International Journal of Difference Equations (IJDE).
ISSN 0973-6069 Volume 2 Number 2 (2007), pp. 139–164
© Research India Publications
http://www.ripublication.com/ijde.htm

# On the Oscillation of Solutions of Stochastic Difference Equations with State-Independent Perturbations*

**John A.D. Appleby**
*School of Mathematical Sciences,*
*Dublin City University, Dublin 9, Ireland*
*E-mail: john.appleby@dcu.ie*

**Alexandra Rodkina**
*Department of Maths/CSci, The University of the West Indies,*
*Kingston 7, Jamaica*
*E-mail: alexandra.rodkina@uwimona.edu.jm*

**Abstract**

This paper considers the pathwise oscillatory behaviour of the scalar nonlinear stochastic difference equation

$$X(n+1) = X(n) - f(X(n)) + \sigma(n)\xi(n+1), \quad n = 0, 1, \dots,$$

where $(\xi(n))_{n\geq 0}$ is a sequence of independent random variables with zero mean and unit variance. The real-valued function $f : \mathbb{R} \to \mathbb{R}$ is presumed to be continuous with $f(0) = 0$ and $xf(x) > 0$ for $x \neq 0$. It is shown that when the stochastic sequence is identically distributed, oscillation occurs if the noise intensity is not square summable, or if the mean reversion is relatively strong sufficiently far from the equilibrium, even in the case when the equilibrium is nonhyperbolic. If the noise intensity is square summable, it can be shown for both linear equations and for equations with a hyperbolic equilibrium that oscillation as well as nonoscillation can occur. This depends on the relation between the rates of decay of the noise intensity and of the solution of the underlying unperturbed deterministic equation.

**AMS subject classification:** Primary: 37H10, 39A11; Secondary: 60H10, 34F05, 65C20.
**Keywords:** Stochastic difference equation, oscillation, state-independent perturbation.

*The first author was partially supported by an Albert College Fellowship awarded by Dublin City University's Research Advisory Panel. The second author was supported by the Mona Research Fellowship Programm awarded by University of the West Indies, Mona.
Received May 25, 2007; Accepted August 20, 2007

## 1. Introduction

The oscillation of the solutions of deterministic difference equations has been discussed in many papers; a comprehensive survey of this literature is contained in [1]. In this paper we concentrate on the oscillation of solutions to scalar stochastic nonlinear difference equations. Aside from results concerning the preservation of oscillation and nonoscillation in solutions of discretised linear stochastic delay differential equations in [3,4] there is little known about the oscillation of the solutions of stochastic nonlinear difference equations.

Global a.s. asymptotic stability of the solutions to stochastic nonlinear difference equations was discussed in many papers, see the most relevant publications: [2, 5–7, 10, 11].

We say that the solution is oscillatory if it changes sign infinitely many times. In this paper we consider the oscillatory behaviour of sample paths of the stochastic difference equation

$$X(n+1) = X(n) - f(X(n)) + \sigma(n)\xi(n+1), \quad n = 0, 1, \dots. \tag{1.1}$$

Here the function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous, and to obey $f(0) = 0$, $xf(x) > 0$, for $x \neq 0$. This equation can be viewed as a stochastic analogue of the deterministic equation

$$x(n+1) = x(n) - f(x(n)) + S(n), \quad n = 0, 1, 2, \dots, \quad S_n \to 0, \tag{1.2}$$

where instead of the fading deterministic perturbation, $S_n \to 0$, a noise term $\sigma(n)\xi(n+1)$ is introduced, in which $\sigma(n)\xi(n+1) \to 0$ as $n \to \infty$ a.s. To analyse the effect of the introduction of the noise term alone, we often find it instructive to compare the oscillatory behaviour of (1.1) with its unperturbed deterministic counterpart

$$x(n+1) = x(n) - f(x(n)), \quad n = 0, 1, 2, \dots. \tag{1.3}$$

In this paper we show that when the noise is persistent or decays slowly in the sense that $\sigma \notin \boldsymbol{\ell_2}$, the solution $X$ of (1.1) oscillates almost surely regardless of the function $f$. Therefore, regardless of whether the solution of (1.3) is oscillatory or not, the introduction of a sufficiently intense noise term will cause almost all sample paths to oscillate. It is interesting to note that the rate at which the tails of the distributions of the random variables $(\xi_n)_{n\geq 0}$ decay does not play an important role in inducing this oscillation, at least in the case of identically distributed $\xi$.

When the intensity of the noise perturbation decays more quickly (i.e., when $\sigma \in \boldsymbol{\ell_2}$) it is possible for the solution of (1.1) to be oscillatory or nonoscillatory according to the form of the mean-reversion which results from the presence of the function $f$. When the function $f$ grows relatively quickly for large departures from the equilibrium level in the sense that

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 2, \tag{1.4}$$

we prove in Theorem 4.1 that for sufficiently large initial values $X(0)$ the solution $(X(n))_{n\geq 0}$ oscillates with probability arbitrarily close to unity, even when $\sigma \in \boldsymbol{\ell_2}$. This oscillation occurs independently on the behaviour of $\sigma$ and $\xi$. Here the presence of the noise perturbation does not change significantly the oscillatory property of the solution, since all solutions of (1.3) with sufficiently large initial values oscillate.

We should not expect all solutions to oscillate if the function $f$ obeys a condition of the form (1.4) for large $|x|$, but is nonhyperbolic at the equilibrium. An example of what can happen in the deterministic case is given by the equation

$$x(n+1) = x(n) - x^3(n) + S(n), \quad n = 1, 2, \ldots, \text{ where } S(n) \to 0 \text{ as } n \to \infty. \tag{1.5}$$

When $S_n \equiv 0$, the behaviour of the solution $(x(n))_{n\geq 0}$ differs according to the initial value $x(0)$; four types of behaviour are possible depending on whether

(i) $x(0) \in (-1, 1)$;

(ii) $x(0) \in (-\sqrt{2}, -1) \cup (1, \sqrt{2})$;

(iii) $x(0) \in \{-\sqrt{2}, \sqrt{2}\}$;

(iv) $x(0) \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

In case (i) the solution tends to zero monotonically; in case (ii) it is eventually monotone, experiencing at most a finite number of sign changes; in case (iii), the solution is the two-cycle $\{-\sqrt{2}, \sqrt{2}\}$; finally in case (iv) $|x(n)| \to \infty$ as $n \to \infty$, with the solution changing sign at each step. Therefore the only possibility for oscillation of solutions arises for sufficiently large initial values $x(0) \in (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$. Notice however that solutions are nonoscillatory if the initial values are sufficiently small. The situation is markedly different for equation (1.5) when $S(n) \not\equiv 0$. Even if the only one value, $S_0$, is big enough, the solution $(x(n))_{n\geq 0}$ to (1.5) with arbitrarily small initial value $x_0$ may oscillate. It can be shown that when $S(0) = 3$ and $S(n) \in (0, 0.3)$ for all $n > 0$, then any solution $(x(n))_{n\geq 0}$ of (1.5) with $x(0) \in (-0.5, 0.5)$ is oscillatory.

Aside from general forms of the function $f$ in this paper we consider the oscillation and nonoscillation of solutions of (1.1) in the case when the equation is linear (i.e., $f(x) = ax$ for $a > 0$) or when the deterministic equation (1.3) has a hyperbolic equilibrium at zero (i.e., $f'(0) \neq 0$). The results which cover these cases are presented in Theorems 5.5 and 5.6 and in Proposition 5.8. Theorem 6.1 furnishes us with a complete picture of the oscillatory behaviour of the linear equation in the important special case when the random process $(\xi(n))_{n\geq 0}$ is Gaussian. Roughly speaking, the results on the linear equation suggest that when $\sigma$ decays sufficiently rapidly in $\boldsymbol{\ell_2}$, and at a rate which depends on the rate of decay of the solutions of the unperturbed equation

$$x(n+1) = x(n) - ax(n), \quad n = 0, 1, \ldots,$$

then solutions are nonoscillatory; but once the rate of decay of $\sigma$ becomes slower than this critical rate, solutions are oscillatory. These results also show that while $\sigma \in \boldsymbol{\ell_2}$ is not sufficient to ensure the nonoscillation of solutions of (1.1), the transition between oscillation and nonoscillation occurs at a critical increasing weight function in a weighted space of $\ell_2$ sequences; in other words, it suggests that as far as oscillation of solutions of (1.1) is concerned, particular attention should be given to noise intensities in $\ell_2$ and related weighted $\ell_2$ spaces. This contrasts with the critical $\sigma$ for stability of (1.1) which can be identified with the condition $\sigma(n)\xi(n+1) \to 0$ a.s. as $n \to \infty$.

