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A Transform Method in Discrete Fractional Calculus

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Abstract

We begin with an introduction to a calculus of fractional finite differences. We extend the discrete Laplace transform to develop a discrete transform method. We define a family of finite fractional difference equations and employ the transform method to obtain solutions.

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1. Introduction

In this article we study discrete fractional calculus. We proceed to consider a family of finite fractional linear difference equations and we shall develop a transform method of solution. Our preliminary definitions follow in the spirit of Miller and Ross [6]. Our goals follow in the spirit of Miller and Ross [6] and Podlubny [7] to develop the theory of linear finite fractional difference equations analogously to the theory of finite difference equations.

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The Riemann–Liouville fractional integral of a function f of order ν has the form

$$D_0^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s) ds.$$

Here it is assumed that Re $\nu > 0$ and it is assumed that f is continuous on $(0, \infty)$ and integrable on bounded subintervals of $[0, \infty)$. The kernel, $(t-s)^{\nu-1}/\Gamma(\nu)$, is a clear generalization of a Cauchy function, $(t-s)^{n-1}/(n-1)!$, for nth order ordinary differential equations. Miller and Ross [6] employ this observation and develop a theory of linear fractional differential equations that is analogous to the classical theory of linear ordinary differential equations.

In [5], Miller and Ross initiated the process to develop the analogous theory for fractional finite differences. In this article, we continue to build on their work, develop properties of fractional finite differences, define a family of linear finite fractional difference equations, and develop a transform method of solution.

In Section 2 we shall provide the elementary definition of a fractional sum of a function f of order ν, and we shall define the fractional difference. These definitions are essentially due to Miller and Ross [5]. We shall also develop and present some of the elementary properties of discrete fractional calculus.

In Section 3, we shall employ a discrete transform to solve the same family of equations. We present examples to illustrate the method. The discrete transform, which is not the z-transform, is the Laplace transform on the time scale of integers [2,4]. We find the discrete Laplace transform to be the transform of convenience.

2. An Introduction to the Calculus of Finite Fractional Differences

Recall the factorial polynomial,

$$t^{(n)} = \prod_{j=0}^{n-1} (t-j) = \Gamma(t+1)/\Gamma(t+1-n),$$

where Γ denotes the special gamma function and if $t+1-j=0$ for some j, we assume the product is zero. We shall employ the convention that division at a pole yields zero. For arbitrary ν, define

$$t^{(\nu)} = \Gamma(t+1)/\Gamma(t+1-\nu).$$

We will list some of the properties of this factorial function with their proofs.

Theorem 2.1. Assume that the following factorial functions are well defined.

(i) $\Delta t^{(\nu)} = \nu t^{(\nu-1)}$, where Δ is the forward difference operator.

(ii) $(t-\mu)t^{(\mu)} = t^{(\mu+1)}$, where $\mu \in \mathbb{R}$.

(iii) $\mu^{(\mu)} = \Gamma(\mu+1)$.

(iv) If $t \le r$, then $t^{(\nu)} \le r^{(\nu)}$ for any $\nu > r$.

(v) If $0 < \nu < 1$, then $t^{(\alpha\nu)} \ge (t^{(\alpha)})^{\nu}$.

(vi) $t^{(\alpha+\beta)} = (t-\beta)^{(\alpha)} t^{(\beta)}$.

Proof. The proofs of (i), (ii), (iii), (vi) are straightforward. The proof of (iv) follows from Euler's infinite product

$$\Gamma(u) = \frac{1}{u} \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^u}{1+(\frac{u}{n})}.$$

The proof of (v) follows from the log-convexity property of the gamma function.

$$\begin{aligned} t^{(\alpha\nu)} &= \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha\nu)} \\ &= \frac{\Gamma(t+1)}{\Gamma(\nu(t+1-\alpha)+(1-\nu)(t+1))} \\ &\ge \frac{\Gamma(t+1)}{(\Gamma(t+1-\alpha))^{\nu}(\Gamma(t+1))^{1-\nu}} \\ &= (t^{(\alpha)})^{\nu}. \end{aligned}$$

