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Asymptotic Stability for 2 × 2 Linear Dynamic Systems on Time Scales

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Abstract

We prove asymptotical stability and instability theorems for 2×2 system of first-order linear dynamic equations on a time scale with complex-valued functions as coefficients. To prove stability estimates and asymptotic stability for a 2×2 system we use the integral representations of the fundamental matrix via asymptotic solutions, the error estimates, and the time scales calculus.

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1. Main Result

In this paper we study asymptotic stability of a system of linear dynamic equations on a time scale $\mathbb{T}_\infty = \mathbb{T} \cap (t_0, \infty)$:

$$u^\nabla(t) = A(t)u(t), \tag{1.1}$$

where u^∇ is the nabla derivative (see [7]), $u(t)$ is a 2-vector function, and

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \tag{1.2}$$

is a 2×2 matrix-function ld-differentiable on $\mathbb{T}_\infty$.

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Exponential decay and stability of solutions of dynamic equations on time scales was investigated in the recent papers [1, 8–11, 16, 17] by using Lyapunov's method. We use a different approach based on integral representations of solutions via asymptotic solutions and error estimates developed in [3, 12, 13, 15]. Denote

$$\operatorname{Tr} A(t) = a_{11}(t) + a_{22}(t), \quad |A(t)| = \det(A(t)). \tag{1.3}$$

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We assume $\sup \mathbb{T} = \infty$. For $t \in \mathbb{T}$ we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{for all} \quad t \in \mathbb{T}. \tag{1.4}$$

The backward graininess function $\nu : \mathbb{T} \to [0, \infty]$ is defined by $\nu(t) = t - \rho(t)$. If $\rho(t) < t$ (or $\nu(t) > 0$) we say that t is left-scattered. If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$ define the nabla derivative of f at t denoted $f^{\nabla}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho - s)| \le \varepsilon|\rho(t) - s| \tag{1.5}$$

for all $s \in U$.

The rest state $u(t) = 0$ of the system (1.1) is called stable if for any $\varepsilon > 0$ there exists $\delta(T, \varepsilon) > 0$ such that if $|u(T)| \le \delta(T, \varepsilon)$, then $|u(t)| \le \varepsilon$ for all $t \ge T$. The rest state $u(t) = 0$ of the system (1.1) is called asymptotically stable if it is stable, and attractive:

$$\lim_{t \to \infty} u(t) = 0. \tag{1.6}$$

To prove asymptotic stability we establish stability estimates for the dynamic system (1.1) by using integral representations of the fundamental matrix of (1.1) via asymptotic solutions, and calculus on time scales [6, 7].

A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous ($f \in C_{\mathrm{ld}}(\mathbb{T})$) provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. $C^k_{\mathrm{ld}}(\mathbb{T})$ is the class of functions for which nabla derivatives of order k exist and are ld-continuous on $\mathbb{T}$. Denote by $L_{\mathrm{ld}}(\mathbb{T})$ the class of functions $f : \mathbb{T} \to \mathbb{R}$ that are ld-continuous on $\mathbb{T}$ and Lebesgue nabla integrable on $\mathbb{T}$. Let

$$\mathbb{R}^+_{\mathrm{ld}} := \{K : \mathbb{T} \to \mathbb{R}, \quad K(t) \ge 0, \quad 1 - \nu K(t) > 0, \quad \text{and} \quad K \in C_{\mathrm{ld}}(\mathbb{T})\}. \tag{1.7}$$

We assume that $A \in C_{\mathrm{ld}}(\mathbb{T}_\infty)$ and $a_{12}(t) \ne 0$ for all $t \in \mathbb{T}_\infty$.

The main idea of this paper is a special construction of the phase functions $\theta_{1,2}$ of asymptotic solutions of the nonautonomous system (1.1).

From a given nontrivial function $\theta \in C^2_{\mathrm{ld}}(\mathbb{T}_\infty)$ we construct the function

$$k(t) = \frac{a_{12}(t)}{2\theta^2(t)} \left(\frac{\theta(t)}{a_{12}(t)}\right)^{\nabla}. \tag{1.8}$$

Here and further in the text we often suppressed dependance on t for simplicity.

Assuming $1 - 2k(t)\theta(t)\nu(t) \neq 0$ for all $t \in \mathbb{T}_\infty$ we choose a phase function $\theta_1(t)$ as a solution of the equation

$$\nu\theta_1^2 - 2\theta_1(1+\nu\theta) + 2\theta + \frac{\operatorname{Tr} A - \nu|A| - 2k\theta}{1 - 2k\theta\nu} = 0, \tag{1.9}$$

which is the version of Liouville's formula. If $\nu > 0$, then θ_1 is the solution of the quadratic equation

$$\theta_1 = \frac{1}{\nu} + \theta + \sqrt{D}, \quad D = \theta^2 + \frac{1 - \nu \operatorname{Tr} A + \nu^2|A|}{(1-2k\theta\nu)\nu^2}, \quad \nu > 0. \tag{1.10}$$

If $\nu = 0$, then (1.9) reduces to the linear equation

$$2\theta_1 - \operatorname{Tr} A + 2\theta(k-1) = \theta_1 + \theta_2 - \operatorname{Tr} A + \frac{a_{12}}{\theta}\left(\frac{\theta}{a_{12}}\right)' = 0,$$

and in this case the function $\theta_1(t)$ is defined by the formula

$$\theta_1(t) = \theta(t) - \frac{a_{12}(t)}{2\theta(t)}\left(\frac{\theta(t)}{a_{12}(t)}\right)' + \frac{\operatorname{Tr} A(t)}{2}, \quad \nu(t) = 0. \tag{1.11}$$