All these results suggest a general principle guiding the oscillatory behaviour of solutions of mean-reverting stochastic difference equations of the form (1.1) where the noise perturbation is independent of the state. The conjectured principle is that the introduction of such a noise perturbation into (1.3) can preserve oscillation if it is already present, and can induce it if oscillation is absent. However, it will not prevent oscillation if it is already present in solutions of (1.2). We have a lot less evidence for this conjecture in the case when the solution starts from a small initial condition or when $f$ is a nonlinear function possessing a nonhyperbolic equilibrium at zero. This question is completely open at present. However, based on a rather complete picture from the linear case, we conjecture that when noise intensity decays quite rapidly ($\sigma \in \ell_2$) the rate of decay of $\sigma$ to zero and the speed of mean reversion to zero given by the behaviour of $f$ local to zero, will interact to produce either oscillation or nonoscillation of the solution $(X(n))_{n\geq 0}$. It does not appear, based on the evidence from the linear case, that the rate of decay of the tails of the distributions of $(\xi(n))_{n\geq 0}$ influences greatly the presence of oscillatory or nonoscillatory solutions. We hope that the connection between the rate of decay of $\sigma$ and the rate of decay of the unperturbed equation will afford us the possibility of determining the rate of decay to zero of the solution $(X(n))_{n\geq 0}$ of (1.1); specifically we conjecture that a change in the pathwise rate of decay of the solution when the decay of $\sigma$ reaches a critical rate is coincident with a change in the behaviour of solutions from nonoscillatory to oscillatory. Some results about the pathwise decay rates of scalar stochastic difference equations with nonhyperbolic equilibria are given in [2, 5].

The oscillation of solutions of equations of the form (1.1) is of interest in economics. We may view (1.1) as a highly schematized and simplified model of a system in an economy which is subject to persistent stochastic shocks and which exhibits reversion to an equilibrium level. Such features prevail in many economic models of dynamic adjustment to equilibrium.

In this context, the oscillation of solutions of (1.1) corresponds to the overshooting of the equilibrium; such overshooting may therefore be thought of as an aperiodic economic cycle. The results of this paper suggest that if overshooting is present in the absence of stochastic disturbances, it will still be present when disturbances are introduced. Moreover, if the mean reversion in the system is sufficiently weak and the intensity of the shock fades sufficiently slow, overshooting around the equilibrium will be induced. Such overshooting without loss of stability of the equilibrium may be interpreted as a robust response of the system to persistent external shocks.

The paper is organized as follows. In Section 2 we give necessary definitions from stochastic analysis and discuss our results. Section 3 is devoted to the case of big noise, $\sigma \notin \ell_2$. In Section 4 we show oscillation for big initial conditions. Section 5 is devoted to the case $\sigma \in \boldsymbol{\ell_2}$ and deals with linear and hyperbolic $f$. In Section 6 we completely describe oscillatory behavior of solutions to linear equations with normal noises.

## 2. Definitions, Assumptions, Results and Discussion

Throughout this paper, we say that a sequence $\nu = \{\nu(n) : n \geq 0\} = (\nu(n))_{n\geq 0}$ is in $\boldsymbol{\ell_2}$ if $\sum_{n=0}^{\infty} \nu^2(n) < \infty$. We say that the equilibrium point 0 of (1.3) is *hyperbolic* if $f'(0) \neq 0$, and is *nonhyperbolic* if $f'(0) = 0$ (see, e.g., Elaydi [9]).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ be a complete filtered probability space.

**Assumption 2.1.** $(\xi(n))_{n\in\mathbb{N}}$ is a sequence of independent random variables with distribution functions $F_n$ and with $\mathbb{E}[\xi(n)] = 0$, $\mathbb{E}\left[\xi^2(n)\right] = 1$.

We suppose that the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is naturally generated, namely that $\mathcal{F}_n = \sigma\{\xi(0), \xi(1), \dots, \xi(n)\}$. Among all sequences $(X(n))_{n\in\mathbb{N}}$ of random variables we distinguish those for which $X(n)$ are $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$. We use the standard abbreviation "a.s." for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text. For more details on stochastic concepts and notations, the reader may consult [12].

In this paper we consider a nonlinear stochastic difference equation (1.1) with arbitrary initial value $X(0) \in \mathbb{R}$. We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and

$$xf(x) > 0, \quad x \neq 0, \quad f(0) = 0. \tag{2.1}$$

We distinguish between the following three classes of $f$:

(i) $f$ is nonlinear and nonhyperbolic at 0;

(ii) $f$ is linear;

(iii) $f$ is nonlinear and hyperbolic at 0;

and the following two classes of noise intensity $\sigma$, which appears in the stochastic perturbation $\sigma(n)\xi(n+1)$ in (1.1):

(i) $\sigma \notin \boldsymbol{\ell_2}$;

(ii) $\sigma \in \boldsymbol{\ell_2}$.

However, in Sections 5 and 6 we show that it is sometimes useful to consider subclasses of class (ii) of noise intensity $\sigma$.

**Definition 2.2.** The solution $(X(n))_{n\in\mathbb{N}}$ of equation (1.1) is said to be *a.s. oscillatory* if

$$\mathbb{P}\{X(n) < 0 \text{ i.o.}\} = 1, \quad \mathbb{P}\{X(n) > 0 \text{ i.o.}\} = 1. \tag{2.2}$$

It is said to be *a.s. nonoscillatory* if

$$\mathbb{P}\{X(n) > 0 \text{ ev.}\} = 1, \quad \text{or} \quad \mathbb{P}\{X(n) < 0 \text{ ev.}\} = 1. \tag{2.3}$$

We say that the solution is oscillatory with positive probability if $\mathbb{P}[\{X(n) < 0 \text{ i.o.}\} \cap \{X(n) > 0 \text{ i.o.}\}] > 0$, and nonoscillatory with positive probability if $\mathbb{P}[\{X(n) < 0 \text{ ev.}\} \cup \{X(n) > 0 \text{ ev.}\}] > 0$. Here "i.o." stands for *infinitely often* and "ev." stands for *eventually*.

When $\sigma \notin \boldsymbol{\ell_2}$, we distinguish between the cases when the $\xi(n)$ are independent and identically distributed and when $\xi(n)$ are simply independent. In the first case we prove that if $(\sigma(n))_{n\geq 0}$ is nondecreasing and

$$\sum_{n=0}^{\infty} \sigma^2(n) = \infty, \tag{2.4}$$

then $\mathbb{P}\{X(n) < 0 \text{ i.o.}\} = 1, \mathbb{P}\{X(n) > 0 \text{ i.o.}\} = 1$. The proof is based on the Law of the Iterated Logarithm for weighted sums of independent and identically distributed random variables (see e.g., [8]). When $\xi(n)$ are just independent but not identically distributed, we apply a different approach, based on the use of the Central Limit Theorem and Kolmogorov's Zero-One Law (see e.g., [12]). It appears that in this situation to guarantee oscillation we need to demand a more stringent restriction on the noise intensities than (2.4). This extra restriction depends on the particular distribution of the $\xi$. For example, in order to ensure the oscillation of solutions when the probability densities of each $\xi(n)$ decay polynomially with degree $m$, it is sufficient that $\sigma(n)$ decays to zero more slowly than $n^{-\frac{m-3}{2(m-1)}}$. Also, when the probability densities of each $\xi(n)$ decay exponentially, it suffices that $\sigma(n)$ decays to zero more slowly than $n^{-\frac{1}{2}}$. The detailed analysis of the main cases is given in Subsection 3.2.

## 3. Oscillation for Nonsquare Summable $\sigma$

In this section we consider equation (1.1) in the case where $\sigma \notin \boldsymbol{\ell_2}$ and where $f$ satisfies only (2.1). No other restrictions are imposed on $f$. This means that the results of this section can be applied to linear $f$ and nonlinear $f$ with a hyperbolic equilibrium at 0, as well as to equations with a unique nonhyperbolic equilibrium at zero, such as given when $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^3$.

**Lemma 3.1.** Let Assumption 2.1 hold, and suppose that $f$ is a continuous function satisfying (2.1). Let $(X(n))_{n\geq 0}$ be a solution of (1.1) and suppose that

$$\begin{aligned} &\mathbb{P}\left\{\limsup_{n\to\infty} \sum_{i=0}^{n-1} \sigma(i)\xi(i+1) = \infty\right\} = 1, \\ &\mathbb{P}\left\{\liminf_{n\to\infty} \sum_{i=0}^{n-1} \sigma(i)\xi(i+1) = -\infty\right\} = 1. \end{aligned} \tag{3.1}$$

Then the solution $(X(n))_{n\geq 0}$ of equation (1.1) oscillates a.s. for every initial condition $X_0 \in \mathbb{R}$, i.e., (2.2) holds.

*Proof.* Suppose that (2.2) is not true, i.e., there exists $\Omega_1 \subset \Omega$, with $\mathbb{P}[\Omega_1] > 0$, and $N(\omega)$ such that $X(n,\omega) > 0$ for all $n \geq N(\omega)$. The case when $X(n,\omega) > 0$ for all $\omega \in \Omega_1$ is analogous.

After summation and rearranging of (1.1), for $n \geq N(\omega)$ and $\omega \in \Omega_1$, we obtain

$$X(n,\omega) + \sum_{i=N(\omega)}^{n-1} f(X(i,\omega)) = X_0 - \sum_{i=0}^{N(\omega)-1} f(X(i,\omega)) + \sum_{i=0}^{n-1} \sigma(i)\xi(i+1,\omega). \quad (3.2)$$

Since $f(u) > 0$ for $u > 0$, the left-hand side of (3.2) is positive. However, due to (3.1) the right-hand side of (3.2) is oscillating. The contradiction thus obtained proves the result. ■

To prove the oscillation of the solution of equation (1.1), we show that (3.1) holds. In Subsection 3.1 we prove (3.1) for the case when $(\xi(n))_{n\in\mathbb{N}}$ are independent and identically distributed random variables and when $(\sigma(n))_{n\in\mathbb{N}}$ is nonincreasing. In Subsection 3.2 we prove results on the oscillation in the case when the random variables $(\xi(n))_{n\in\mathbb{N}}$ are not presumed to be identically distributed.