This completes the proof. ■

Define iteratively the operator $\Delta^j = \Delta(\Delta^{j-1})$, where j is a nonnegative integer, Δ^0 denotes the identity operator and $\Delta^1 f(t) = \Delta f(t) = f(t+1) - f(t)$. It is the case that the solution of an initial value problem of the form

$$\begin{aligned} \Delta^n u(t) &= f(t), \quad t = a, a+1, \dots, \\ u(a+j-1) &= 0, \quad j = 1, \dots n, \end{aligned}$$

is the function

$$\Delta^{-n} f(t) = u(t) = \sum_{s=a}^{t-1} \frac{(t-\sigma(s))^{(n-1)}}{(n-1)!} f(s).$$

Here, $\sigma(s) = s+1$. We shall use this notation throughout the article. It is the intention that some of this material will generalize to a study of fractional calculus on time scales.

Note that the Cauchy function, $\dfrac{(t-\sigma(s))^{(n-1)}}{(n-1)!}$ vanishes at $s = t-(n-1), \dots, t-1$. So

$$\begin{aligned} \sum_{s=a}^{t-1} \frac{(t-\sigma(s))^{(n-1)}}{(n-1)!} f(s) &= \sum_{s=a}^{t-n} \frac{(t-\sigma(s))^{(n-1)}}{(n-1)!} f(s) \\ &= \frac{1}{(n-1)!} \sum_{s=a}^{t-n} \frac{\Gamma(t-s)}{\Gamma(t-s-(n-1))} f(s). \end{aligned}$$

With this observation define (as done in [5]) the νth fractional sum of f by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \frac{\Gamma(t-s)}{\Gamma(t-s-(\nu-1))} f(s), \tag{2.1}$$

where $\nu > 0$. Note that f is defined for $s = a, \mod (1)$ and $\Delta^{-\nu} f$ is defined for $t = a+\nu, \mod (1)$. For calculations, we shall assume that $s = a, \mod (1)$, $s - a \geq 0$. This observation is important to determine lower limits throughout the paper.

Next we shall state and prove the law of exponents for fractional sums.

Theorem 2.2. Let f be a real-valued function, and let $\mu, \nu > 0$. Then for all t such that $t = \mu + \nu, \mod (1)$,

$$\Delta^{-\nu}[\Delta^{-\mu} f(t)] = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu}[\Delta^{-\nu} f(t)].$$

Proof. By definition of fractional sum, we have

$$\begin{aligned}
\Delta^{-\mu}(\Delta^{-\nu} f(t)) &= \frac{1}{\Gamma(\nu)} \Delta^{-\mu} \sum_{r=0}^{t-\nu} (t-\sigma(r))^{(\nu-1)} f(r) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{s=\nu}^{t-\mu} (t-\sigma(s))^{(\mu-1)} \sum_{r=0}^{s-\nu} (s-\sigma(r))^{(\nu-1)} f(r) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{s=\nu}^{t-\mu} \sum_{r=0}^{s-\nu} (t-\sigma(s))^{(\mu-1)} (s-\sigma(r))^{(\nu-1)} f(r) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=0}^{t-(\mu+\nu)} \sum_{s=r+\nu}^{t-\mu} (t-\sigma(s))^{(\mu-1)} (s-\sigma(r))^{(\nu-1)} f(r) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=0}^{t-(\mu+\nu)} \left(\sum_{x=\nu-1}^{t-\sigma(r)-\mu} (t-\sigma(r)-\sigma(x))^{(\mu-1)} x^{(\nu-1)} \right) f(r) \\
&= \frac{1}{\Gamma(\nu)} \sum_{r=0}^{t-(\mu+\nu)} (\Delta^{-\mu} (t-\sigma(r))^{(\nu-1)}) f(r) \\
&= \frac{1}{\Gamma(\nu)} \sum_{r=0}^{t-(\mu+\nu)} \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} (t-\sigma(r))^{(\nu+\mu-1)} f(r) \\
&= \Delta^{-(\mu+\nu)} f(t).
\end{aligned}$$

This completes the proof. ■

Next, we obtain a power rule. Miller and Ross [5] obtained the following lemma in the case that μ is a positive integer by induction and $\mu = 0$ with a straightforward calculation.

