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Positive Periodic Solutions for Higher-order Functional Difference Equations*

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Abstract

In this paper, we apply a fixed point theorem to obtain sufficient conditions for the existence of positive periodic solutions for two classes of higher-order functional difference equations.

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1. Introduction

The existence of positive periodic solutions of discrete mathematical models has been studied extensively in recent years, see [1, 5–12], for example,

(i) discrete model of blood cell production:

$$\Delta x(n) = -a(n)x(n) + b(n)\frac{1}{1 + x^k(n - \tau(n))}, \ k \in \mathbb{N}, \tag{1.1}$$

$$\Delta x(n) = -a(n)x(n) + b(n)\frac{x(n - \tau(n))}{1 + x^k(n - \tau(n))}, \ k \in \mathbb{N}, \tag{1.2}$$

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(ii) the periodic Michaelis–Menton discrete model:

$$\Delta x(n) = a(n)x(n)\left[1 - \sum_{j=1}^{k} \frac{a_j(n)x(n-\tau_j(n))}{1 + c_j(n)x(n-\tau_j(n))}\right], \tag{1.3}$$

(iii) the single species discrete periodic population model:

$$\Delta x(n) = x(n)\left[a(n) - \sum_{j=1}^{k} b_j(n)x(n-\tau_j(n))\right]. \tag{1.4}$$

Jiang, O'Regan and Agarwal in [4] have obtained the optimal existence theorem for single and multiple positive periodic solutions to general functional difference equations

$$\Delta x(n) = x(n)[a(n) - g(n, x(n-\tau_1(n)), \ldots, x(n-\tau_k(n)))], \tag{1.5}$$

$$\Delta x(n) = -a(n)x(n) + g(n, x(n-\tau(n))). \tag{1.6}$$

Note that the equations (1.1)–(1.6) are first-order functional difference equations. Our aim of this paper is to study existence of positive periodic solutions for the higher-order difference equations

$$x(n+m) = a(n)x(n) + f(n, x(n-\tau(n))), \tag{1.7}$$

$$x(n+m) = a(n)x(n) - f(n, x(n-\tau(n))), \tag{1.8}$$

where $a(n) = a(n+\omega)$, $f(n+\omega, u) = f(n, u)$, $\tau : \mathbb{Z} \to \mathbb{Z}$, $\tau(n+\omega) = \tau(n)$ and $\mathbb{Z}$ denotes the set of integers, $\omega, m \in \mathbb{N}$. By using a fixed point theorem in a cone, we obtain existence results for single and multiple positive periodic solutions to the equation (1.7) and (1.8).

To prove our main results, we present an existence theorem.

Theorem 1.1. [2,3] Let X be a Banach space and K be a cone in X. Suppose Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$ and suppose that

$$\Phi : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$$

is a completely continuous operator such that

(i) $\|\Phi u\| \le \|u\|$ for $u \in K \cap \partial\Omega_1$, and there exists $\psi \in K \backslash \{0\}$ such that $u \neq \Phi u + \lambda\psi$ for $u \in K \cap \partial\Omega_2$ and $\lambda > 0$, or

(ii) $\|\Phi u\| \le \|u\|$ for $u \in K \cap \partial\Omega_2$, and there exists $\psi \in K \backslash \{0\}$ such that $u \neq \Phi u + \lambda\psi$ for $u \in K \cap \partial\Omega_1$ and $\lambda > 0$.

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

2. Positive Periodic Solutions of (1.7)

In this section we establish the existence of positive periodic solutions of equation (1.7). We always assume the following condition throughout this section:

(H) $0 < a(n) < 1$ for all $n \in [0, \omega - 1]$ and $f : \mathbb{Z} \times [0, \infty) \to [0, \infty)$ is continuous.