### 3.1. Independent and Identically Distributed $\xi(n)$

To show (3.1) in the case where $(\xi(n))_{n\geq 0}$ are independently and identically distributed, we apply the law of the iterated logarithm for weighted averages (see Chow and Teicher [8]).

**Theorem 3.2. (Law of the Iterated Logarithm)** Let $(\xi(n))_{n\geq 0}$ be independent identically distributed random variables with $\mathbb{E}[\xi(n)] = 0$, $\mathbb{E}[\xi^2(n)] = 1$. Let $(\sigma(n))_{n\geq 0}$ be such that

$$D^2(n) := \sum_{i=1}^{n} \sigma^2(i-1) \to \infty, \quad \text{as} \quad n \to \infty, \quad (3.3)$$

$$n\sigma^2(n) = \mathrm{O}(D^2(n+1)), \quad \text{as} \quad n \to \infty. \quad (3.4)$$

Define

$$S(n) = \sum_{i=1}^{n} \sigma(i-1)\xi(i). \quad (3.5)$$

Then

$$\limsup_{n\to\infty} \frac{S(n)}{\sqrt{2D^2(n)\log\log D^2(n)}} = 1, \quad \liminf_{n\to\infty} \frac{S(n)}{\sqrt{2D^2(n)\log\log D^2(n)}} = -1, \quad \text{a.s.}$$

**Theorem 3.3.** Let $f$ be a continuous function satisfying (2.1). Let $(\xi(n))_{n\geq 0}$ be identically distributed random variables satisfying Assumption 2.1. If $(X(n))_{n\geq 0}$ is a solution of (1.1) and $(|\sigma(n)|)_{n\geq 0}$ is a nonincreasing sequence such that

$$\sum_{n=0}^{\infty} \sigma^2(n) = \infty, \tag{3.6}$$

then $(X(n))_{n\geq 0}$ oscillates a.s.

*Proof.* We note that the condition (3.3) follows from condition (3.6) and condition (3.4) follows from the fact that $|\sigma|$ is nonincreasing: $D^2(n+1) = \sum_{i=1}^{n} \sigma^2(i) \geq n\sigma^2(n)$. The result then follows from Theorem 3.2 and Lemma 3.1. ■

### 3.2. Independent $\xi(n)$

Assume that the random variables $\xi(n)$ are independent but not identically distributed. To show oscillation of solutions, it is not sufficient merely to assume $\sum_{n=0}^{\infty} \sigma^2(n) = \infty$. We need to impose more restrictions on how slowly $\sigma(n)$ tends to zero as $n \to \infty$. These extra restrictions depend on the behaviour of the tails of the distributions of $\xi(n)$.

We state the central limit theorem and Kolmogorov's zero-one law (see e.g., [12]) which are applied in this section.

**Theorem 3.4. (Central Limit Theorem)** Suppose the sequence $(\xi(n))_{n\in\mathbb{N}}$ obeys Assumption 2.1. Define $S(n)$ as in (3.5) and $D^2(n)$ as in (3.3). Suppose that the Lindeberg condition holds true, namely that for every $\varepsilon > 0$,

$$\frac{1}{D^2(n)} \sum_{k=1}^{n} \int_{x:|x|\geq \varepsilon D(n)} x^2\, dF_k(x) \to 0, \quad n \to \infty. \tag{3.7}$$

Then

$$\frac{S(n)}{\sqrt{D^2(n)}} \xrightarrow{d} \mathcal{N}(0,1).$$

Let $\mathcal{F}_n^{\infty} = \sigma\{\xi(n), \xi(n+1), \ldots\}$. We define the tail algebra $\mathcal{X}$ by $\mathcal{X} = \bigcap_{n=1}^{\infty} \mathcal{F}_n^{\infty}$.

**Theorem 3.5. (Kolmogorov's Zero-One Law)** Let $(\xi(n))_{n\in\mathbb{N}}$ be a sequence of independent random variables and let $A \in \mathcal{X}$. Then $\mathbb{P}[A]$ can assume only the values one or zero.

**Lemma 3.6.** Suppose Assumption 2.1 and condition (3.7) hold. If $S(n)$ is as defined in (3.5) and $D^2(n)$ as in (3.3), then

$$\limsup_{n\to\infty} \frac{S(n)}{\sqrt{D^2(n)}} = \infty, \quad \liminf_{n\to\infty} \frac{S(n)}{\sqrt{D^2(n)}} = -\infty, \quad \text{a.s.} \tag{3.8}$$

*Proof.* Let $\Phi$ be the distribution function of a standardised normal random variable and $\Phi_n$ be a probability distribution function of the variable $\sum_{k=1}^{n} \sigma(k-1)\xi(k)/\sqrt{D^2(n)}$. From Theorem 3.4 we conclude that $\Phi_n(x) \to \Phi(x)$ for every $x \in \mathbb{R}$.

Fix some $c > 0$. Then $1 - \Phi(c) > 0$ and we can take $\varepsilon < (1 - \Phi(c))/2$. Let $N = N(c)$ be such that $|\Phi_n(x) - \Phi(x)| \le \varepsilon$ for $n \ge N(c)$. Then

$$1 - \Phi_n(c) = [1 - \Phi(c)] + [\Phi(c) - \Phi_n(c)] \ge 1 - \Phi(c) - \varepsilon \ge \frac{1 - \Phi(c)}{2} > 0.$$

The event

$$A = \left\{ \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} \sigma(k-1)\xi(k)}{\sqrt{D^2(n)}} > c \right\}$$

is a tail event, i.e., $A \in \mathcal{X}$. By Theorem 3.5, to prove that $\mathbb{P}[A] = 1$ for each $c > 0$, we need only to show that $\mathbb{P}[A] > 0$ for each $c > 0$. By [12, Problem 5, page 383], we have

$$\mathbb{P}\left\{ \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} \sigma(k-1)\xi(k)}{\sqrt{D^2(n)}} > c \right\} \ge \limsup_{n\to\infty} \mathbb{P}\left\{ \frac{\sum_{k=1}^{n} \sigma(k-1)\xi(k)}{\sqrt{D^2(n)}} > c \right\}. \tag{3.9}$$

Since for $n \ge N(c)$,

$$\mathbb{P}\left\{ \frac{\sum_{k=1}^{n} \sigma(k-1)\xi(k)}{\sqrt{D^2(n)}} > c \right\} = 1 - \Phi_n(c) \ge \frac{1 - \Phi(c)}{2} > 0,$$

the result follows. The second relation in (3.8) can be proved similarly. ∎

We denote by $g_n$ the probability density function of the random variable $\xi(n)$. For simplicity we suppose that $g_n$ is a nonincreasing and even function. The next lemma connects the rate of decay of $\sigma(n)$ and the tails of $g_n$ so that condition (3.7) is fulfilled. By Lemma 3.1 and Lemma 3.6, condition (3.7) implies the oscillation of the solution $(X(n))_{n\ge 0}$ of (1.1) with corresponding $\xi(n)$ and $\sigma(n)$. In the examples which follow Lemma 3.7, it is shown that the quicker the density $g_n$ decays the slower $\sigma(n)$ must decay to zero to ensure the oscillation of the solution.

**Lemma 3.7.** Let Assumption 2.1 hold. Let $g_k$ be the density function of the random variable $\xi(k)$. Suppose $g_k$ is a nonincreasing and even function for sufficiently large $|x|$. If moreover for every $\varepsilon > 0$,

$$\lim_{n\to\infty} nD(n) g_k(\varepsilon D(n)) = 0 \quad \text{uniformly in} \quad k \in \mathbb{N}, \tag{3.10}$$

then (3.7) holds.

*Proof.* Fix some $\varepsilon > 0$ and $\gamma > 0$. Let $\delta < \dfrac{3\gamma}{2\varepsilon^3}$. By condition (3.10), we can find $N(\delta)$ such that for $n \ge N(\delta)$ and all $k \in \mathbb{N}$ we have $nD_n g_k(\varepsilon D(n)) \le \delta$. To prove the Lindeberg condition (3.7), we estimate for $n \ge N(\delta)$

$$\begin{aligned}
&\frac{1}{D_n^2}\sum_{k=1}^{n}\int_{x:|x|\ge \varepsilon D_n} x^2 dF_k(x) = \frac{1}{D_n^2}\sum_{k=1}^{n}\int_{x:|x|\ge \varepsilon D_n} x^2 g_k(x)dx \\
&\le \frac{1}{D_n^2}\sum_{k=1}^{n} g_k(\varepsilon D_n)\int_{x:|x|\ge \varepsilon D_n} x^2 dx \le \frac{2\varepsilon^3}{3} D_n \sum_{k=1}^{n} g_k(\varepsilon D_n) \\
&\le \frac{2\varepsilon^3}{3} D_n \sum_{k=1}^{n} \frac{\delta}{nD_n} = \frac{2\varepsilon^3\delta}{3} \le \gamma,
\end{aligned}$$

completing the proof. ■

Below we present examples of densities with polynomially and exponentially decaying tails. The reader is invited to check that the required conditions are valid.