Lemma 2.3. Let $\mu \in \mathbb{R} \setminus \{\dots, -2, -1\}$.

$$\Delta^{-\nu} t^{(\mu)} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)} t^{(\mu+\nu)}.$$

Proof. Set

$$g_1(t) = \frac{\Gamma(\mu + 1) t^{(\mu+\nu)}}{\Gamma(\mu + \nu + 1)}.$$

Set

$$\begin{aligned} g_2(t) = \Delta^{-\nu} t^{(\mu)} &= \frac{1}{\Gamma(\nu)} \sum_{s=\mu}^{t-\nu} \frac{\Gamma(t - s)}{\Gamma(t - s - \nu + 1)} s^{(\mu)} \\ &= \frac{1}{\Gamma(\nu)} \sum_{s=\mu}^{t-\nu} \frac{\Gamma(t - s)\Gamma(s + 1)}{\Gamma(t - s - \nu + 1)\Gamma(s - \mu + 1)}. \end{aligned}$$

We show g_1, g_2 each satisfy

$$\begin{aligned} (t - (\mu - \nu) + 1)\Delta g(t) &= (\mu + \nu) g(t), \\ g(\mu + \nu) &= \Gamma(\mu + 1). \end{aligned}$$

In the next steps of the proof, the formulas (ii) and (iii) in Theorem 2.1 are used repeatedly. First, consider g_1. Apply (iii) to see that

$$g_1(\mu + \nu) = \frac{\Gamma(\mu + 1)(\mu + \nu)^{(\mu+\nu)}}{\Gamma(\mu + \nu + 1)} = \frac{\Gamma(\mu + 1)\Gamma(\mu + \nu + 1)}{\Gamma(\mu + \nu + 1)} = \Gamma(\mu + 1).$$

Apply (ii) to see that

$$\begin{aligned} (t - (\mu + \nu) + 1)\Delta g_1(t) &= \frac{(\mu + \nu)\Gamma(\mu + 1)(t - (\mu + \nu) + 1) t^{(\mu+\nu-1)}}{\Gamma(\mu + \nu + 1)} \\ &= \frac{(\mu + \nu)\Gamma(\mu + 1) t^{(\mu+\nu)}}{\Gamma(\mu + \nu + 1)} = (\mu + \nu) g_1(t). \end{aligned}$$

Second, consider g_2. Apply (iii) to see that

$$g_2(\mu + \nu) = \frac{(\nu - 1)^{(\nu-1)} \mu^{(\mu)}}{\Gamma(\nu)} = \frac{\Gamma(\nu)\Gamma(\mu + 1)}{\Gamma(\nu)} = \Gamma(\mu + 1).$$

We now present a tedious calculation to show g_2 satisfies the difference equation. Use (ii) and add and subtract μ to write

$$\begin{aligned}
g_2(t) &= \frac{1}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-1)}s^{(\mu)} \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-\sigma(s)-(\nu-2))(t-\sigma(s))^{(\nu-2)}s^{(\mu)} \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-s-\nu-\mu+1+\mu)(t-\sigma(s))^{(\nu-2)}s^{(\mu)} \\
&= \frac{(t-(\nu+\mu)+1)}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-2)}s^{(\mu)} \\
&\quad - \frac{1}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-2)}(s-\mu)s^{(\mu)}.
\end{aligned}$$

Consider only the first term

$$\frac{(t-(\nu+\mu)+1)}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-2)}s^{(\mu)}.$$