Define auxiliary functions

$$\theta_2(t) = \theta_1(t) - 2\theta(t), \quad \Psi(t) = \begin{pmatrix} \widehat{e}_{\theta_1}(t,t_0) & \widehat{e}_{\theta_2}(t,t_0) \\ \dfrac{(\theta_1 - a_{11})\widehat{e}_{\theta_1}(t,t_0)}{a_{12}(t)} & \dfrac{(\theta_2 - a_{11})\widehat{e}_{\theta_2}(t,t_0)}{a_{12}(t)} \end{pmatrix}, \tag{1.12}$$

$$\operatorname{Hov}_j = \theta_j^2 - \theta_j \operatorname{Tr}(A) + |A| - a_{12}(1-\nu\theta_j)\left(\frac{a_{11}-\theta_j}{a_{12}}\right)^\nabla, \quad j = 1,2, \tag{1.13}$$

$$Q_0(t) = \frac{\operatorname{Hov}_1(t) - \operatorname{Hov}_2(t)}{2\theta(t)}, \tag{1.14}$$

$$M_j = \|(1 - \nu\Psi^{-1}\Psi^\nabla)^{-1}\| \cdot \left|\frac{\widehat{e}_j \operatorname{Hov}_j}{2\theta \cdot \widehat{e}_{3-j}}\right|, \quad j = 1,2, \tag{1.15}$$

$$K(t) = c\left(\left|\frac{\operatorname{Hov}_1}{\theta}\right| + \sigma\left|\frac{a_{12}Q_0}{\theta}\right|\right)\left[1 + \nu\left(\|A\| + \left|\frac{\operatorname{Hov}_1}{a_{12}}\right| + \sigma|Q_0|\right)\right](t), \tag{1.16}$$

where $\|\cdot\|$ is the Euclidean matrix norm $\|A\| = \sqrt{\sum_{k,j=1}^{n} A_{kj}^2}$, and $\widehat{e}_\theta(t,t_0)$ is the nabla exponential function on a time scale (see [7, 10]).

Note that θ_1 and θ_2 can be used to form the approximate fundamental matrix Ψ of system (1.1) in form (1.12).

Theorem 1.1. Assume $a_{12}(t) \neq 0$, and there exists a nontrivial function $\theta \in C^2_{\mathrm{ld}}(\mathbb{T}_\infty)$ such that $M_j \in \mathbb{R}^+_{\mathrm{ld}}$, $1 - \nu \operatorname{Tr} A + \nu^2 |A|(t) \neq 0$, $1 - 2k\nu\theta(t) \neq 0$ for all $t \in \mathbb{T}_\infty$, and

$$\lim_{t\to\infty} \widehat{e}_{M_j}(t, t_0) < \infty, \quad j = 1, 2. \tag{1.17}$$

Then equation (1.1) is asymptotically stable if and only if the condition

$$\lim_{t\to\infty} \left(\frac{\theta_j - a_{11}}{a_{12}}\right)^{k-1} \widehat{e}_{\theta_j}(t, t_0) = 0, \quad k, j = 1, 2, \tag{1.18}$$

is satisfied.

Remark 1.2. If one can find two different phase functions θ_j, $j = 1, 2$ such that the generalized characteristic equations $\operatorname{Hov}_j(t) = 0$ are satisfied, then from (1.15) we get $M_j \equiv 0$, condition (1.17) disappears, and formula (1.12) with the above phase functions defines the exact fundamental solution of (1.1). Note also that for a constant matrix A, equations $\operatorname{Hov}_j(t) = 0$ turn to the usual characteristic equations of system (1.1).

Condition (1.17) of Theorem 1.1 is complicated and it is very restrictive when one of the functions $\left|\dfrac{\widehat{e}_{\theta_j}(t, t_0)}{\widehat{e}_{\theta_{3-j}}(t, t_0)}\right|$ has exponential growth as $t \to \infty$. In Theorem 1.3 below we replace the condition (1.17) by the less restrictive and simple condition (1.19) under some additional conditions.

Theorem 1.3. Assume $a_{12}(t) \neq 0$, $1 - \nu \operatorname{Tr} A + \nu^2 |A|(t) \neq 0$, and there exists a nontrivial function $\theta \in C^2_{\mathrm{ld}}(\mathbb{T}_\infty)$ such that $K \in \mathbb{R}^+_{\mathrm{ld}}$, $1 - 2k\nu\theta(t) \neq 0$, for all $t \in \mathbb{T}_\infty$, and there exist some constants $\beta > 0$ and $\sigma > 1$ such that

$$\lim_{t\to\infty} \widehat{e}_K(t, t_0) < \infty, \tag{1.19}$$

$$2\mathrm{Re}[\theta_j(t)] \leq \nu(t)|\theta_j(t)|^2, \quad j = 1, 2, \quad t \in \mathbb{T}_\infty, \tag{1.20}$$

$$\left|1 - \nu\left(\operatorname{Tr} A + Q_0\right) + \nu^2(|A| + \theta_1 Q_0 - \operatorname{Hov}_1)\right| \geq \beta > 0, \tag{1.21}$$

$$1 + \left|\frac{(\theta_j - a_{11})(t)}{a_{12}}\right| \leq \sigma, \quad j = 1, 2, \quad t \in \mathbb{T}_\infty, \tag{1.22}$$

$$\lim_{t\to\infty} |\widehat{e}_{\theta_j}(t, t_0)| = 0, \quad j = 1, 2. \tag{1.23}$$

Then equation (1.1) is asymptotically stable.

Note that if $\nu = 0$, then condition (1.20) becomes the classical stability condition $\mathrm{Re}[\theta_j(t)] \leq 0$. Condition (1.19) means that the error of the chosen asymptotic solution is small enough (compare with the well-known Levinson integrability condition from [15]).

The next three lemmas (see [1,10,17]) are useful tools for checking condition (1.23).