**Example 3.8.** Suppose Assumption 2.1 holds. Let $g_k$ be the density function of the random variable $\xi(k)$, where $g_k$ is a nonincreasing and even function for large $|x|$. Then (3.7) is fulfilled if:

(i) $g_k$ decays more quickly than some polynomial: for some $m > 3$

$$x^m g_k(x) \to 0, \text{ as } |x| \to \infty, \quad \text{uniformly in} \quad k \in \mathbb{N}, \tag{3.11}$$

and

$$nD^{-m+1}(n) \to \text{const}, \quad n \to \infty; \tag{3.12}$$

(ii) $g_k$ decays more quickly than some exponential: for some $\alpha > 0$

$$e^{x^\alpha} g_k(x) \to 0, \text{ as } |x| \to \infty, \quad \text{uniformly in} \quad k \in \mathbb{N}, \tag{3.13}$$

and

$$nD(n)e^{-(\varepsilon D(n))^\alpha} \to \text{const}, \quad n \to \infty. \tag{3.14}$$

Condition (3.12) holds when $\sigma^2(n) \ge n^{-\frac{m-3}{m-1}+\varepsilon}$ for some $\varepsilon > 0$. Condition (3.14) holds when $\alpha > 2$ and $\sigma^2(n) \ge n^{-1}$.

## 4. Oscillation for Sufficiently Large Initial Conditions

In this section we show that when

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 2, \tag{4.1}$$

the solution $(X(n))_{n\geq 0}$ of (1.1) oscillates with probability arbitrarily close to unity for all sufficiently large initial values $X_0$.

We notice by a simple Borel–Cantelli argument that $\sigma \in \boldsymbol{\ell_2}$ and Assumption 2.1 imply that $\sigma(n)\xi(n+1) \to 0$ a.s. as $n \to \infty$. Scrutiny of the proofs of the results below reveals that all that is used is the condition $\sigma(n)\xi(n+1) \to 0$ a.s. as $n \to \infty$. However, as all solutions of (1.1) oscillate a.s. when $\sigma \notin \boldsymbol{\ell_2}$ (and $\sigma$ is nonincreasing), we prefer to concentrate on the contrary case when $\sigma \in \boldsymbol{\ell_2}$. In this case, the oscillation is a consequence of the overshooting present in the original deterministic equation (1.3) rather than of a slowly fading stochastic intensity.

**Theorem 4.1.** Suppose that $f$ is a continuous function obeying (2.1) and (4.1). Suppose that $\sigma \in \boldsymbol{\ell_2}$. Let $(X(n))_{n\geq 0}$ be a solution of (1.1) with initial condition $X_0 \in \mathbb{R}$. Then for all $\gamma \in (0,1)$ there exist an event $\Omega_\gamma \subseteq \Omega$ with $\mathbb{P}[\Omega_\gamma] > 1-\gamma$, and a number $d(\gamma) > 0$, such that for all $|X_0| > d(\gamma)$ we have

$$\liminf_{n\to\infty} X(n,\omega) = -\infty, \quad \limsup_{n\to\infty} X(n,\omega) = \infty, \quad \text{for a.s.} \quad \omega \in \Omega_\gamma.$$

The proof of this theorem is a corollary of two results: the corresponding lemma for the deterministic equation

$$x(n+1) = x(n) - f(x(n)) + S(n), \quad n = 0, 1, \dots, \tag{4.2}$$

and a result about the boundedness of the noise term with probability close to 1, see Lemma 4.3.

**Lemma 4.2.** Suppose that $f$ is a continuous function obeying (2.1) and (4.1). Suppose also that $|S(n)| \leq \bar{S}$ for some $\bar{S} > 0$ and for all $n \in \mathbb{N}$. Let $(x(n))_{n\geq 0}$ be a solution of (4.2) with initial value $x_0 \in \mathbb{R}$. Then there is a $d > 0$ such that $\limsup_{n\to\infty} x(n) = \infty$ and $\liminf_{n\to\infty} x(n) = -\infty$ when $|x_0| > d$.

*Proof.* Condition (4.1) implies that for every $\gamma > 0$ there exists a $d_1 > 0$ such that for $|u| > d_1$,

$$|f(u)| \geq (2+\gamma/2)|u|.$$

We define $d = \max\left\{d_1, 4\bar{S}/\gamma\right\}$. For all $x(n) > d$ we have

$$\begin{aligned} x(n+1) &< x(n) - \left(2+\frac{\gamma}{2}\right)x(n) + S(n) = -\left(1+\frac{\gamma}{4}\right)x(n) - \frac{\gamma}{4}x(n) + S(n) \\ &\leq -\left(1+\frac{\gamma}{4}\right)x(n) - \frac{\gamma d}{4} + \bar{S} \leq -\left(1+\frac{\gamma}{4}\right)x(n), \end{aligned}$$

while for $x(n) < -d$ we can similarly show that $x(n+1) > -\left(1+\frac{\gamma}{4}\right)x(n)$. Thus in both cases $|x(n+1)| \geq (1+\gamma/4)|x(n)|$, which implies that $\lim_{n\to\infty} |x(n)| = \infty$. From the above estimates we also conclude that $x(n)$ changes sign at each step. ■

To prove our oscillation result, we must show that on an event of arbitrary probability less than unity, a uniform bound can be placed on $\sigma(n)\xi(n+1)$ which, on that event, depends only on the probability of the event. Lemma 4.3 holds true if $\sigma(n)\xi(n+1)$ is only bounded, i.e., there is a nonrandom $C > 0$ and $N(C, \omega)$ such that $|\sigma(n)\xi(n+1)| \le C$ for all $n \ge N(C, \omega)$ a.s.

**Lemma 4.3.** Let $\sigma \in \boldsymbol{\ell_2}$. Then for all $\gamma \in (0, 1)$ there exist $\Omega_\gamma \subseteq \Omega$ and $j(\gamma) > 0$ such that

$$\max_{n\in\mathbb{N}} |\sigma(n)\xi(n+1, \omega)| < j(\gamma), \quad \omega \in \Omega_\gamma, \quad \text{where } \mathbb{P}[\Omega_\gamma] > 1 - \gamma. \tag{4.3}$$

*Proof.* Since $\sigma \in \boldsymbol{\ell_2}$, it follows that $U$ defined by $U(n) = \sigma(n)\xi(n+1)$ obeys $U(n) \to 0$ a.s. We fix some $\delta_0 \in (0, 1)$. Then for all $\omega \in \Omega$ there is $N(\delta_0, \omega)$ such that $|U(n, \omega)| \le \delta_0$, $n \ge N(\delta_0, \omega)$. For all $\omega \in \Omega$ we set

$$\theta(\omega) = \max_{i=1,\dots,N(\delta_0,\omega)} \{|U(i, \omega)|\},$$

$$\underline{\Omega}_j = \{\omega : j - 1 \le \theta(\omega) < j\}, \quad \Omega_j = \{\omega : \theta(\omega) < j\} = \bigcup_{i=1}^{j} \underline{\Omega}_i.$$

Then $\underline{\Omega}_j \cap \underline{\Omega}_i = \emptyset$ when $j \ne i$, and $\Omega = \bigcup_{j=1}^{\infty} \underline{\Omega}_j$. Therefore $1 = \mathbb{P}[\Omega] = \sum_{i=1}^{\infty} \mathbb{P}[\underline{\Omega}_i]$, and for every $\gamma \in (0, 1)$ we can find $j(\gamma)$ such that for all $j > j(\gamma)$,

$$\mathbb{P}[\Omega_j] = \sum_{i=1}^{j} \mathbb{P}[\underline{\Omega}_i] > 1 - \gamma.$$

We let $\Omega_\gamma = \Omega_{j(\gamma)}$ and observe that $\mathbb{P}[\Omega_\gamma] > 1 - \gamma$ and $\theta(\omega) < j(\gamma)$ when $\omega \in \Omega_\gamma$. Since $\delta_0 < 1$, we also have $\max_{n\in\mathbb{N}} |\sigma(n)\xi(n+1, \omega)| < j(\gamma)$ a.s. for $\omega \in \Omega_\gamma$. ■

## 5. Linear Equations and Equations with Hyperbolic Equilibrium

Throughout this section we consider the case when $\sigma \in \boldsymbol{\ell_2}$. This case was not covered by the results in Section 3. We show that for both linear $f$ and hyperbolic $f$ both oscillation and nonoscillation of the solution $(X(n))_{n\ge0}$ of equation (1.1) may occur.

### 5.1. Oscillation and Nonoscillation in the Linear Case

Consider the linear stochastic difference equation

$$X(n+1) = X(n) - aX(n) + \sigma(n)\xi(n+1), \quad n = 0, 1, \dots, \quad X_0 \in \mathbb{R}. \tag{5.1}$$

We are going to distinguish between the cases when

$$\sum_{j=0}^{\infty} (1 - a)^{-2j} \sigma^2(j) < \infty, \tag{5.2}$$

and

$$\sum_{j=0}^{\infty}(1-a)^{-2j}\sigma^2(j)=\infty. \tag{5.3}$$

We now define an auxiliary process $M$ by

$$M(n)=\sum_{j=0}^{n-1}(1-a)^{-j}\sigma(j)\xi(j+1), \quad n=1,2,\ldots. \tag{5.4}$$

The following lemmas are required in the forthcoming proofs in this section and in Section 6.