Note that

$$\begin{aligned}
\Delta g_2(t) &= \frac{\nu-1}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-2)}s^{(\mu)} + \frac{(\nu-1)^{(\nu-1)}(t+1-\nu)^{(\mu)}}{\Gamma(\nu)} \\
&= \frac{\nu-1}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-2)}s^{(\mu)} + (t+1-\nu)^{(\mu)}.
\end{aligned}$$

So

$$\begin{aligned}
&\frac{(t-(\nu+\mu)+1)}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-2)}s^{(\mu)} \\
&= (t-(\nu+\mu)+1)\left(\frac{\Delta g_2-(t+1-\nu)^{(\mu)}}{\nu-1}\right) \\
&= \frac{(t-(\nu+\mu)+1)\Delta g_2}{\nu-1} - \frac{(t+1-\nu)^{(\mu+1)}}{\nu-1}.
\end{aligned}$$

Consider now the second term

$$\frac{1}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-2)}(s-\mu)s^{(\mu)}.$$

Sum by parts and note that

$$\Delta((t-s)^{(\nu-1)}s^{(\mu+1)}) = (t-\sigma(s))^{(\nu-1)}(\mu+1)s^{(\mu)} - (\nu-1)(t-\sigma(s))^{(\nu-2)}s^{(\mu+1)}.$$

Also recall $s^{(\mu+1)} = (s-\mu)s^{(\mu)}$. Thus,

$$\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-2)}s^{(\mu+1)} = \sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-2)}(s-\mu)s^{(\mu)}$$
$$= \frac{1}{\nu-1}\left((\mu+1)\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-1)}s^{(\mu)} - (\nu-1)^{(\nu-1)}(t+1-\nu)^{(\mu+1)}\right).$$

In particular,

$$-\frac{1}{\Gamma(\nu)}\sum_{s=\mu}^{t-\nu}(t-\sigma(s))^{(\nu-2)}(s-\mu)s^{(\mu)} = -\frac{\mu+1}{\nu-1}g_2(t) + \frac{(t+1-\nu)^{(\mu+1)}}{\nu-1}.$$

In summary we have shown

$$g_2(t) = \frac{(t-(\nu+\mu)+1)\Delta g_2(t)}{\nu-1} - \frac{(t+1-\nu)^{(\mu+1)}}{\nu-1} - \frac{(\mu+1)g_2(t)}{\nu-1}$$
$$+ \frac{(t+1-\nu)^{(\mu+1)}}{\nu-1}.$$

Thus, $(t-(\mu-\nu)+1)\Delta g_2(t) = (\mu+\nu)g_2(t)$. The proof is complete. ■

The definition of a fractional difference is analogous to the definition of a fractional derivative [5]. Let $\mu > 0$ and assume that $m-1 < \mu < m$, where m denotes a positive integer. Set $-\nu = \mu - m$. Define

$$\Delta^{\mu}u(t) = \Delta^{m-\nu}u(t) = \Delta^{m}(\Delta^{-\nu}u(t)).$$

Lemma 2.3 gives indication that fractional difference equations can make sense. We argue that the difference equation

$$\Delta^{4/3}u = 0$$

makes sense in the fact that it generates two linearly independent solutions on the set $\left\{-\frac{2}{3}, \frac{1}{3}, \frac{4}{3}, \dots\right\}$. Consider $u_1(t) = t^{(1/3)}$ and $u_2(t) = t^{(-2/3)}$,

$$\Delta^{4/3}u_1(t) = \Delta^2(\Delta^{-2/3}t^{(1/3)}) = K\Delta^2 t^{(2/3+1/3)} = 0.$$

Moreover, $\Delta^{-2/3}t^{(1/3)}$ implies $t = \frac{2}{3} + \frac{1}{3}, \mod (1)$. Similarly, u_2 satisfies

$$\Delta^{4/3}u_2(t) = 0$$

for $t = \frac{2}{3} - \frac{2}{3}, \mod (1)$.

