \quad n = 0, 1, \ldots. \tag{12.1}$$

For this equation we conjecture that every solution has a finite limit but we can only confirm it when

$$A \geq \alpha. \tag{12.2}$$

Note that the function

$$f(x, y) = \frac{\alpha + xy}{A + xy}$$

increases in both variables when (12.2) holds and decreases in both variables when

$$A < \alpha. \tag{12.3}$$

Also every solution of Eq. (12.1) is bounded from above and from below by positive constants. Indeed for all $n \geq 0$,

$$\frac{\min\{\alpha, 1\}}{\max\{A, 1\}} < x_{n+1} = \frac{\alpha + x_n x_{n-1}}{A + x_n x_{n-1}} < \frac{\max\{\alpha, 1\}}{\min\{A, 1\}}. \tag{12.4}$$

When (12.2) holds, Eq. (12.1) has one, two, or three equilibrium points. In view of Theorems 1.6–1.8 and the fact that Eq. (12.1) has no prime period-two solutions, we see that every solution of Eq. (12.1) has a finite limit.

When (12.3) holds, Eq. (12.1) has a unique equilibrium point and we conjecture that it is globally asymptotically stable.

Conjecture 12.1. Assume that (12.3) holds. Show that the equilibrium of Eq. (12.1) is globally asymptotically stable.

13. Equation #12:

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1}}{1 + x_{n-1}}, \quad n = 0, 1, \ldots. \tag{13.1}$$

The main result for this equation is the following:

Theorem 13.1.

(a) Assume that

$$\beta \leq 1.$$

Then every solution of Eq. (13.1) has a finite limit.

(b) Assume that

$$\beta > 1. \tag{13.2}$$

Then Eq. (13.1) has unbounded solutions.

The proof of this theorem will be a consequence of the following lemmas.

Lemma 13.2. Assume that

$$\beta < 1.$$

Then every solution of Eq. (13.1) has a finite limit.

Proof. First we claim that every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (13.1) is bounded from above by $\frac{\alpha}{\beta}$.

Otherwise for some $N \geq 0$, which we can choose as large as we please,

$$x_{N+1} \geq \frac{\alpha}{\beta}.$$

Hence,

$$\frac{\alpha + \beta x_N x_{N-1}}{1 + x_{N-1}} \geq \frac{\alpha}{\beta}$$

which implies that

$$x_N > \frac{1}{\beta}\left(\frac{\alpha}{\beta}\right).$$

Similarly this implies that

$$x_{N-1} > \frac{1}{\beta^2}\left(\frac{\alpha}{\beta}\right)$$

and eventually this process leads to a contradiction.

One can see that the function

$$f(x, y) = \frac{\alpha + \beta xy}{1 + y}, \quad \text{for} \quad x < \frac{\alpha}{\beta}$$

increases in x and decreases in y.

The result follows by applying Theorem 1.2 in the interval $\left[0, \dfrac{\alpha}{\beta}\right]$. ∎

Lemma 13.3. Assume that
$$\beta = 1.$$
Then every solution of Eq. (13.1) converges monotonically to the equilibrium $\bar{x} = \alpha$.

Proof. Note that the following two identities hold from which the result follows:

$$x_{n+1} - \alpha = \frac{x_{n-1}}{1 + x_{n-1}}(x_n - \alpha), \quad \text{for all} \quad n \geq 0.$$

$$x_{n+1} - x_n = \frac{1}{1 + x_{n-1}}(\alpha - x_n), \quad \text{for all} \quad n \geq 0.$$

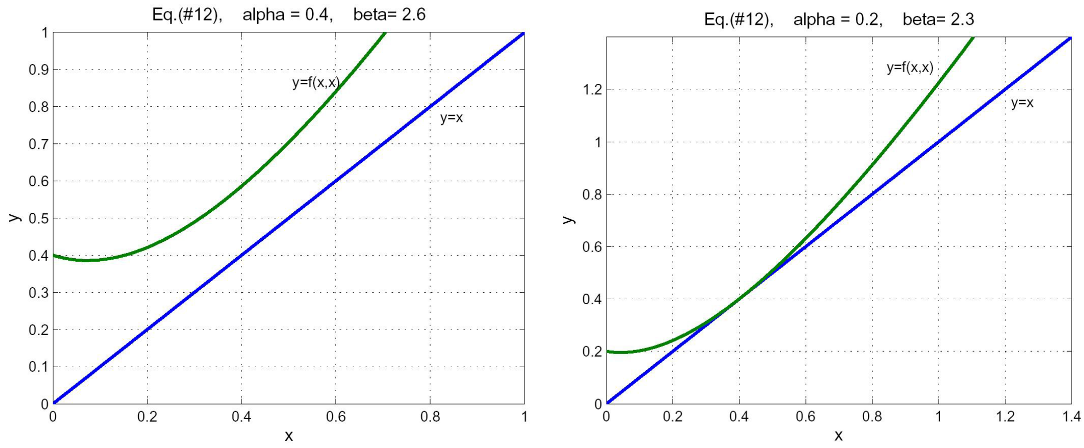
This completes the proof. ∎

Lemma 13.4. Assume that
$$\beta > 1.$$
Then every positive solution of Eq. (13.1) is bounded from below by $\dfrac{\alpha}{\beta}$.