**Lemma 5.1.** Let $(\xi(n))_{n\in\mathbb{N}}$ be independent random variables with continuous distributions. Suppose that $X$ obeys (5.1) and that $a>1$. Suppose for every $m\in\mathbb{N}$ that

$$\prod_{n=m}^{\infty}\mathbb{P}[\sigma(n-1)\xi(n)>0]=0, \quad \prod_{n=m}^{\infty}\mathbb{P}[\sigma(n-1)\xi(n)<0]=0. \tag{5.5}$$

Then $X$ is a.s. oscillatory.

The proof of this result is deferred to Section 7.

**Lemma 5.2.** Suppose that $\xi$ are identically and independently distributed random variables with zero means and continuous distributions. Then condition (5.5) holds.

*Proof.* We consider each of the terms in the infinite products in (5.5). Since the $\xi$ are iid we have $\mathbb{P}[\sigma(n-1)\xi(n)>0]=\mathbb{P}[\sigma(n)\xi>0]$. Suppose now that $\sigma(n-1)=0$. Then $\mathbb{P}[\sigma(n-1)\xi>0]=0$. If $\sigma(n-1)>0$, then $\mathbb{P}[\sigma(n-1)\xi>0]=\mathbb{P}[\xi>0]=:p$. Finally, if $\sigma(n-1)<0$, then $\mathbb{P}[\sigma(n-1)\xi<0]=\mathbb{P}[\xi<0]=1-p$. Thus

$$\mathbb{P}[\sigma(n-1)\xi(n)>0]=\begin{cases}0, & \sigma(n-1)=0,\\ p, & \sigma(n-1)>0,\\ 1-p, & \sigma(n-1)<0.\end{cases}$$

Thus $\mathbb{P}[\sigma(n-1)\xi(n)>0]\le\max\{p,1-p\}$. Similarly, $\mathbb{P}[\sigma(n-1)\xi(n)<0]\le\max\{p,1-p\}$. Note that $\max\{p,1-p\}=\alpha<1$ unless $p=0$ or $p=1$, last can occur only if $\mathbb{P}[\xi>0]=1$ or $\mathbb{P}[\xi<0]=1$, but both are incompatible with $\xi$ having zero mean. Hence (5.5) holds. ■

The following lemma is a corollary of Lemmas 5.1 and 5.2.

**Lemma 5.3.** Suppose that $a>1$ and $\xi$ are identically and independently distributed random variables with zero means and continuous distributions. If $X$ is a solution of (5.1), then $X$ is a.s. oscillatory.

The proof of the following result is deferred to Section 7.

**Lemma 5.4.** Suppose that $\xi$ are identically and independently distributed random variables with zero means and continuous distributions. If $X_n = \sigma(n-1)\xi(n)$ for all $n \in \mathbb{N}$, then $X$ is a.s. oscillatory.

**Theorem 5.5.** Suppose that the sequence $(\xi(n))_{n\in\mathbb{N}}$ satisfies Assumption 2.1. Suppose that $a \in (0,2) \setminus \{1\}$ and $\sigma$ obeys (5.2). If $(X(n))_{n\geq 0}$ is a solution of (5.1), then

$$\lim_{n\to\infty} (1-a)^{-n} X(n) \quad \text{exists and is finite a.s.} \tag{5.6}$$

(a) If $a \in (0,1)$, then $X$ is nonoscillatory with positive probability.

(b) If $a \in (0,1)$ and $\sum_{j=0}^{\infty}(1-a)^{-j}\sigma(j)\xi(j+1)$ is a continuous random variable, then $X$ is a.s. nonoscillatory.

(c) If $a \in (1,2)$, then $X$ is oscillatory with positive probability.

(d) If $a \in (1,2)$ and $\sum_{j=0}^{\infty}(1-a)^{-j}\sigma(j)\xi(j+1)$ is a continuous random variable, then $X$ is a.s. oscillatory.

*Proof.* Applying the variation of constants formula pathwise, we get

$$(1-a)^{-n}X(n) = X(0) + M(n), \quad n = 1, 2, \dots, \tag{5.7}$$

where $M$ is defined by (5.4). By the martingale convergence theorem (see, e.g., [12, Theorem 1, page 384], (5.2) implies that $\lim_{n\to\infty} M(n) = M^*$ exists and is finite a.s., and moreover that $\mathbb{E}[(M(n) - M^*)^2] \to 0$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty}(1-a)^{-n}X(n) = X(0) + \sum_{j=0}^{\infty}(1-a)^{-j}\sigma(j)\xi(j+1) =: L, \quad \text{a.s.,} \tag{5.8}$$

where $L$ is an almost surely finite random variable. In fact $\mathbb{E}[L] = X(0)$ and $\text{Var}[L] = \sum_{j=0}^{\infty}(1-a)^{-2j}\sigma^2(j)$. Clearly $\text{Var}[L] > 0$ unless $\sigma(n) = 0$ for all $n \in \mathbb{N}$, in which case the equation is deterministic, and the we have oscillation or nonoscillation according to whether $a \in (1,2)$ or $a \in (0,1)$.

To prove part (a), we notice that in the case when $a \in (0,1)$, $\text{Var}[L] > 0$ implies that $L$ cannot be atomic and obeys $\mathbb{P}[L = 0] = 0$. This implies that $\mathbb{P}[L \neq 0] > 0$, and so there is probability greater than $\mathbb{P}[L \neq 0]$ of nonoscillation. The proof of part (c) is similar; for each $\omega$ in the event $\{L \neq 0\}$, which has positive probability, the sign of $X(\omega)$ changes each step after some time $N(\omega)$, because the factor $(1-a)^{-n}$ alternates in sign. Hence $X$ is oscillatory with positive probability.

In case (b), when the right-hand side of (5.8) has a continuous density, it follows that $\mathbb{P}[L = 0] = 0$, and so $L \neq 0$ a.s. Therefore in this case we can prove that solutions of (5.1) are a.s. nonoscillatory. Case (d) follows similarly, by making the additional observation that the sign of $(1-a)^{-n}$ alternates for $a \in (1, 2)$. ■

When $a = 1$ and $\xi(n)$ are iid, we have $X(n+1) = \sigma(n)\xi(n+1)$ and condition (5.2) reduces to $\sigma \in \boldsymbol{\ell_2}$. Oscillation of $X$ in this case is proved in Lemma 5.4.

We now show that (5.2) is an essentially optimal condition for the intensity of noise that can be added into the equation without changing the oscillatory behaviour of solutions.

Suppose that

$$\left((1-a)^{-2n}\sigma^2(n)\right)_{n\geq 0} \text{ is nonincreasing and } \sum_{n=0}^{\infty}(1-a)^{-2n}\sigma^2(n) = \infty. \tag{5.9}$$

(5.9) is nothing but a special case of (5.3). An example of $\sigma$ for which (5.9) holds is $\sigma(n) = C(1-a)^n/(n+1)^\alpha$ for some $C > 0$ and $\alpha \in (0, 1/2]$. Notice however that if $\sigma(n) = C(1-a)^n/(n+1)^\alpha$ for $\alpha > 1/2$, then (5.2) holds, in which case $X$ is nonoscillatory.

At various stages in the paper it is convenient to employ the sequence $\tilde{\sigma}$ defined by

$$\tilde{\sigma}(n) = (1-a)^{-n}\sigma(n), \quad n \geq 0. \tag{5.10}$$

If (5.9) holds, then $(|\tilde{\sigma}(n)|)_{n\geq 0}$ is nonincreasing and $\sum_{n=0}^{\infty}\tilde{\sigma}^2(n) = \infty$.

**Theorem 5.6.** Let $(\xi(n))_{n\in\mathbb{N}}$ be identically distributed and satisfy Assumption 2.1. Suppose that $a \in (0, 1)$ and $\sigma$ obeys (5.9). If $(X(n))_{n\geq 0}$ is a solution of (5.1), then $X$ is a.s. oscillatory. Moreover, if $a \geq 1$, and in addition $\xi(n)$ is continuously distributed for each $n \in \mathbb{N}$, then $X$ is a.s. oscillatory.

*Proof.* Under condition (5.9) we may apply Theorem 3.2 to conclude that the weighted sum in (5.4) obeys

$$\limsup_{n\to\infty} M(n) = \infty, \quad \liminf_{n\to\infty} M(n) = -\infty. \tag{5.11}$$

Hence,

$$\limsup_{n\to\infty}(1-a)^{-n}X(n) = \infty, \quad \liminf_{n\to\infty}(1-a)^{-n}X(n) = -\infty, \quad \text{a.s.}$$

Therefore in the case when $a \in (0, 1)$, $X$ is a.s. oscillatory. The cases $a > 1$ and $a = 1$ follow from Lemmas 5.3 and 5.4 respectively. ■

### 5.2. Oscillation and Nonoscillation in the Nonlinear Case

Consider the nonlinear equation (1.1) where the function $f$, obeying (2.1), is differentiable in an open interval $I$ containing 0 and obeys

$$f'(0) = a \in (0, 2). \tag{5.12}$$

Suppose also that $f$ obeys a global linear bound with a "small" Lipschitz constant, in the sense that

$$\text{there exists } \gamma \in (0, 2) \text{ such that } |f(x)| \le (2 - \gamma)|x| \text{ for all } x \in \mathbb{R}. \tag{5.13}$$

This condition therefore covers cases which are not covered by Theorem 4.1, as (5.13) is inconsistent with (4.1).