3. A Discrete Transform Method of Solution

We first introduce the reader to a discrete transform which is the Laplace transform on the time scale of integers [2,4]. This is not the more common z-transform. We choose a Laplace transform to employ the common analogous properties of the common Laplace transform.

Define the discrete transform (R-transform) by

$$R_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} f(t). \tag{3.1}$$

$R_0(f(t))(s)$ is the Laplace transform on the time scale of integers [2,4].

Lemma 3.1. For any $\nu \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$,

(i) $R_{\nu-1}(t^{(\nu-1)})(s) = \dfrac{\Gamma(\nu)}{s^{\nu}}$

(ii) $R_{\nu-1}(t^{(\nu-1)}\alpha^t)(s) = \dfrac{\alpha^{\nu-1}\Gamma(\nu)}{(s+1-\alpha)^{\nu}}$.

Proof. (i) For $\nu = 1$, one employs a straightforward calculation with the geometric series. The cases $\nu = 2, 3, \dots$ follow by induction once we apply Theorem 2.1 (i) and summation by parts. For $0 < \nu < 1$,

$$\begin{aligned} R_{\nu-1}(t^{(\nu-1)})(s) &= \sum_{t=\nu-1}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} t^{(\nu-1)} \\ &= \sum_{u=0}^{\infty} \left(\frac{1}{s+1}\right)^{u+\nu} (u+\nu-1)^{(\nu-1)} \\ &= \left(\frac{1}{s+1}\right)^{\nu} \sum_{u=0}^{\infty} \left(\frac{1}{s+1}\right)^{u} \frac{\Gamma(u+\nu)}{\Gamma(u+1)}. \end{aligned}$$

Apply [1, Theorem 2.2.1] to the right-hand side and introduce the hypergeometric function ${}_2F_1$ in the following way:

$$\begin{aligned} \sum_{t=\nu-1}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} t^{(\nu-1)} &= \frac{1}{(s+1)^{\nu}} \Gamma(\nu)_2F_1\left(1, \nu; 1; \frac{1}{s+1}\right) \\ &= \frac{1}{(s+1)^{\nu}} \frac{\Gamma(\nu)\Gamma(1)}{\Gamma(\nu)\Gamma(1-\nu)} \int_0^1 t^{\nu-1}(1-t)^{-\nu} \left(1 - \frac{t}{s+1}\right)^{-1} dt \\ &= \frac{1}{(s+1)^{\nu-1}} \frac{1}{\Gamma(1-\nu)} \int_0^1 \frac{(1-u)^{\nu-1}u^{-\nu}}{s+u} du \end{aligned}$$

$$= \frac{1}{(s+1)^{\nu-1}} \frac{1}{\Gamma(1-\nu)} B(\nu, 1-\nu)(1+s)^{\nu-1} s^{-\nu}$$
$$= \frac{\Gamma(\nu)}{s^{\nu}},$$

where B is the beta function. Here we used [3, Exercise 4.2.10]. This completes the proof for $0 < \nu < 1$. The equality

$$R_{\nu-1}(t^{(\nu-1)})(s) = \frac{\nu}{s} R_{\nu}(t^{(\nu)}),$$

where $\nu \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, follows from Theorem 2.1 (i) and summation by parts. The proof of (i) is complete.

The proof of (ii) follows from the proof of (i). ■

We shall also need to make use of a convolution theorem. Bohner and Peterson [2] have developed the convolution on general time scales; it has been developed on the time scale of integers in [4].

Let $h(t) = t^{(\nu-1)}$ and $g(t) = \alpha^t$. Define the convolution

$$(h * g)(t) = \sum_{s=0}^{t-\nu} (t - \sigma(s))^{(\nu-1)} g(s),$$

where $\nu \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$.

We now obtain a standard property for $R_{\nu}((h * g)(t))(s)$.