Let X be the set of all real ω-periodic sequences. When endowed with the maximum norm $\|x\| = \max_{n\in[0,\omega-1]} |x(n)|$, X is a Banach space. Set $(m, \omega) = l$, $\omega/l = h$. From (1.7), we have that for any $x \in X$

$$
\begin{aligned}
&\frac{1}{a(n)}x(n+m) - x(n) = \frac{1}{a(n)} f(n, x(n-\tau(n))),\\
&\frac{1}{a(n)a(n+m)}x(n+2m) - \frac{1}{a(n)}x(n+m)\\
&\quad = \frac{1}{a(n)a(n+m)} f(n+m, x(n+m-\tau(n+m))),\\
&\quad \cdots\cdots\cdots\cdots\\
&\left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)}\right) x(n+hm) - \left(\prod_{i=0}^{h-2} \frac{1}{a(n+im)}\right) x(n+(h-1)m)\\
&\quad = \left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)}\right) f(n+hm, x(n+(h-1)m - \tau(n+(h-1)m))).
\end{aligned}
$$

By summing the above equations and using periodicity of x, we obtain the following result.

Lemma 2.1. $x \in X$ is a solution of equation (1.7) if and only if

$$
\begin{aligned}
x(n) = \left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right)&\\
\times f(n+im, x(n+im-\tau(n+im))).& \qquad (2.1)
\end{aligned}
$$

Put

$$
M^* = \max\left\{\prod_{i=0}^{h-1} a(n+im) : \ 0 \le n \le \omega - 1\right\},
$$

$$
M_* = \min\left\{\prod_{i=0}^{h-1} a(n+im) : \ 0 \le n \le \omega - 1\right\},
$$

$$
\delta = \frac{M_*^2(1-M^*)}{M^*(1-M_*)}.
$$

Clearly, $\delta \in (0, 1)$. We define a cone by

$$P = \{y \in X : y(n) \geq 0, n \in \mathbb{Z},\ y(n) \geq \delta\|y\|\},$$

and a mapping $T : X \to X$ by

$$(Tx)(n) = \left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) \times f(n+im, x(n+im-\tau(n+im))).$$

By the nonnegativity of f and a, $Tx(n) \geq 0$ on $[0, \omega-1]$. It is clear that $(Tx)(n+\omega) = (Tx)(n)$ and T is completely continuous on bounded subsets of P. Noting that

$$\begin{aligned}(Tx)(n) &= \left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) \\ &\quad \times f(n+im, x(n+im-\tau(n+im))) \\ &\leq \left(\frac{M_*}{M^*} - M_*\right)^{-1} \sum_{i=0}^{h-1} f(n+im, x(n+im-\tau(n+im))),\end{aligned}$$

$$(Tx)(n) \geq \left(\frac{1}{M_*} - 1\right)^{-1} \sum_{i=0}^{h-1} f(n+im, x(n+im-\tau(n+im))),$$

we easily obtain that $Tx(n) \geq \delta\|Tx\|$, that is, $T(P) \subset P$.

For convenience, we denote

$$\varphi(s) = \max\left\{\frac{f(n,u)}{1-a(n)} : n \in [0, \omega-1],\ u \in [\delta s, s]\right\},$$

$$\psi(s) = \min\left\{\frac{f(n,u)}{(1-a(n))u} : n \in [0, \omega-1],\ u \in [\delta s, s]\right\},$$

$$\varphi_0 = \lim_{s\to 0^+} \frac{\varphi(s)}{s},\quad \varphi_\infty = \lim_{s\to\infty} \frac{\varphi(s)}{s},$$

$$\psi_0 = \lim_{s\to 0^+} \psi(s),\quad \psi_\infty = \lim_{s\to\infty} \psi(s).$$

Theorem 2.2. Assume that (H) holds, and there exist two positive constants a, b with $a \neq b$ such that $\varphi(a) \leq a$ and $\psi(b) \geq 1$. Then the equation (1.7) has at least one positive solution $x \in X$ with $\min\{a, b\} \leq \|x\| \leq \max\{a, b\}$.

Proof. Without loss of generality, we assume that $a < b$. Let $\Omega_1 = \{x \in X : \|x\| < a\}$ and $\Omega_2 = \{x \in X : \|x\| < b\}$. We claim that

(i) $\|Tx\| \le \|x\|, x \in P \cap \partial\Omega_1$.