Under the conditions stated above we can conclude that for all initial conditions $X_0 \in \mathbb{R}$ equation (1.1) has a solution $X(n)$ which tends to zero a.s. for $n \to \infty$. In other words we can prove the following result (see [5]).

**Theorem 5.7.** Let $f$ obey (2.1) and (5.13). Suppose that $\sigma \in \boldsymbol{\ell_2}$ and that $(\xi(n))_{n\in\mathbb{N}}$ obeys Assumption 2.1. If $(X(n))_{n\ge 0}$ is a solution to equation (1.1) with arbitrary initial value $X_0 \in \mathbb{R}$, then $\lim_{n\to\infty} X(n) = 0$ a.s.

Prior to determining the oscillatory behaviour of solutions of (1.1) under the above conditions, we first need to determine the asymptotic behaviour of solutions of the nonlinear equation. We believe moreover that this result is of independent interest. In the following proposition we show the fact that $\sigma \in \boldsymbol{\ell_2}$ is not enough for nonoscillation.

**Proposition 5.8.** Suppose that $f$ satisfies conditions (2.1), (5.12) and (5.13). Suppose that $a \neq 1$, and that $(\xi(n))_{n\in\mathbb{N}}$ obeys Assumption 2.1.

(i) If $\sigma$ obeys (5.2), then

$$\limsup_{n\to\infty} \frac{1}{n} \log |X(n)| \le \log |1 - a| < 0, \quad \text{a.s.} \tag{5.14}$$

Moreover, if $f$ is twice differentiable in some zero-neighborhood, then

$$\lim_{n\to\infty} (1 - a)^{-n} X(n) \quad \text{exists and is finite a.s.} \tag{5.15}$$

(ii) If $\sigma$ obeys (5.9), then (5.14) holds and

$$\limsup_{n\to\infty} (1 - a)^{-n} X(n) = \infty, \quad \liminf_{n\to\infty} (1 - a)^{-n} X(n) = -\infty. \tag{5.16}$$

The proof of Proposition 5.8 is deferred to Section 7.

**Theorem 5.9.** Suppose that $f$ satisfies conditions (2.1), (5.12) and (5.13). Suppose that $a \in (0, 1)$ and that $(\xi(n))_{n\in\mathbb{N}}$ obeys Assumption 2.1.

(i) If $\sigma$ obeys (5.2), $f$ is twice differentiable in some zero-neighborhood, and either

$$X(0) > 0 \quad \text{and} \quad ax - f(x) > 0 \quad \text{for all} \quad x \in \mathbb{R}, \tag{5.17}$$

or

$$X(0) < 0 \quad \text{and} \quad ax - f(x) < 0 \quad \text{for all} \quad x \in \mathbb{R}, \tag{5.18}$$

then $X$ is nonoscillatory with positive probability.

(ii) If $\sigma$ obeys (5.9), then $X$ is a.s. oscillatory.

*Proof.* In part (i), we have already shown by Proposition 5.8 that $\lim_{n\to\infty}(1-a)^{-n}X(n) = L'$, a.s., where

$$L' = X(0) + \sum_{j=0}^{\infty}(1-a)^{-j}\sigma(j)\xi(j+1) + \sum_{j=0}^{\infty}(1-a)^{-j}(aX(j) - f(X(j))).$$

Suppose (5.17) holds. If $a \in (0, 1)$, and $ax - f(x) \geq 0$, we can see that $L' \geq X(0) + \sum_{j=0}^{\infty}(1-a)^{-j}\sigma(j)\xi(j+1)$, and therefore, as $X(0) > 0$, that $\mathbb{E}[L'] > 0$. Hence $\mathbb{P}[L' > 0] > 0$, and so $X$ must be nonoscillatory with positive probability. The proof in the case (5.18) is similar.

In part (ii), we already note that (5.16) holds. Since $a \in (0, 1)$, and the sign of $(1-a)^{-n}$ does not change, it follows from (5.16) that the sign of $X$ must ultimately alternate infinitely often with probability one. ■

## 6. Linear Equations with Gaussian Noise

In this section we suppose that in the equation

$$X(n+1) = X(n) - aX(n) + \sigma(n)\xi(n+1), \quad n = 0, 1, \ldots, \quad X_0 \in \mathbb{R}, \tag{6.1}$$

$(\xi(n))_{n\in\mathbb{N}}$ are independent normal random variables, and $a \geq 0$. Therefore (6.1) is in the form (1.1); we do not analyse (6.1) when $a < 0$ because in this case $f(x) = ax$ does not obey (2.1). In this section we completely describe the oscillatory behaviour of the solutions to (6.1).

We assume that $\sigma \not\equiv 0$, for otherwise we are studying a deterministic equation.

**Theorem 6.1.** Suppose that $(\xi(n))_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed standard normal random variables. Let $(X(n))_{n\geq 0}$ be a solution of (6.1).

If $\sigma \notin \boldsymbol{\ell_2}$, then $X$ oscillates a.s.

Suppose $\sigma \in \boldsymbol{\ell_2}$.

(i) Let $1 \leq a$. Then $X$ is a.s. oscillatory.

(ii) Let $0 \le a < 1$.

(a) If $\sum_{j=0}^{\infty}(1-a)^{-2j}\sigma^2(j) < \infty$, then $X$ is a.s. nonoscillatory.

(b) If $\sum_{j=0}^{\infty}(1-a)^{-2j}\sigma^2(j) = \infty$, then $X$ is a.s. oscillatory.

*Proof.* Consider first the case when $\sigma \notin \boldsymbol{\ell_2}$. We recall from Lemma 3.1 that if (3.1) holds, then $X$ is a.s. oscillatory. Note that $S(n) = \sum_{j=0}^{n-1}\sigma(j)\xi(j+1)$ is normally distributed with mean 0 and variance $D^2(n) = \sum_{j=0}^{n-1}\sigma^2(j)$. Therefore $S(n)/\sqrt{D^2(n)}$ is a standardised normal random variable with distribution function $\Phi$. Hence for every $c > 0$, we have

$$\mathbb{P}\left[\frac{S(n)}{\sqrt{D^2(n)}} > c\right] = 1 - \Phi(c),$$

so by (3.9), we have

$$\mathbb{P}\left[\limsup_{n\to\infty}\frac{S(n)}{\sqrt{D^2(n)}} > c\right] \ge \limsup_{n\to\infty}\mathbb{P}\left[\frac{S(n)}{\sqrt{D^2(n)}} > c\right] = 1 - \Phi(c) > 0.$$

Therefore for every $c > 0$ we have

$$\limsup_{n\to\infty}\frac{S(n)}{\sqrt{D^2(n)}} > c, \quad \text{a.s.},$$

so $\limsup_{n\to\infty} S(n) = \infty$, a.s. Similarly, we can prove that $\liminf_{n\to\infty} S(n) = -\infty$, a.s., which gives (3.1).

Oscillation of $X$ in the case (i) follows from Lemma 5.3, when $a > 1$, and from Lemma 5.4, when $a = 1$.

Concerning (ii), we write the solution in the form

$$(1-a)^{-n}X(n) = X(0) + M(n), \quad n = 1, 2, \dots, \tag{6.2}$$

where $M$ is given by (5.4). We note now that $M(n)$ is normally distributed with mean zero and variance

$$\Delta^2(n) = \sum_{j=0}^{n-1}(1-a)^{-2j}\sigma^2(j). \tag{6.3}$$

For (a), we apply Theorem 5.5 (b), as the random variable $\sum_{j=0}^{\infty}(1-a)^{-j}\sigma(j)\xi(j+1)$ is normally distributed with nonzero variance, and hence continuously distributed, except in the case when $\sigma(n) \equiv 0$, which is ruled out by hypothesis. The existence of this random variable is guaranteed by the finiteness of $\sum_{j=0}^{\infty}(1-a)^{-2j}\sigma^2(j)$.