Lemma 3.2. For any $\nu \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$

$$R_{\nu}((h * g)(t))(s) = R_{\nu-1}(h(t))(s) R_0(g(t))(s).$$

Proof. We have

$$\begin{aligned} R_{\nu}((h * g)(t))(s) &= \sum_{t=\nu}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} \sum_{r=0}^{t-\nu} (t - \sigma(r))^{(\nu-1)} g(r) \\ &= \sum_{r=0}^{\infty} \sum_{t=r+\nu}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} (t - \sigma(r))^{(\nu-1)} g(r) \\ &= \sum_{r=0}^{\infty} \sum_{\tau=\nu-1}^{\infty} \left(\frac{1}{s+1}\right)^{\tau+r+2} \tau^{(\nu-1)} g(r) \\ &= \left(\sum_{\tau=\nu-1}^{\infty} \left(\frac{1}{s+1}\right)^{\tau+1} \tau^{(\nu-1)} \right) \left(\sum_{r=0}^{\infty} \left(\frac{1}{s+1}\right)^{r+1} g(r) \right) \\ &= R_{\nu-1}(h(t))(s) R_0(g(t))(s). \end{aligned}$$

This completes the proof. ■

We introduce a few more properties of the R-transform before we solve some fractional difference equations.

Treat $\Delta^{-\nu} f(t)$ as a convolution and apply Lemma 3.2 to obtain

$$R_{a+\nu}(\Delta^{-\nu} f(t))(s) = s^{-\nu} R_a(f(t))(s). \tag{3.2}$$

Lemma 3.3. For $0 < \nu < 1$ and the function f defined for $\nu - 1, \nu, \nu + 1, \ldots$,

$$R_0(\Delta^{\nu} f(t))(s) = s^{\nu} R_{\nu-1}(f(t)) - f(\nu - 1).$$

Proof. It is already shown [2,4], that if m is a positive integer, then

$$R_0(\Delta^m f)(s) = s^m R_0(f)(s) - \sum_{k=0}^{m-1} s^{m-k-1} \Delta^k f|_{t=0}.$$

For $m = 1$, we have

$$\begin{aligned}
R_0(\Delta^{\nu} f(t)) = R_0(\Delta\Delta^{-(1-\nu)} f(t)) &= \sum_{t=0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} \Delta\Delta^{-(1-\nu)} f(t) \\
&= s \sum_{t=0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} \Delta^{-(1-\nu)} f(t) - \Delta^{-(1-\nu)} f(t)|_{t=0} \\
&= s \sum_{t=0}^{\infty} \left(\frac{1}{s+1}\right)^{t+1} \Delta^{-(1-\nu)} f(t) - f(\nu - 1) \\
&= \frac{s}{\Gamma(1-\nu)} \sum_{\tau=-\nu}^{\infty} \left(\frac{1}{s+1}\right)^{\tau+1} t^{(-\nu)} \sum_{r=\nu-1}^{\infty} \left(\frac{1}{s+1}\right)^{r+1} f(r) - f(\nu - 1) \\
&= \frac{s}{\Gamma(1-\nu)} \frac{\Gamma(1-\nu)}{s^{1-\nu}} R_{\nu-1}(f(t)) - f(\nu - 1) \\
&= s^{\nu} R_{\nu-1}(f(t))(s) - f(\nu - 1).
\end{aligned}$$

This completes the proof. ∎

One can easily generalize this result to higher order. So if $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer and f is defined for $\mu - m, \mu - m + 1, \ldots$, then

$$R_0(\Delta^{\mu} f)(s) = s^{\mu} R_{\mu-m}(f)(s) - \sum_{k=0}^{m-1} s^{m-k-1} \Delta^{k-m+\mu} f|_{t=0}. \tag{3.3}$$

Example 3.4. Consider the problem $\Delta^{4/3} y(t) = 0$ for $t = 0, 1, 2, \ldots$. We shall look for a solution $y(t)$ defined on $\left\{-\frac{2}{3}, \frac{1}{3}, \frac{4}{3}, \ldots\right\}$. Assume that $\Delta^{1/3} y(t)$ is defined and finite.