Proof. Otherwise for some $N \geq 0$, which we can choose as large as we please,
$$x_{N+1} \leq \frac{\alpha}{\beta}.$$
Hence,
$$\frac{\alpha + \beta x_N x_{N-1}}{1 + x_{N-1}} \leq \frac{\alpha}{\beta}$$
which implies that
$$x_N < \frac{1}{\beta}\left(\frac{\alpha}{\beta}\right).$$
Similarly this implies that
$$x_{N-1} < \frac{1}{\beta^2}\left(\frac{\alpha}{\beta}\right)$$
and eventually this process leads to a contradiction. ∎

In view of Lemma 13.4, the function
$$f(x, y) = \frac{\alpha + \beta x y}{1 + y}$$
is increasing in both variables.

Part (b) of Theorem 13.1 is now a consequence of Theorems 1.6–1.8.



Note that Eq. (13.1) has no equilibrium points when

$$\beta > 1 + \frac{1}{4\alpha}, \tag{13.3}$$

exactly one equilibrium point, namely,

$$\bar{x} = 2\alpha$$

when

$$\beta = 1 + \frac{1}{4\alpha}, \tag{13.4}$$

and exactly the two equilibrium points

$$\bar{x}_1 = \frac{1 - \sqrt{1 - 4\alpha(\beta - 1)}}{2(\beta - 1)} \quad \text{and} \quad \bar{x}_2 = \frac{1 + \sqrt{1 - 4\alpha(\beta - 1)}}{2(\beta - 1)}$$

when

$$1 < \beta < 1 + \frac{1}{4\alpha}. \tag{13.5}$$

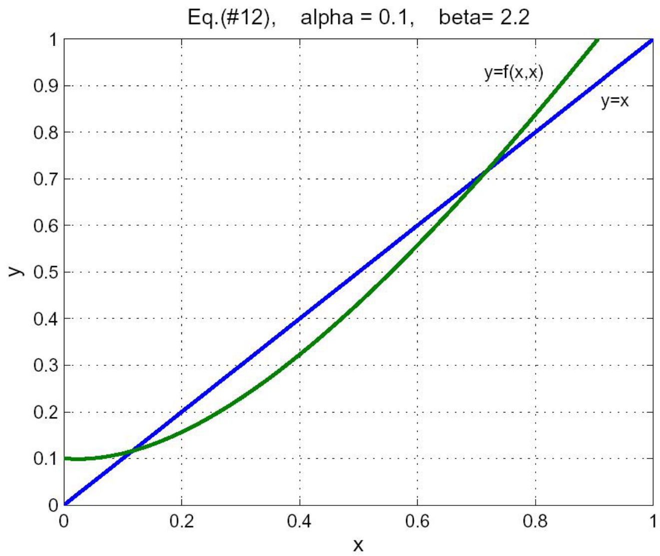
The long-term behavior of solutions of Eq. (13.1) resembles that of Eq. (7.1) and further details will be omitted.

Appendix A

Table of the Global Character of the 30 nontrivial special cases of

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}.$$

In this table we use the following abbreviations:



ESC	stands for "**every solution has a finite limit**".
∃US	stands for "**there exist unbounded solutions**".
ESB	stands for "**every solution of the equation is bounded**".
ESC*	stands for "**we conjecture that every solution has a finite limit**".
ESCP$_2$	stands for "**every solution of the equation converges to a not necessarily prime period-two solution**".

#1 : $x_{n+1} = \dfrac{\alpha}{1+x_n x_{n-1}}$;	**ESB** **ESC***; This conjecture has been confirmed for $\alpha \le 2$
#2 : $x_{n+1} = \dfrac{\alpha}{(1+x_n)x_{n-1}}$;	This equation possesses the **invariant**: $x_{n-1} + x_n + x_{n-1}x_n + \alpha\left(\dfrac{1}{x_{n-1}} + \dfrac{1}{x_n}\right) = \text{constant}$ **ESB**
#3 : $x_{n+1} = \dfrac{\beta x_n x_{n-1}}{1+x_n x_{n-1}}$;	**ESC** Local Stability $\not\Rightarrow$ Global Stability
#4 : $x_{n+1} = \dfrac{\beta x_n x_{n-1}}{1+x_{n-1}}$;	$\beta \le 1 \;\Rightarrow\;$ **ESC** $\beta > 1 \;\Rightarrow\;$ **∃US** See Theorem 5.1 Local Stability $\not\Rightarrow$ Global Stability