To prove part (b), we note that $\Delta^2(n) \to \infty$ as $n \to \infty$. By the argument in the case where $\sigma \notin \boldsymbol{\ell_2}$ above, we find that for every $c > 0$ we have

$$\limsup_{n\to\infty} \frac{M(n)}{\sqrt{\Delta^2(n)}} > c, \quad \text{a.s.},$$

and so $\limsup_{n\to\infty} M(n) = \infty$ a.s., and similarly $\liminf_{n\to\infty} M(n) = -\infty$ a.s. Therefore (6.2) implies that

$$\limsup_{n\to\infty}(1-a)^{-n}X(n) = \infty, \quad \liminf_{n\to\infty}(1-a)^{-n}X(n) = -\infty, \quad \text{a.s.} \tag{6.4}$$

In this case we cannot have $a = 0$, since $\sigma \in \boldsymbol{\ell_2}$. Therefore, as $a \in (0, 1)$, the factor $(1-a)^{-n}$ is positive, and so it must follow from (6.4) that $X$ changes sign infinitely often, almost surely. ■

## 7. Proofs

### 7.1. Proof of Lemma 5.1

Suppose that contrary to the desired result, $X$ is not oscillatory with positive probability. For simplicity we assume that $\mathbb{P}[A^*] > 0$, where

$$A^* = \{\omega : \text{ there exists } N_1(\omega) \in \mathbb{N} \text{ such that } X(n,\omega) \geq 0 \text{ for all } n \geq N_1(\omega)\}. \tag{7.1}$$

Since each $\xi$ has a continuous distribution, the random variable $X(0) + M(n)$ has a nonatomic and continuous distribution for each $n \in \mathbb{N}$, and hence $\mathbb{P}[X(0)+M(n) \neq 0] = 1$. Since $(1-a)^{-n}X(n) = X(0) + M(n)$, and $a > 1$, it follows that $\mathbb{P}[X(n) \neq 0] = 1$ for each $n \in \mathbb{N}$, and therefore

$$\mathbb{P}[X(n) \neq 0 \text{ for all } n \geq 1] = 1.$$

Therefore, we may consider an event $A$ with equal probability to $A^*$ defined by

$$A = \{\omega : \text{ there exists } N_1(\omega) \in \mathbb{N} \text{ such that } X(n,\omega) > 0 \text{ for all } n \geq N_1(\omega)\}. \tag{7.2}$$

We want to show that $\mathbb{P}[A] = 0$.

Since $a > 1$, the left-hand side of $(1-a)^{-n}X(n,\omega) = X(0)+M(n)$ changes sign at each step when $n \geq N_1(\omega)$ and $\omega \in A$. We set $\tilde{M}(n) = X(0) + M(n)$, with $M$ defined as in (5.4), and notice that $\tilde{M}(n)$ changes sign at each step for $n \geq N_1(\omega)$ and that

$$\tilde{M}(n+1) = \tilde{M}(n) + (1-a)^{-n}\sigma(n+1)\xi(n+2).$$

Then $(1-a)^{-n}\sigma(n+1)\xi(n+2) = \tilde{M}(n+1) - \tilde{M}(n)$ also changes sign at each step for $n \geq N_1(\omega)$. Without loss of generality we can suppose that $N_1(\omega)+1$ is odd. If $\tilde{M}(N_1(\omega)) > 0$, then

$$(1-a)^{-N_1}\sigma(N_1+1)\xi(N_1+2) < 0, \quad (1-a)^{-(N_1+1)}\sigma(N_1+2)\xi(N_1+3) > 0, \quad \ldots.$$

Since $1 - a < 0$, this implies that $\sigma(n-1)\xi(n) > 0$ for $n \geq N_1(\omega)$ and $\omega \in A$. Similarly, if $\tilde{M}(N_1(\omega)) < 0$, then $\sigma(n-1)\xi(n) < 0$ for $n \geq N_1(\omega)$ and $\omega \in A$. Then

$$\begin{aligned} 0 < \mathbb{P}\{A\} &= \mathbb{P}\{A \cup [\tilde{M}(N_1) > 0]\} + \mathbb{P}\{A \cup [\tilde{M}(N_1) < 0]\} \\ &\leq \mathbb{P}\{\omega \in A : \sigma(n-1)\xi(n) > 0,\ n \geq N_1(\omega)\} \\ &\quad + \mathbb{P}\{\omega \in A : \sigma(n-1)\xi(n) < 0,\ n \geq N_1(\omega)\}. \end{aligned}$$

The above means that either $\mathbb{P}\{\omega \in A : \sigma(n-1)\xi(n) > 0,\ n \geq N_1(\omega)\} > 0$ or $\mathbb{P}\{\omega \in A : \sigma(n-1)\xi(n) < 0,\ n \geq N_1(\omega)\} > 0$. Suppose that the first is true: $\mathbb{P}\{\omega \in A : \sigma(n-1)\xi(n) > 0,\ n \geq N_1(\omega)\} > 0$, and show that it is impossible.

Let $B = \{\omega \in A : \sigma(n-1)\xi(n) > 0,\ n \geq N_1(\omega)\}$, and $B' = \{\omega \in A : \sigma(n-1)\xi(n) < 0,\ n \geq N_1(\omega)\}$. We set $B_m = \{\omega \in A : N_1(\omega) = m\}$, $m = 1, 2, \ldots$. Then $B = \bigcup_{m=1}^{\infty} B_m$, and since the sets $B_m$ are disjoint, $\mathbb{P}[B] = \sum_{m=1}^{\infty} \mathbb{P}[B_m] > 0$. The last implies that there exists $m^* \in \mathbb{N}$ such that $\sum_{m=1}^{m^*} \mathbb{P}[B_m] > 0$. Therefore, on one side,

$$0 < \sum_{m=1}^{m^*} \mathbb{P}[B_m] = \mathbb{P}\left[\bigcup_{m=1}^{m^*} B_m\right].$$

On the other side, $\bigcup_{m=1}^{m^*} B_m = \bigcup_{m=1}^{m^*} \{\omega \in A : \sigma(n-1)\xi(n) > 0,\ n \geq m\}$. But

$$\begin{aligned} \mathbb{P}\{\omega \in A : \sigma(n-1)\xi(n) > 0,\ n \geq m\} &\leq \mathbb{P}\{\sigma(n-1)\xi(n) > 0,\ n \geq m\} \\ &= \prod_{n=m}^{\infty} \mathbb{P}\{\sigma(n-1)\xi(n) > 0\} = 0, \end{aligned}$$

by assumption. So

$$0 < \mathbb{P}\left[\bigcup_{m=1}^{m^*} B_m\right] = \mathbb{P}\left[\bigcup_{m=1}^{m^*}\{\omega \in A : \sigma(n-1)\xi(n) > 0,\ n \geq m\}\right]$$
$$\leq \sum_{m=1}^{m^*} \mathbb{P}\{\sigma(n-1)\xi(n) > 0,\ n \geq m\} = 0,$$

a contradiction. Hence $\mathbb{P}[B] = 0$. Similarly it can be proven that $\mathbb{P}[B'] = 0$. Therefore, $\mathbb{P}[A] = 0$.

### 7.2. Proof of Lemma 5.4

Suppose that $X$ is not oscillatory with positive probability, e.g. $X(n)$ is nonnegative for big $n$. Then, reasoning as in Lemma 5.1, we conclude that there is $A \in \Omega$, and $N_1(\omega)$, defined as in (7.1), such that

$$0 < \mathbb{P}[\omega \in A : \sigma(n-1)\xi(n) > 0,\ n \geq N_1(\omega)].$$

We set $B_m = \{\omega \in A : N_1(\omega) = m\}$, $m = 1, 2, \ldots$, and note that $A = \bigcup_{m=1}^{\infty} B_m$, $\mathbb{P}[A] = \sum_{m=1}^{\infty} \mathbb{P}[B_m] > 0$. Then there exists $m^* \in \mathbb{N}$ such that $\sum_{m=1}^{m^*} \mathbb{P}[B_m] > 0$ and

$$0 < \sum_{m=1}^{m^*} \mathbb{P}\{\sigma(n-1)\xi(n) > 0,\ n \geq m\}.$$

However, by Lemma 5.2, condition (5.5) holds, and then

$$\mathbb{P}\{\sigma(n-1)\xi(n) > 0,\ n \geq m\} = \prod_{n=m}^{\infty} \mathbb{P}\{\sigma(n-1)\xi(n) > 0\} = 0,$$

which leads to a contradiction.

### 7.3. Proof of Proposition 5.8

Theorem 5.7 implies that $\lim_{n\to\infty} X(n) = 0$ a.s. We rewrite (1.1) in the form

$$X(n+1) = b(n)X(n) + \sigma(n)\xi(n+1), \quad n = 1, 2, \ldots, \tag{7.3}$$

where

$$b(n) = \begin{cases} 1 - f(X(n))/X(n), & X(n) \neq 0, \\ 1 - a, & X(n) = 0. \end{cases}$$

We note that $b(n) \to 1-a$ as $n \to \infty$ a.s. since $X(n) \to 0$ as $n \to \infty$ a.s. Since $a \neq 1$, it follows that $b(n) \neq 0$ for all $n > N_1$, where $N_1$ is a random variable. Let $\phi(n) = \sigma(n)\xi(n+1)$. From (7.3), $X(n)$ can be written as

$$X(n) = \prod_{j=N_1}^{n} b(j) \left[ X(N_1) + \sum_{j=N_1}^{n-1} \left( \prod_{k=N_1}^{j} b(k) \right)^{-1} \phi(j) \right]$$

or

$$\left( \prod_{j=N_1}^{n} b(j) \right)^{-1} X(n) = X(N_1) + \sum_{j=N_1}^{n-1} \left( \prod_{k=N_1}^{j} b(k) \right)^{-1} \phi(j). \tag{7.4}$$

Now,

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \prod_{j=N_1}^{n} b(j) \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=N_1}^{n} \log |b(j)| = \log |1-a|.$$