Lemma 1.4. [1, 10] Let θ be a complex valued function from $C_{\rm ld}(\mathbb{T})$ such that $1 - \theta(t)\nu(t) \neq 0$ for all $t \in \mathbb{T}_\infty$. Then

$$\lim_{t\to\infty} \widehat{e}_{\theta(t)}(t, t_0) = 0 \tag{1.24}$$

if and only if

$$\lim_{T\to\infty} \int_{t_0}^{T} \lim_{p\searrow\nu(s)} \frac{\mathrm{Log}|1 - p\theta(s)|}{-p} \nabla s = -\infty. \tag{1.25}$$

The following lemma gives simpler sufficient conditions of decay of the nabla exponential function.

Lemma 1.5. [1, 10] Assume $\theta \in C_{\rm ld}(\mathbb{T})$, and for some $\varepsilon > 0$

$$\lim_{t\to\infty} \int_{t_0}^{t} \mathrm{Re}[\theta(s)]\nabla s = -\infty, \quad \text{if} \quad \nu = 0, \tag{1.26}$$

$$|1 - \theta\nu(t)| \geq e^{\varepsilon} > 1, \quad \int_{t_0}^{\infty} \frac{\nabla s}{\nu(s)} = \infty, \quad \text{if} \quad \nu > 0. \tag{1.27}$$

Then (1.24) is satisfied.

Remark 1.6. [1] The first condition (1.27) for $\nu > 0$ means that values of $\theta(t)$ are located in the exterior of the ball with the center $\frac{1}{\nu_*}$ and radius $\frac{e^{\varepsilon}}{\nu_*}$:

$$\left\{ z : \left| z - \frac{1}{\nu_*} \right| > \frac{e^{\varepsilon}}{\nu_*} \right\}, \quad \nu_* = \inf[\nu(t)], \tag{1.28}$$

and it may be written in the form

$$2\mathrm{Re}[\theta(t)] < \nu(t)|\theta(t)|^2. \tag{1.29}$$

Remark 1.7. In view of Lemma 1.5, conditions (1.20), (1.23) of Theorem 1.3 can be replaced by

$$\int_{t_0}^{\infty} \frac{ds}{\nu(s)} = \infty, \quad \text{for} \quad \nu > 0. \tag{1.30}$$

$$2\mathrm{Re}[\theta_j(t)] < \nu(t)|\theta_j(t)|^2, \quad t \in \mathbb{T}_\infty, \quad j = 1, 2. \tag{1.31}$$

The scalar equation

$$x^{\nabla}(t) = \theta(t)x(t) \tag{1.32}$$

is called *exponentially stable* if there exists a constant $\alpha > 0$ such that for every $t_0 \in \mathbb{T}$ there exist a $N = N(t_0) \geq 1$ with

$$\|\widehat{e}_\theta(t, t_0)\| \leq N(t_0)e^{-\alpha(t-t_0)}, \quad \text{for} \quad t \geq t_0. \tag{1.33}$$

If the constant $N(t_0)$ from (1.33) can be chosen independent of t_0, then equation (1.32) is called *uniformly exponentially stable*.

Lemma 1.8. [17] Equation (1.32) is exponentially stable if and only if one of following conditions is satisfied for arbitrary $t_1 \in \mathbb{T}$:

$$\gamma(\theta) := \limsup_{T\to\infty} \frac{1}{T-t_1} \int_{t_1}^{T} \lim_{p\searrow\nu(s)} \frac{(\log|1-p\theta(s)|)}{-p} \nabla s < 0, \tag{1.34}$$

$$\text{for every } \tau \in \mathbb{T}: \text{ there exist } t \in \mathbb{T} \text{ with } t > \tau \text{ such that } 1-\nu(t)\theta(t) = 0, \tag{1.35}$$

where we use the convention $\log 0 = -\infty$ in (1.34).

Remark 1.9. In order to apply Theorem 1.3 to the study of exponential stability of the dynamic system (1.1), one can replace condition (1.23) by the necessary and sufficient conditions of exponential stability of an exponential function on a time scale given in Lemma 1.8.

2. Fundamental Matrix and Error Estimates

If we seek a solution of (1.1) in the form

$$u = \Psi v, \tag{2.1}$$

then from (1.1) we get

$$\Psi^\nabla v + \Psi v^\nabla - \nu\Psi^\nabla v^\nabla = A\Psi v,$$

$$\Psi(1-\nu\Psi^{-1}\Psi^\nabla)v^\nabla = (A\Psi - \Psi^\nabla)v,$$

or

$$v^\nabla(t) = H(t)v(t), \tag{2.2}$$

where

$$H = (1-\nu\Psi^{-1}\Psi^\nabla)^{-1}\Psi^{-1}(A\Psi - \Psi^\nabla). \tag{2.3}$$

Assume we can find an exact solution of an auxiliary system

$$\psi^\nabla(t) = A_1(t)\psi(t), \quad t \in \mathbb{T}_\infty, \tag{2.4}$$

with a matrix-function A_1 close to the matrix-function A, which means that condition (2.6) below is satisfied. Note that if $A = A_1$, then $H \equiv 0$ and (2.6) is satisfied.

Let $\Psi(t)$ be the fundamental matrix of the auxiliary system (2.4). If the matrix-function A_1 is regressive and ld-continuous, then $\Psi(t)$ exists ([6]). The solutions of (1.1) can be represented in the form

$$u(t) = \Psi(t)(C + \varepsilon(t)), \tag{2.5}$$

where $u(t), \varepsilon(t), C$ are the 2-vector columns: $u(t) = \text{column}(u_1(t), u_2(t))$, $\varepsilon(t) = \text{column}(\varepsilon_1(t), \varepsilon_2(t))$, $C = \text{column}(C_1, C_2)$, C_j are arbitrary constants. We can consider (2.5) as a definition of the error vector-function $\varepsilon(t)$.