First we take the R-transform of each side of the equation to obtain

$$R_0(\Delta^{4/3}y(t)) = s^2 R_0(\Delta^{-2/3}y(t)) - s\Delta^{-2/3}y(t)|_{t=0} - \Delta^{1/3}y(t)|_{t=0} = 0.$$

Since $R_0(\Delta^{-2/3}y(t)) = s^{-2/3}R_{-2/3}(y(t))$ by (3.2), it follows by Lemma 3.1(i) that

$$\begin{aligned} R_{-2/3}(y(t)) &= \frac{\Delta^{1/3}y(t)|_{t=0} + sy\left(-\frac{2}{3}\right)}{s^{4/3}} \\ &= \frac{\Delta^{1/3}(y(t))|_{t=0}}{\Gamma\left(\frac{4}{3}\right)} R_{1/3}(t^{(1/3)}) + \frac{y\left(-\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} R_{-2/3}(t^{(-2/3)}). \end{aligned}$$

Note that $R_{1/3}(t^{(1/3)}) = R_{-2/3}(t^{(1/3)})$, and so we conclude that

$$y(t) = \frac{\Delta^{1/3}(y(t))|_{t=0}}{\Gamma\left(\frac{4}{3}\right)} t^{(1/3)} + \frac{y\left(-\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} t^{(-2/3)}$$

for $t \in \left\{-\frac{2}{3}, \frac{1}{3}, \frac{4}{3}, \dots\right\}$.

Example 3.5. Consider the problem $\Delta^{1/2}y(t) = 5$ for $t = 0, 1, 2, \dots$. We shall look for a solution $y(t)$ defined on $\left\{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots\right\}$.

Applying the R-transform for each side of the given equation, we have

$$\begin{aligned} R_0(\Delta^{1/2}y(t)) &= R_0(5), \\ s^{1/2}R_{-1/2}(y(t)) - y\left(-\frac{1}{2}\right) &= \frac{5}{s}, \\ R_{-1/2}(y(t)) &= \frac{y\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} R_{-1/2}(t^{(-1/2)}) + \frac{5}{\Gamma\left(-\frac{3}{2}\right)} R_{1/2}(t^{(1/2)}). \end{aligned}$$

Since $R_{1/2}(t^{(1/2)}) = R_{-1/2}(t^{(1/2)})$, we have the solution

$$y(t) = \frac{y(-1/2)}{\Gamma(1/2)} t^{(-1/2)} + \frac{5}{\Gamma(-3/2)} t^{(1/2)}$$

for $t \in \left\{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots\right\}$.

Example 3.6. Consider the problem $\Delta^{5/3}y(t) + (1-\alpha)\Delta^{2/3}y(t) = 0$ for $t = 0, 1, 2, \dots$. We shall look for a solution $y(t)$ defined on $\left\{-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \dots\right\}$.

Applying the R-transform to each side of the equation, we have

$$\begin{aligned} R_{-1/3}(y(t)) = {} & \frac{y(-\frac{1}{3})}{\Gamma(-\frac{1}{3})} R_{-1/3}(t^{(-4/3)} * \alpha^t) \\ & + \frac{(1-\alpha)y(-1/3) + \Delta^{2/3}(y(t))|_{t=0}}{\Gamma(\frac{2}{3})} R_{2/3}(t^{(-1/3)} * \alpha^t). \end{aligned}$$

Using the step function $u_a(t)$ defined in [2, Example 3.93, page 125], we see the relation

$$R_a(f(t)u_a(t)) = R_{a+1}(f(t)).$$

Then the solution of the fractional difference equation is

$$y(t) = \frac{y(-1/3)}{\Gamma(-\frac{1}{3})}(t^{(-4/3)} * \alpha^t) + \frac{(1-\alpha)y(-\frac{1}{3}) + \Delta^{2/3}(y(t))|_{t=0}}{\Gamma(\frac{2}{3})}(\Delta^{-2/3}\alpha^t)u_{-1/3}(t)$$

for $t \in \left\{-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \ldots\right\}$.

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