Hence, with

$$F(n) = \left( \prod_{j=N_1}^{n} b(j) \right)^{-1} \phi(n),$$

we get

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log |F(n)| &= \limsup_{n\to\infty} \left[ -\frac{1}{n} \log \sum_{k=N_1}^{n} |b(k)| + \frac{1}{n} \log |\phi(n)| \right] \\ &= -\log|1-a| + \limsup_{n\to\infty} \frac{1}{n} \log |\phi(n)|. \end{aligned}$$

Now,

$$\begin{aligned} &\mathbb{P}\left[ |\phi(n)| > \varepsilon |1-a|^n \right] \le \frac{1}{\varepsilon^2} \frac{\mathbb{E}\left[\phi^2(n)\right]}{(1-a)^{2n}} \\ &= (1-a)^{-2n} \frac{\mathbb{E}\left[\sigma^2(n)\xi^2(n+1)\right]}{\varepsilon^2} = \frac{1}{\varepsilon^2}(1-a)^{-2n}\sigma^2(n). \end{aligned}$$

At this point, we use (5.2) to get

$$\sum_{n=0}^{\infty} \mathbb{P}\left[ |\phi(n)| > \varepsilon |1-a|^n \right] < \infty,$$

so by the Borel–Cantelli lemma we have

$$\limsup_{n\to\infty} \frac{|\phi(n)|}{(1-a)^n} = 0, \quad \text{a.s.}$$

Hence, a.s.,

$$\limsup_{n\to\infty} \frac{1}{n} \log |\phi(n)| \le \log |1-a|.$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \log |F(n)| \le -\log |1-a| + \log |1-a| = 0. \tag{7.5}$$

If, on the other hand, (5.9) holds, we have $\phi(n) = \sigma(n)\xi(n+1) = (1-a)^n \tilde{\sigma}(n)\xi(n+1)$. As $|\tilde{\sigma}|$ is nonincreasing, using Chebyshev's inequality, we get for arbitrary $\varepsilon > 0$ that

$$\mathbb{P}[|\phi(n)| > (1-a)^n e^{\varepsilon n}] \le \frac{1}{e^{2\varepsilon n}} \tilde{\sigma}^2(n) \le \tilde{\sigma}^2(0) e^{-2\varepsilon n}.$$

Therefore, by the Borel–Cantelli lemma, for every $\varepsilon \in (0, 1)$ there exists an almost sure event $\Omega_\varepsilon$ and an almost surely finite random variable $N_\varepsilon$ such that

$$|\phi(n)| \le (1-a)^n e^{\varepsilon n}, \quad n > N_\varepsilon(\omega),$$

which implies

$$\limsup_{n\to\infty} \frac{1}{n} \log |\phi(n)| \le \log |1-a| + \varepsilon, \quad \text{a.s. on} \quad \Omega_\varepsilon.$$

Letting $\varepsilon \to 0^+$ through the rational numbers, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log |\phi(n)| \le \log |1-a|, \quad \text{a.s.},$$

and so (7.5) holds in this case also.

Thus for every $\varepsilon > 0$ there is an almost sure event $\Omega^\varepsilon$ and a random variable $N_2(\varepsilon) > 0$ such that

$$|F(n)| \le e^{\varepsilon n}, \quad n > N_2(\varepsilon),$$

and, therefore, for $n > \max\{N_2(\varepsilon) + 1, N_1 + 1\}$,

$$\left| \sum_{j=N_1}^{n-1} F(j) \right| \le \sum_{j=N_1}^{N_2} |F(j)| + \sum_{j=N_2}^{n-1} e^{\varepsilon j} \le \sum_{j=N_1}^{N_2} |F(j)| + \frac{e^{\varepsilon n} - 1}{e^{\varepsilon} - 1}.$$

Hence, for all $\varepsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \left| \sum_{j=N_1}^{n-1} F(j) \right| \le \varepsilon, \quad \text{a.s. on} \quad \Omega^\varepsilon,$$

and then

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \sum_{j=N_1}^{n-1} F(j) \right| \le 0, \quad \text{a.s.} \tag{7.6}$$

Relation (7.4) and inequality (7.6) imply that a.s.

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log |X(n)| \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \left| X(n) \left( \prod_{j=N_1}^{n} b(j) \right)^{-1} \right| + \lim_{n\to\infty} \frac{1}{n} \log \left| \prod_{j=N_1}^{n} b(j) \right| \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \left| X(N_1) + \sum_{j=N_1}^{n-1} F(j) \right| + \log |1-a| \le \log |1-a|.
\end{aligned}$$

This proves the first part of both items, (i) and (ii), of the proposition.

We now consider the case when $f$ is twice differentiable in some open interval $I$ containing 0. If $g(x) = ax - f(x)$, then we have

$$g(x) = ax - \left[ f(0) + f'(0) + \frac{f''(\theta(x))}{2} x^2 \right] = -\frac{f''(\theta(x))}{2} x^2, \quad x \in I.$$

Then for $x \in I$ and for some $C > 0$, which does not depend on $x$, we have $|g(x)| \le Cx^2$.

We rearrange (1.1) and get

$$X(n+1) = X(n) - aX(n) + g(X(n)) + \sigma(n)\xi(n+1), \quad n = 0, 1, \dots .$$

Therefore, with $M$ defined as in (5.4), we get

$$X(n) = (1-a)^n X(0) + (1-a)^n M(n) + \sum_{j=0}^{n-1} (1-a)^{n-j} g(X(j)), \quad n = 0, 1, \dots . \tag{7.7}$$

Now, by (7.7), with $\tilde{g}(n) = (1-a)^{-n} g(X(n))$, we get for $n = 0, 1, \dots$

$$(1-a)^{-n} X(n) = X(0) + M(n) + \sum_{j=0}^{n-1} \tilde{g}(j). \tag{7.8}$$

Since

$$\limsup_{n\to\infty} \frac{1}{n} \log |g(X(n))| \le \limsup_{n\to\infty} \frac{1}{n} \log |X(n)|^2 = 2|1-a|,$$

we have $\limsup_{n\to\infty} \frac{1}{n} \log |\tilde{g}(n)| \le \log |1-a| < 0$, a.s., which implies that

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \tilde{g}(j) \quad \text{is finite a.s.} \tag{7.9}$$

Consider (i), when (5.2) holds. By the argument in Theorem 5.5, $\lim_{n\to\infty} M(n)$ is finite a.s. Therefore,

$$\lim_{n\to\infty}(1-a)^{-n}X(n) = X(0) + \sum_{j=0}^{\infty}(1-a)^{-j}\sigma(j)\xi(j+1) + \sum_{j=0}^{\infty}(1-a)^{-j}g(X(j)),$$

proving the result in this case.

To deal with part (ii), in which (5.9) holds, we note that (7.9) holds. The conditions of Theorem 5.6 now apply to $M$ defined by (5.4) and we have (5.11), which together with (7.9) and the representation (7.8), completes the proof of (ii).

## References


[1] Ravi P. Agarwal, Martin Bohner, Said R. Grace, and Donal O'Regan, *Discrete oscillation theory*, Hindawi Publishing Corporation, New York, 2005.

[2] John A.D. Appleby, Gregory Berkolaiko, and Alexandra Rodkina, Nonexponential stability and decay rates in nonlinear stochastic difference equations with unbounded noises, 2007. Submitted for publication.

[3] John A.D. Appleby and Cónall Kelly, Oscillation in a nonuniform discretisation of linear stochastic differential equations with vanishing delay, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 13B(suppl.):535–550, 2006.

[4] John A.D. Appleby and Cónall Kelly, Spurious oscillation in a uniform Euler discretisation of linear stochastic differential equations with vanishing delay, *J. Comput. Appl. Math.*, 205(2):923–935, 2007.

[5] John A.D. Appleby, D. MacKey, and Alexandra Rodkina, Almost sure polynomial asymptotic stability of stochastic difference equations, *Contemporary Mathematics. Fundamental Directions*, 17:110–128, 2006.

[6] John A.D. Appleby, Xuerong Mao, and Alexandra Rodkina, On stochastic stabilization of difference equations, *Discrete Contin. Dyn. Syst.*, 15(3):843–857, 2006.

[7] Gregory Berkolaiko and Alexandra Rodkina, Almost sure convergence of solutions to nonhomogeneous stochastic difference equation, *J. Difference Equ. Appl.*, 12(6):535–553, 2006.

[8] Y.S. Chow and H. Teicher, Iterated logarithm laws for weighted averages, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 26:87–94, 1973.

[9] Saber Elaydi, *An introduction to difference equations*, Undergraduate Texts in Mathematics. Springer, New York, third edition, 2005.

[10] Alexandra Rodkina and Henri Schurz, Global asymptotic stability of solutions of cubic stochastic difference equations. *Adv. Difference Equ.*, (3):249–260, 2004.

[11] Alexandra Rodkina and Henri Schurz, On global asymptotic stability of solutions of some in-arithmetic-mean-sense monotone stochastic difference equations in $\mathbb{R}^1$, *Int. J. Numer. Anal. Model.*, 2(3):355–366, 2005.

[12] A.N. Shiryaev, *Probability*, volume 95 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, second edition, 1996. Translated from the first (1980) Russian edition by R. P. Boas.