In [12, 14] the following theorem was proved.

Theorem 2.1. Assume there exists a matrix function $\Psi \in C^1_{\text{ld}}(\mathbb{T}_\infty)$ such that $\|H\| \in \mathbb{R}^+_{\text{ld}}$, the matrix function $\Psi - \nu\Psi^\nabla$ is invertible, and the following exponential function is bounded:

$$\widehat{e}_{\|H\|}(\infty, t) = \exp \int_t^\infty \lim_{p \searrow \nu(s)} \frac{\text{Log}(1 - p\|H(s)\|)\nabla s}{-p} < \infty. \tag{2.6}$$

Then every solution of (1.1) can be represented in the form (2.5) and the error vector-function $\varepsilon(t)$ can be estimated by

$$\|\varepsilon(t)\| \le \|C\| \left(\widehat{e}_{\|H\|}(\infty, t) - 1\right), \tag{2.7}$$

where $\|\cdot\|$ is the Euclidean vector (or matrix) norm.

To find the fundamental matrix function let us seek solutions of equation (1.1)

$$u_1^\nabla = a_{11}u_1 + a_{12}u_2, \quad u_2^\nabla = a_{21}u_1 + a_{22}u_2, \tag{2.8}$$

of the form

$$u_1(t) = C_1\widehat{e}_{\theta_1}(t, t_0) + C_2\widehat{e}_{\theta_2}(t, t_0), \tag{2.9}$$

where

$$\widehat{e}_{\theta_j}(t, t_0) = \exp\left(\int_{t_0}^t \lim_{p \searrow \nu(\tau)} \frac{\text{Log}(1 - p\theta_j(\tau))}{-p}\nabla\tau\right), \quad j = 1, 2. \tag{2.10}$$

By differentiation

$$u_1^\nabla = C_1\theta_1\widehat{e}_{\theta_1}(t, t_0) + C_2\theta_2\widehat{e}_{\theta_2}(t, t_0), \tag{2.11}$$

and from (2.8) assuming $a_{12} \neq 0$ we get

$$u_2 = \frac{u_1^\nabla - a_{11}u_1}{a_{12}} = C_1U_1\widehat{e}_{\theta_1}(t, t_0) + C_2U_2\widehat{e}_{\theta_2}(t, t_0), \tag{2.12}$$

$$U_1 = \frac{\theta_1 - a_{11}}{a_{12}}, \quad U_2 = \frac{\theta_2 - a_{11}}{a_{12}}, \tag{2.13}$$

and

$$u(t) = \Psi(t)C, \tag{2.14}$$

where the fundamental matrix $\Psi(t)$ of system (1.1) is defined by

$$\Psi(t) = \begin{pmatrix} \widehat{e}_{\theta_1}(t, t_0) & \widehat{e}_{\theta_2}(t, t_0) \\ U_1\widehat{e}_{\theta_1}(t, t_0) & U_2\widehat{e}_{\theta_2}(t, t_0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ U_1 & U_2 \end{pmatrix} \begin{pmatrix} \widehat{e}_{\theta_1}(t, t_0) & 0 \\ 0 & \widehat{e}_{\theta_2}(t, t_0) \end{pmatrix}. \tag{2.15}$$

Denote

$$B_j = (a_{11} - \theta_j)(1 - \nu\theta_j)\frac{U_j^\nabla}{U_j} = a_{12}(1 - \nu\theta_j)\left(\frac{a_{11} - \theta_j}{a_{12}}\right)^\nabla, \quad j = 1, 2. \tag{2.16}$$

Then from (1.13) we get

$$\mathrm{Hov}_j = E_j - B_j, \quad E_j = \theta_j^2 - \theta_j \operatorname{Tr}(A) + |A|, \quad j = 1, 2, \tag{2.17}$$

where E_j is the usual characteristic polynomial of (1.1).

Lemma 2.2. Assume $a_{12} \neq 0$, $1 - \nu \operatorname{Tr} A + \nu^2|A| \neq 0$, $1 - 2k\nu\theta \neq 0$ for all $t \in \mathbb{T}_\infty$, $\Psi \in C_{\mathrm{ld}}(\mathbb{T})$ is invertible and nabla differentiable. Then the following formulas are true:

$$|\Psi(t)| = \det[\Psi(t)] = -\frac{2\theta}{a_{12}}\widehat{e}_{\theta_1}(t, t_0)\widehat{e}_{\theta_2}(t, t_0), \tag{2.18}$$

$$\theta_1 + \theta_2 - \operatorname{Tr} A + \frac{B_2 - B_1}{2\theta} = \frac{\mathrm{Hov}_1 - \mathrm{Hov}_2}{2\theta}, \tag{2.19}$$

$$\Psi^\nabla(t)\Psi^{-1}(t) = \begin{pmatrix} a_{11} & a_{12} \\ Q_1 + a_{21} & Q_0 + a_{22} \end{pmatrix}, \tag{2.20}$$

where

$$Q_0 = \frac{\Lambda_2 U_2 - \Lambda_1 U_1}{U_2 - U_1} - a_{22}, \quad Q_1 = \frac{(\Lambda_1 - \Lambda_2)U_1 U_2}{U_2 - U_1} - a_{21}, \tag{2.21}$$

or

$$Q_0 = \frac{\mathrm{Hov}_1 - \mathrm{Hov}_2}{2\theta}, \quad Q_1 = \frac{U_1 \mathrm{Hov}_2 - U_2 \mathrm{Hov}_1}{2\theta} = \frac{\mathrm{Hov}_1}{a_{12}} - U_1 Q_0, \tag{2.22}$$

$$\Psi^{-1}(A\Psi - \Psi^\nabla)(t) = \frac{1}{2\theta}\begin{pmatrix} -\mathrm{Hov}_1 & -\frac{\widehat{e}_{\theta_2}}{\widehat{e}_{\theta_1}}\mathrm{Hov}_2 \\ \frac{\widehat{e}_{\theta_1}}{\widehat{e}_{\theta_2}}\mathrm{Hov}_1 & \mathrm{Hov}_1 \end{pmatrix}, \tag{2.23}$$

$$\operatorname{Tr} A - \nu|A| = 2k\theta + (\theta_1 + \theta_2 - \nu\theta_1\theta_2)(1 - 2k\theta\nu). \tag{2.24}$$

Note that (2.24) is the version of Liouville's formula.