(ii) $x \ne Tx + \lambda, x \in P \cap \partial\Omega_2$ and $\lambda > 0$.

From $\varphi(a) \le a$ and $\psi(b) \ge 1$, we have that

$$f(n, x) \le (1 - a(n))a,\ 0 \le n \le \omega - 1,\ a\delta \le x \le a, \tag{2.2}$$

$$f(n, x) \ge (1 - a(n))x,\ 0 \le n \le \omega - 1,\ b\delta \le x \le b. \tag{2.3}$$

To justify (i), let $x \in P \cap \partial\Omega_1$. Then $\|x\| = a$ and $\delta a \le x(n) \le a$ for $0 \le n \le \omega - 1$. It follows that

$$\begin{aligned} Tx(n) &\le \left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) \\ &\quad \times f(n+im, x(n+im-\tau(n+im))) \\ &\le \left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) \\ &\quad \times (1 - a(n+im))x(n+im-\tau(n+im)) \\ &\le \left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) (1 - a(n+im))a \\ &\le \|x\|. \end{aligned}$$

This means that

$$\|Tx\| \le \|x\| \quad \text{for all} \quad x \in P \cap \partial\Omega_1.$$

Next, we prove (ii). If not, there exist $x^* \in P \cap \partial\Omega_2$ and $\lambda_0 > 0$ such that

$$x^* = Tx^* + \lambda_0.$$

Since $x^* \in P \cap \partial\Omega_2$, we have $\|x^*\| = b$ and $\delta b \le x^*(n) \le b$. Put $\chi = \min\{x(n), 0 \le n \le \omega - 1\}$. Then we have $\chi = x(n)$ for some $n \in [0, \omega - 1]$. Thus it follows that

$$
\begin{aligned}
x^*(n) &= (Tx^*)(n) + \lambda_0 \\
&= \left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) \\
&\quad \times f(n+im, x(n+im-\tau(n+im))) + \lambda_0 \\
&\geq \left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) \\
&\quad \times (1-a(n+im))x(n+im-\tau(n+im)) + \lambda_0 \\
&\geq \chi \left(\prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) \\
&\quad \times (1-a(n+im)) + \lambda_0 = \chi + \lambda_0,
\end{aligned}
$$

and this implies $\chi > \chi$, a contradiction.

Therefore, by Theorem 1.1, it follows that T has a fixed point $x \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$. Furthermore, $a \leq \|x\| \leq b$ and $x(n) \geq \delta a$, which means that x is a positive ω-periodic solution of (1.7). The proof is complete. ■

Corollary 2.3. Assume that (H) holds, and one of the following conditions holds:

(i) $\varphi_0 < 1$ and $\psi_\infty > 1$,

(ii) $\varphi_\infty < 1$ and $\psi_0 > 1$.

Then the equation (1.7) has at least one positive solution $x \in X$ with $\|x\| > 0$.

Theorem 2.4. Assume that (H) holds. There exist $N+1$ positive constants $p_1 < p_2 < \cdots < p_N < p_{N+1}$ such that one of the following conditions is satisfied:

(i) $\varphi(p_{2k-1}) < p_{2k-1},\ k = 1, 2, \ldots, [(N+2)/2]$,
$\psi(p_{2k}) > 1,\ \ k = 1, 2, \ldots, [(N+1)/2]$,

(ii) $\psi(p_{2k-1}) > 1,\ k = 1, 2, \ldots, [(N+2)/2]$,
$\psi(p_{2k}) < p_{2k},\ \ k = 1, 2, \ldots, [(N+1)/2]$,

where $[d]$ denotes the integer part of d. Then the equation (1.7) has at least N positive solutions $x_k \in X, k = 1, 2, \ldots, N$ with $p_k < \|x_k\| < p_{k+1}$.