#5 :	$x_{n+1} = \dfrac{\gamma x_{n-1}}{1 + x_n x_{n-1}};$	$\gamma < 1 \Rightarrow$ **ESC** $\gamma = 1 \Rightarrow$ **ESCP$_2$** $\gamma > 1 \Rightarrow$ **∃US** Has Period-Two Trichotomy See Theorem 6.1
#6 :	$x_{n+1} = \alpha + x_n x_{n-1};$	**∃US** See Theorem 7.1
#7 :	$x_{n+1} = \beta + \dfrac{1}{x_n x_{n-1}};$	**ESB** **ESC***; This equation can be transformed to Eq. # 1
#8 :	$x_{n+1} = \beta x_n + \dfrac{1}{x_{n-1}};$	**ESB** $\Leftrightarrow \beta < 1$ $\beta < 1 \Rightarrow$ **ESC** See Theorem 9.1
#9 :	$x_{n+1} = \dfrac{\alpha + x_{n-1}}{x_n x_{n-1}};$	**ESC** See Theorem 10.1
#10 :	$x_{n+1} = (\gamma + x_n)x_{n-1};$	**∃US** See Theorem 11.1
#11 :	$x_{n+1} = \dfrac{\alpha + x_n x_{n-1}}{A + x_n x_{n-1}};$	**ESB** **ESC***; This conjecture has been confirmed for $\alpha \leq A$

#12 : $x_{n+1} = \dfrac{\alpha + \beta x_n x_{n-1}}{1 + x_{n-1}}$;	$\beta \leq 1 \Rightarrow$ **ESC** $\beta > 1 \Rightarrow$ **∃US** See Theorem 13.1
#13 : $x_{n+1} = \dfrac{\alpha + x_n x_{n-1}}{(A + x_n) x_{n-1}}$;	**ESB** **ESC***; This conjecture has been confirmed for $\alpha \leq A$ See Theorem 2.1 in Part 2
#14 : $x_{n+1} = \dfrac{\alpha + x_{n-1}}{A + x_n x_{n-1}}$;	**ESB** **ESC***; This conjecture has been confirmed for $\alpha \leq A$ See Theorem 3.1 in Part 2
#15 : $x_{n+1} = \dfrac{\alpha + x_{n-1}}{(1 + B x_n) x_{n-1}}$;	**ESB** **ESC***; This conjecture has been confirmed for $\alpha B \leq 1$ See Theorem 4.1 in Part 2
#16 : $x_{n+1} = \dfrac{(1 + \beta x_n) x_{n-1}}{A + x_n x_{n-1}}$;	$A > 1 \Rightarrow$ **ESC** $A = 1 \Rightarrow$ **ESCP$_2$** $A < 1 \Rightarrow$ **∃US** Has Period-Two Trichotomy See Theorem 5.1 in Part 2
#17 : $x_{n+1} = \dfrac{(1 + \beta x_n) x_{n-1}}{A + x_{n-1}}$;	$\beta \leq 1 \Rightarrow$ **ESC** $\beta > 1$ and $A \neq 1 \Rightarrow$ **∃US** See Theorem 6.1 in Part 2

#18 :	$x_{n+1} = \dfrac{\alpha}{1 + x_n x_{n-1} + C x_{n-1}};$	**ESB** **ESC***; This conjecture has been confirmed for $(\alpha - C)^2 \leq 4$
#19 :	$x_{n+1} = \dfrac{\beta x_n x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}};$	**ESC** Local Stability $\not\Rightarrow$ Global Stability See Theorem 8.1 in Part 2
#20 :	$x_{n+1} = \dfrac{\gamma x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}};$	$\gamma \leq 1 \;\Rightarrow$ **ESC** $\gamma > 1 \;\Rightarrow$ **ESCP$_2$** See Theorem 9.1 in Part 2
#21 :	$x_{n+1} = \alpha + x_n x_{n-1} + \gamma x_{n-1};$	$\exists$**US**
#22 :	$x_{n+1} = \dfrac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}};$	**ESB** **ESC***
#23 :	$x_{n+1} = \dfrac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_{n-1}};$	$\beta < 1 \;\Rightarrow$ **ESC** $\beta \geq 1 \;\Rightarrow$ Every solution increases to ∞ See Theorem 12.1 in Part 2
#24 :	$x_{n+1} = \dfrac{\alpha + \beta x_n x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}};$	**ESB** **ESC***
#25 :	$x_{n+1} = \dfrac{\alpha + \gamma x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}};$	**ESB** **ESC***

#26 : $x_{n+1} = \dfrac{\beta x_n x_{n-1} + \gamma x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}};$	**ESB** **ESC***
#27 : $x_{n+1} = \dfrac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{A + x_n x_{n-1}};$	**ESB** **ESC***
#28 : $x_{n+1} = \dfrac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{A + x_{n-1}};$	**∃US** See Theorem 13.1 in Part 2
#29 : $x_{n+1} = \dfrac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{B x_n x_{n-1} + x_{n-1}};$	**ESB** **ESC***
#30 : $x_{n+1} = \dfrac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{A + B x_n x_{n-1} + x_{n-1}};$	**ESB** **ESC***

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