Proof. From (2.13) we have

$$U_2 - U_1 = \frac{\theta_2 - \theta_1}{a_{12}} = -\frac{2\theta}{a_{12}}, \tag{2.25}$$

and formula (2.18) follows from (2.15) and (2.25). From (2.15) we get the inverse matrix

$$\Psi^{-1}(t) = \frac{1}{U_2 - U_1}\begin{pmatrix} 1/\widehat{e}_{\theta_1}(t, t_0) & 0 \\ 0 & 1/\widehat{e}_{\theta_2}(t, t_0) \end{pmatrix}\begin{pmatrix} U_2 & -1 \\ -U_1 & 1 \end{pmatrix}.$$

Formula (2.19) follows from (2.17). From the time scales calculus we have

$$(ab)^\nabla = a^\nabla b + b^\nabla a - \nu a^\nabla b^\nabla, \quad \widehat{e}^\nabla_{\theta_j}(t,t_0) = \theta_j \widehat{e}_{\theta_j}(t,t_0).$$

The nabla derivative of the Ψ matrix function is given by the formula

$$\Psi^\nabla(t) = \begin{pmatrix} \theta_1 & \theta_2 \\ \Lambda_1 U_1 & \Lambda_2 U_2 \end{pmatrix} \begin{pmatrix} \widehat{e}_{\theta_1}(t,t_0) & 0 \\ 0 & \widehat{e}_{\theta_2}(t,t_0) \end{pmatrix}, \tag{2.26}$$

where

$$\Lambda_j = \frac{(U_j \widehat{e}_{\theta_j})^\nabla}{U_j \widehat{e}_{\theta_j}} = \theta_j + \frac{U_j^\nabla}{U_j}(1 - \nu\theta_j) = \theta_j - \frac{B_j}{a_{12} U_j}, \quad j = 1, 2. \tag{2.27}$$

Formulas (2.20), (2.21) are proved by direct calculations:

$$\begin{aligned} \Psi^\nabla(t)\Psi^{-1}(t) &= \frac{1}{U_2 - U_1} \begin{pmatrix} \theta_1 & \theta_2 \\ \Lambda_1 U_1 & \Lambda_2 U_2 \end{pmatrix} \begin{pmatrix} U_2 & -1 \\ -U_1 & 1 \end{pmatrix} \\ &= \frac{1}{U_2 - U_1} \begin{pmatrix} \theta_1 U_2 - \theta_2 U_1 & \theta_2 - \theta_1 \\ (\Lambda_1 - \Lambda_2) U_1 U_2 & \Lambda_2 U_2 - \Lambda_1 U_1 \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ Q_1 + a_{21} & Q_0 + a_{22} \end{pmatrix}. \end{aligned}$$

From (2.13), (2.17) we get

$$\begin{aligned} \frac{U_2 E_1 - U_1 E_2}{2\theta} &= \frac{(\theta_2 - a_{11})(\theta_1^2 - \theta_1 \operatorname{Tr} A + |A|) - (\theta_1 - a_{11})(\theta_2^2 - \theta_2 \operatorname{Tr} A + |A|)}{2\theta a_{12}} \\ &= \frac{a_{11}(\theta_2^2 - \theta_1^2 + (\theta_1 - \theta_2)\operatorname{Tr} A) + \theta_1\theta_2(\theta_1 - \theta_2) + |A|(\theta_2 - \theta_1)}{2\theta a_{12}} = -\frac{P}{a_{12}}, \end{aligned}$$

where

$$P = |A| + a_{11}(\theta_1 + \theta_2 - \operatorname{Tr} A) - \theta_1\theta_2. \tag{2.28}$$

Further we prove formulas (2.22):

$$\begin{aligned} Q_1 + a_{21} &= \frac{(\Lambda_1 - \Lambda_2) U_1 U_2}{U_2 - U_1} = \left(2\theta + \frac{B_2 U_1 - B_1 U_2}{a_{12} U_1 U_2}\right) \frac{U_1 U_2 a_{12}}{-2\theta} \\ &= -U_1 U_2 a_{12} - \frac{B_2 U_1 - B_1 U_2}{2\theta} = \frac{(a_{11} - \theta_1)(\theta_2 - a_{11})}{a_{12}} - \frac{B_2 U_1 - B_1 U_2}{2\theta} \\ &= \frac{|A| - \theta_1\theta_2 + a_{11}(\theta_1 + \theta_2 - a_{11} - a_{22})}{a_{12}} \\ &\quad + a_{21} + \frac{U_2(E_1 - \operatorname{Hov}_1) - U_1(E_2 - \operatorname{Hov}_2)}{2\theta} \\ &= \frac{P}{a_{12}} + a_{21} + \frac{U_1 \operatorname{Hov}_2 - U_2 \operatorname{Hov}_1}{2\theta} + \frac{U_2 E_1 - U_1 E_2}{2\theta} \\ &= \frac{U_1 \operatorname{Hov}_2 - U_2 \operatorname{Hov}_1}{2\theta} + a_{21}, \end{aligned}$$