Proof. It is enough to prove case (i). Since $\phi, \psi : (0, \infty) \to [0, \infty)$ are continuous, there exist $p_k < a_k < b_k < p_{k+1},\ k = 1, 2, \ldots, N$ such that

$$\varphi(a_{2k-1}) \leq a_{2k-1},\ \phi(b_{2k-1}) \geq 1,\ k = 1, 2, \ldots, [(N+1)/2],$$

$$\phi(a_{2k}) \geq 1, \ \varphi(b_{2k}) \leq b_{2k}, \ k = 1, 2, \ldots, [(N+1)/2].$$

It follows by Theorem 2.2 that equation (1.7) has at least one positive periodic solution $x_k \in X$ for every pair of numbers $\{a_k, b_k\}$ with $p_k < a_k \leq \|x_k\| \leq b_k < p_{k+1}$. The proof is complete. ■

Corollary 2.5. Assume that (H) holds, and the following conditions are satisfied:

(i) $\varphi_0 < 1$ and $\varphi_\infty < 1$,

(ii) there exists a positive constant b such that $\psi(b) > 1$.

Then the equation (1.7) has at least two positive solutions $x_1, x_2 \in X$ with

$$0 < \|x_1\| < b < \|x_2\| < \infty.$$

Corollary 2.6. Assume that (H) holds, and the following conditions are satisfied:

(i) $\psi_0 > 1$ and $\psi_\infty > 1$,

(ii) there exists a positive constant a such that $\varphi(a) < a$.

Then the equation (1.7) has at least two positive solutions $x_1, x_2 \in X$ with

$$0 < \|x_1\| < a < \|x_2\| < \infty.$$

3. Positive Periodic Solutions of (1.8)

In this section we establish the existence of positive periodic solutions of equation (1.8). We always assume the following condition throughout this section:

(H^*) $a(n) > 1$ for all $n \in [0, \omega - 1]$ and $f : \mathbb{Z} \times [0, \infty) \to [0, \infty)$ is continuous.

The proofs of the results presented in this section are similar to those given in Section 2 and hence are omitted.

Lemma 3.1. $x \in X$ is a solution of equation (1.8) if and only if

$$x(n) = \left(1 - \prod_{i=0}^{h-1} \frac{1}{a(n+im)}\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) \times f(n+im, x(n+im-\tau(n+im))),$$

where X and h are defined in Section 2.

Put

$$M^* = \max\left\{\prod_{i=0}^{h-1} a(n+im) : \ 0 \leq n \leq \omega - 1\right\},$$

$$M_* = \min\left\{\prod_{i=0}^{h-1} a(n+im) : \ 0 \le n \le \omega - 1\right\},$$

$$\delta^* = \frac{M_* - 1}{M_*(M^* - 1)}.$$

Clearly, $\delta^* \in (0, 1)$. We define a cone by

$$P = \{y \in X : y(n) \ge 0, n \in \mathbb{Z},\ y(n) \ge \delta^* \|y\|\},$$

and a mapping $T : X \to X$ by

$$(Tx)(n) = \left(1 - \prod_{i=0}^{h-1} \frac{1}{a(n+im)}\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) \times f(n+im, x(n+im-\tau(n+im))).$$

It is not difficult to verify that $T(P) \subset P$ is completely continuous. Let

$$\widetilde{\varphi}(s) = \max\left\{\frac{f(n,u)}{a(n)-1} : \ n \in [0, \omega-1],\ u \in [\delta^* s, s]\right\},$$

$$\widetilde{\psi}(s) = \min\left\{\frac{f(n,u)}{(a(n)-1)u} : \ n \in [0, \omega-1],\ u \in [\delta^* s, s]\right\},$$

$$\widetilde{\varphi}_0 = \lim_{s\to 0^+} \frac{\widetilde{\varphi}(s)}{s},\quad \widetilde{\varphi}_\infty = \lim_{s\to\infty} \frac{\widetilde{\varphi}(s)}{s},$$

$$\widetilde{\psi}_0 = \lim_{s\to 0^+} \widetilde{\psi}(s),\quad \widetilde{\psi}_\infty = \lim_{s\to\infty} \widetilde{\psi}(s).$$

Theorem 3.2. Assume that (H^*) holds, and there exist two positive constants a, b with $a \ne b$ such that $\widetilde{\varphi}(a) \le a$ and $\widetilde{\psi}(b) \ge 1$. Then the equation (1.8) has at least one positive solution $x \in X$ with $\min\{a, b\} \le \|x\| \le \max\{a, b\}$.