and

$$
\begin{aligned}
Q_0 + a_{22} = \frac{\Lambda_2 U_2 - \Lambda_1 U_1}{U_2 - U_1} &= \frac{a_{11}-\theta_2}{2\theta}\left[\theta_2 - \frac{B_2}{U_2 a_{12}}\right] - \frac{a_{11}-\theta_1}{2\theta}\left[\theta_1 - \frac{B_1}{U_1 a_{12}}\right] \\
&= \frac{a_{11}\theta_2 - \theta_2^2 - a_{11}\theta_1 + \theta_1^2}{2\theta} + \frac{B_2 - B_1}{2\theta} \\
&= a_{22} + \theta_1 + \theta_2 - \operatorname{Tr} A + \frac{B_2 - B_1}{2\theta} \\
&= \frac{\mathrm{Hov}_1 - \mathrm{Hov}_2}{2\theta} + a_{22},
\end{aligned}
$$

in view of (2.19). For $j = 1, 2$ we have

$$
\frac{a_{12}}{2\theta}(Q_1 + Q_0 U_j) = \frac{a_{12}}{2\theta}\left(\frac{U_1 \mathrm{Hov}_2 - U_2 \mathrm{Hov}_1}{2\theta} + \frac{(\mathrm{Hov}_1 - \mathrm{Hov}_2)U_j}{2\theta}\right) = \frac{\mathrm{Hov}_j}{2\theta},
$$

from which formula (2.23) is deduced:

$$
\begin{aligned}
\Psi^{-1}(A\Psi - \Psi^{\nabla})(t) &= \frac{a_{12}}{2\theta}\begin{pmatrix} -Q_1 - Q_0 U_1 & -(Q_1 + Q_0 U_2)\dfrac{\widehat{e}_{\theta_2}}{\widehat{e}_{\theta_1}} \\ (Q_1 + Q_0 U_1)\dfrac{\widehat{e}_{\theta_1}}{\widehat{e}_{\theta_2}} & Q_1 + Q_0 U_2 \end{pmatrix} \\
&= \frac{1}{2\theta}\begin{pmatrix} -\mathrm{Hov}_1 & -\dfrac{\widehat{e}_{\theta_2}}{\widehat{e}_{\theta_1}}\mathrm{Hov}_2 \\ \dfrac{\widehat{e}_{\theta_1}}{\widehat{e}_{\theta_2}}\mathrm{Hov}_1 & \mathrm{Hov}_2 \end{pmatrix}.
\end{aligned}
$$

Formula (2.24) is proved by using the well-known Liouville formula for (1.1) (see [7, Theorem 3.9.4]):

$$
\frac{|\Psi(t)|^{\nabla}}{|\Psi(t)|} = \operatorname{Tr} A(t) - \nu|A(t)|, \tag{2.29}
$$

or in view of (2.18):

$$
\frac{a_{12}}{\theta e_1 e_2}\left(\frac{\theta e_1 e_2}{a_{12}}\right)^{\nabla} = \operatorname{Tr} A - \nu|A|.
$$

From this formula using notation (1.8) we get formula (2.24):

$$
\begin{aligned}
\operatorname{Tr} A - \nu|A| &= \left(\frac{\theta}{a_{12}}\right)^{\nabla}\frac{a_{12}}{\theta} + \frac{(\widehat{e}_{\theta_1}\widehat{e}_{\theta_2})^{\nabla}}{\widehat{e}_{\theta_1}\widehat{e}_{\theta_2}} - \nu\left(\frac{\theta}{a_{12}}\right)^{\nabla}\frac{a_{12}}{\theta}\frac{(\widehat{e}_{\theta_1}\widehat{e}_{\theta_2})^{\nabla}}{\widehat{e}_{\theta_1}\widehat{e}_{\theta_2}} \\
&= 2k\theta + \frac{(\widehat{e}_{\theta_1}\widehat{e}_{\theta_2})^{\nabla}}{\widehat{e}_{\theta_1}\widehat{e}_{\theta_2}}(1 - 2k\theta\nu) = 2k\theta + (\theta_1 + \theta_2 - \nu\theta_1\theta_2)(1 - 2k\theta\nu).
\end{aligned}
$$

This completes the proof. ∎

Proof of Theorem 1.1. First note that from the assumption $M_j \in \mathbb{R}^+_{\text{ld}}$ it follows that the exponential functions $\widehat{e}_{M_j}(t, t_0)$ exist ([7]). From assumptions $1 - \nu \operatorname{Tr} A + \nu^2 |A|(t) \neq 0$, $1 - 2k\nu\theta(t) \neq 0$ it follows that $1 - \nu\theta_j \neq 0$ and the exponential functions $\widehat{e}_{\theta_j}(t, t_0)$ exist. Consider the system (1.1) or the equivalent system (2.2). From (2.3), (1.15) and (2.23) it follows that $\|H\| \leq \|(1 - \nu\Psi^{-1}\Psi^{\nabla})^{-1}\Psi^{-1}(A\Psi - \Psi^{\nabla})\| \leq \max_{j=1,2} |M_j|$, and from condition (1.17) of Theorem 1.1 it follows that condition (2.6) of Theorem 2.1 is satisfied. Applying Theorem 2.1 we obtain representation (2.5) for the solutions of (1.1) and the estimate (2.7) for the error function ε. From (2.5), (2.7) we get the stability inequality

$$\|u(t)\| \leq \text{const}\|\Psi(t)\|. \tag{2.30}$$

From (1.18) it follows that $\|\Psi(t)\| \to 0$ as $t \to \infty$, and using (2.30) we obtain asymptotic stability of (1.1). ∎