Corollary 3.3. Assume that (H^*) holds, and one of the following conditions holds:

(i) $\widetilde{\varphi}_0 < 1$ and $\widetilde{\psi}_\infty > 1$,

(ii) $\widetilde{\varphi}_\infty < 1$ and $\widetilde{\psi}_0 > 1$.

Then the equation (1.8) has at least one positive solution $x \in X$ with $\|x\| > 0$.

Theorem 3.4. Assume that (H^*) holds, and there exist $N+1$ positive constants $p_1 < p_2 < \cdots < p_N < p_{N+1}$ such that one of the following conditions is satisfied:

(i) $\widetilde{\varphi}(p_{2k-1}) < p_{2k-1},\ k = 1, 2, \ldots, [(N+2)/2]$,
$\widetilde{\psi}(p_{2k}) > 1,\ k = 1, 2, \ldots, [(N+1)/2]$,

(ii) $\widetilde{\psi}(p_{2k-1}) > 1,\ k = 1, 2, \ldots, [(N+2)/2]$,
$\widetilde{\varphi}(p_{2k}) < p_{2k},\ k = 1, 2, \ldots, [(N+1)/2]$,

where $[d]$ denotes the integer part of d. Then the equation (1.8) has at least N positive solutions $x_k \in X, k = 1, 2, \ldots, N$ with $p_k < \|x_k\| < p_{k+1}$.

Corollary 3.5. Assume that (H^*) holds, and the following conditions are satisfied:

(i) $\widetilde{\varphi}_0 < 1$ and $\widetilde{\varphi}_\infty < 1$,

(ii) there exists a positive constant b such that $\widetilde{\psi}(b) > 1$.

Then the equation (1.8) has at least two positive solutions $x_1, x_2 \in X$ with

$$0 < \|x_1\| < b < \|x_2\| < \infty.$$

Corollary 3.6. Assume that (H^*) holds, and the following conditions are satisfied:

(i) $\widetilde{\psi}_0 > 1$ and $\widetilde{\psi}_\infty > 1$,

(ii) there exists a positive constant a such that $\widetilde{\varphi}(a) < a$.

Then the equation (1.8) has at least two positive solutions $x_1, x_2 \in X$ with

$$0 < \|x_1\| < a < \|x_2\| < \infty.$$

4. Some Examples

In this section, we apply the main results obtained in the previous sections to several examples.

Example 4.1. Consider the difference equation

$$x(n+2) = a(n)x(n) + \frac{1}{1 + x(n-2)}, \tag{4.1}$$

where a is an ω-periodic function with $0 < a(n) < 1$ for all $n \in [1, \omega]$. Obviously $f(n, x) = 1/(x+1)$ and $\varphi_\infty = 0,\ \psi_0 = \infty$. By Corollary 2.3, (4.1) has at least one positive ω-periodic solution.

Example 4.2. Consider the difference equation

$$x(n+3) = a(n)x(n) + x^{100}(n-5) + \frac{101}{100}\sin x(n-5), \tag{4.2}$$

where a is an ω-periodic function with $0 < a(n) < 0.01$ for all $n \in [1, \omega]$. It is clear that $\psi_\infty = \infty,\ \psi_0 = 1.01 > 1$. Put $a = \pi/6$. Then

$$f(n, x) = x^{100} + \frac{101}{100}\sin x < \frac{99}{100}a,\ 0 < x \leq a.$$

By Corollary 2.5, (4.2) has at least two positive ω-periodic solutions.

Example 4.3. Consider the difference equation

$$x(n+5) = a(n)x(n) - b(n)x^2(n+\tau(n)), \tag{4.3}$$

where a, b, τ are ω-periodic functions with $a(n) > 1, b(n) > 0$ for all $n \in [1, \omega]$ and $\tau : \mathbb{Z} \to \mathbb{Z}$. By Corollary 3.3, (4.2) has at least one positive ω-periodic solution.

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