Lemma 2.3. If the conditions of Lemma 2.2 are satisfied and for some number $\sigma > 1$

$$1 + \left| \frac{\theta_j - a_{11}}{a_{12}} \right| \leq \sigma, \quad j = 1, 2, \quad t \in \mathbb{T}_\infty, \tag{2.31}$$

$$|1 - \nu\theta_j| = \sqrt{(1 - \nu \text{Re}[\theta_j])^2 + (\nu\Im[\theta_j])^2} \geq 1, \quad j = 1, 2, \quad t \in \mathbb{T}_\infty, \tag{2.32}$$

then

$$\|\Psi(s)\| \leq \text{const}, \tag{2.33}$$

$$\|\Psi(t)\Psi^{-1}(s)\| \leq C \left| \frac{a_{12}(s)}{\theta(s)} \right|, \quad s \leq t, \tag{2.34}$$

$$\|A - \Psi^{\nabla}\Psi^{-1}\| \leq |Q_0| + |Q_1| \leq \left| \frac{\text{Hov}_1}{a_{12}} \right| + \sigma|Q_0|. \tag{2.35}$$

Proof. From (2.32) it follows that the functions $|\widehat{e}_{\theta_j}(t, t_0)|$, $j = 1, 2$, are nonincreasing. Indeed, if $\nu > 0$, then from (2.32) it follows that

$$\frac{\text{Log}|1 - \nu(t)\theta_j(t)|}{-\nu(t)} \leq 0, \tag{2.36}$$

so the functions

$$\widehat{e}_{\theta_j}(t, t_0)| = \exp\left(\int_{t_0}^{t} \lim_{p \searrow \nu(\tau)} \frac{\text{Log}|1 - p\theta_j(\tau)|}{-p} \nabla\tau \right), \quad j = 1, 2$$

are nonincreasing. If $\nu \equiv 0$, then the functions $|\widehat{e}_{\theta_j}(t, t_0)|$ are nonincreasing in view of

$$\frac{|\widehat{e}_{\theta_j}(t, t_0)|^{\nabla}}{|\widehat{e}_{\theta_j}(t, t_0)|} = \frac{|\widehat{e}_{\theta_j}(t, t_0)|'}{|\widehat{e}_{\theta_j}(t, t_0)|} = \text{Re}[\theta_j] \leq 0. \tag{2.37}$$

Because the functions $|\widehat{e}_{\theta_j}(t,t_0)|$, $j=1,2$, are nonincreasing we get

$$|\widehat{e}_{\theta_j}(t,t_0)| \le |\widehat{e}_{\theta_j}(t_0,t_0)| = 1, \qquad \left|\frac{\widehat{e}_{\theta_j}(t,t_0)}{\widehat{e}_{\theta_j}(s,t_0)}\right| \le 1, \quad t \ge s \ge t_0. \tag{2.38}$$

From condition (2.31) it follows that $|U_j| \le C$ and inequality (2.33) is true. Inequality (2.34) follows from the formula

$$\Psi(t)\Psi^{-1}(s) = \frac{1}{(U_2-U_1)(s)} \times \begin{bmatrix} \frac{\widehat{e}_{\theta_1}(t,t_0)}{\widehat{e}_{\theta_1}(s,t_0)}U_2(s) - \frac{\widehat{e}_{\theta_2}(t,t_0)}{\widehat{e}_{\theta_2}(s,t_0)}U_1(s) & \frac{\widehat{e}_{\theta_2}(t,t_0)}{\widehat{e}_{\theta_2}(s,t_0)} - \frac{\widehat{e}_{\theta_1}(t,t_0)}{\widehat{e}_{\theta_1}(s,t_0)} \\ \frac{\widehat{e}_{\theta_1}(t,t_0)}{\widehat{e}_{\theta_1}(s,t_0)}U_1(t)U_2(s) - \frac{\widehat{e}_{\theta_2}(t,t_0)}{\widehat{e}_{\theta_2}(s,t_0)}U_1(s)U_2(t) & \frac{\widehat{e}_{\theta_2}(t,t_0)}{\widehat{e}_{\theta_2}(s,t_0)}U_2(t) - \frac{\widehat{e}_{\theta_1}(t,t_0)}{\widehat{e}_{\theta_1}(s,t_0)}U_1(t) \end{bmatrix}. \tag{2.39}$$

Further using (2.20), (2.22) we get estimate (2.35):

$$\|A - \Psi^{\nabla}\Psi^{-1}\| \le |Q_1| + |Q_0| = \left|\frac{\mathrm{Hov}_1}{a_{12}} - U_1 Q_0\right| + |Q_0| \le \left|\frac{\mathrm{Hov}_1}{a_{12}}\right| + \sigma|Q_0|. \tag{2.40}$$

This completes the proof. ■

Lemma 2.4. If the conditions of Lemma 2.3 and (1.21) are satisfied, then

$$\|\Psi(t)H(s)\Psi^{-1}(s)\| \le K(s), \qquad s \in \mathbb{T}_\infty \cap [t_0, t], \tag{2.41}$$

where $K(s)$ is defined in (1.16).

Proof. Denote

$$\Omega(s) \equiv \Psi(1 - \nu\Psi^{-1}\Psi^{\nabla})\Psi^{-1}(s) = 1 - \nu\Psi^{\nabla}\Psi^{-1} = 1 - \nu A - \nu \begin{pmatrix} 0 & 0 \\ Q_1 & Q_0 \end{pmatrix}. \tag{2.42}$$

Then

$$\|\Omega\| \le 1 + \nu(\|A\| + |Q_1| + |Q_0|),$$
$$\|\Omega^{\mathrm{co}}\| \le 1 + \nu(\|A\| + |Q_1| + |Q_0|),$$

where Ω^{co} is the adjoint of the matrix Ω. Using (2.22) we have

$$a_{11}Q_0 - a_{12}Q_1 = a_{11}Q_0 - a_{12}\left(\frac{\mathrm{Hov}_1}{a_{12}} - U_1 Q_0\right) = \theta_1 Q_0 - \mathrm{Hov}_1\,. \tag{2.43}$$

From (2.20) we get

$$|\det(\Omega)| = \left|\det[1 - \nu\Psi^{\nabla}\Psi^{-1}]\right| = \left|\det\begin{bmatrix} 1-\nu a_{11} & -\nu a_{12} \\ -\nu(Q_1 + a_{21}) & 1 - \nu(Q_0 + a_{22}) \end{bmatrix}\right|$$
$$= |1 - \nu(Q_0 + \mathrm{Tr}\,A) + \nu^2(|A| + a_{11}Q_0 - a_{12}Q_1)|.$$

In view of (2.43)

$$|\det(\Omega)| = |1 - \nu(Q_0 + \operatorname{Tr} A) + \nu^2(|A| + \theta_1 Q_0 - \mathrm{Hov}_1)|,$$

and from assumption (1.21) we have

$$|\det(\Omega)| \geq \beta > 0. \tag{2.44}$$

Further

$$\|\Omega^{-1}\| = \frac{\|\Omega^{\mathrm{co}}\|}{|\det(\Omega)|} \leq \frac{1 + \nu(\|A\| + |Q_1| + |Q_0|)}{|\det(\Omega)|} \leq \frac{1 + \nu(\|A\| + |Q_0| + |Q_1|)}{\beta}$$

and using (2.34), (2.35) we get

$$\begin{aligned}
&\|\Psi(t)H(s)\Psi^{-1}(s)\| \\
&\quad\leq \|\Psi(t)\Psi^{-1}(s)\| \cdot \|\Psi(1 - \nu\Psi^{-1}\Psi^{\nabla})^{-1}\Psi^{-1}(A - \Psi^{\nabla}\Psi^{-1})\|(s) \\
&\quad\leq C\left|\frac{a_{12}(s)}{\theta(s)}\right| \|\Omega^{-1}(A - \Psi^{\nabla}\Psi^{-1})\|(s) \\
&\quad\leq C\left|\frac{a_{12}}{\theta}\right| (|Q_1| + |Q_0|)(1 + \nu(\|A\| + |Q_1| + |Q_0|)) \\
&\quad\leq c\left|\frac{a_{12}(s)}{\theta(s)}\right| \left(\sigma|Q_0| + \frac{|\mathrm{Hov}_1|}{|a_{12}|}\right)\left(1 + \nu(\|A\| + \sigma|Q_0| + \frac{|\mathrm{Hov}_1|}{|a_{12}|}\right) = K(s).
\end{aligned}$$

The proof is complete. ■

Lemma 2.5. [6,12] Assume $y, f \in C_{\mathrm{ld}}(\mathbb{T})$, $f, y \geq 0$, $K \in \mathbb{R}_\nu^+$. Then

$$y(t) \leq f(t) + \int_{t_0}^{t} K(s)y(s)\nabla s \quad \text{for all} \quad t \in \mathbb{T}_\infty \tag{2.45}$$

implies

$$y(t) \leq f(t) + \int_{t_0}^{t} \widehat{e}_K(t, \rho(s))K(s)f(s)\nabla s \quad \text{for all} \quad t \in \mathbb{T}_\infty. \tag{2.46}$$

Proof of Theorem 1.3. Integrating both sides of (2.2) we get

$$v(t) = C + \int_{t_0}^{t} H(s)v(s)\nabla s. \tag{2.47}$$

Multiplying by $\Psi(t)$ we have

$$\Psi(t)v(t) = \Psi(t)C + \int_{t_0}^{t} \Psi(t)H(s)v(s)\nabla s, \tag{2.48}$$

and using $u(t) = \Psi(t)v(t)$ we get

$$u(t) = \Psi(t)C + \int_{t_0}^{t} \Psi(t)H(s)\Psi^{-1}(s)u(s)\nabla s. \tag{2.49}$$

In view of (2.41) we get

$$\|u(t)\| \leq \|\Psi(t)C\| + \int_{t_0}^{t} K(s)\|u(s)\|\nabla s. \tag{2.50}$$

Applying Gronwall's inequality (2.46) to (2.50) we get the stability estimate

$$\|u(t)\| \leq \|\Psi(t)C\| + \int_{t_0}^{t} \widehat{e}_K(t, \rho(s))K(s)\|\Psi(t)C\|\nabla s, \quad t \in \mathbb{T}_\infty. \tag{2.51}$$

From (1.22), (1.23) it follows that

$$\lim_{t\to\infty} \|\Psi(t)C\| = 0, \tag{2.52}$$

so for any $\varepsilon > 0$ there exists t_0 such that for $t \in \mathbb{T}_\infty$ we have

$$\|\Psi(t)C\| \leq \varepsilon. \tag{2.53}$$

Hence it follows from (2.51) that

$$\|u(t)\| = \varepsilon \left(1 + \int_{t_0}^{t} \widehat{e}_K(t, \rho(s))K(s)\nabla s\right), \quad t \in \mathbb{T}_\infty. \tag{2.54}$$

Further

$$\int_{t_0}^{t} \widehat{e}_K(t, \rho(s))K(s)\nabla s = \widehat{e}_K(t, t_0) - \widehat{e}_K(t, t), \tag{2.55}$$

and so

$$\|u(t)\| \leq \varepsilon\widehat{e}_K(t, t_0) \leq C\varepsilon, \tag{2.56}$$

from which we get asymptotic stability of the dynamic system (1.1). ■

Example 2.6. Consider the system (1.1) with

$$A = \begin{bmatrix} 0 & 1 \\ -\bar{q} & -\bar{p} \end{bmatrix}, \quad \bar{q} = \frac{b}{t\rho}, \quad \bar{p} = \frac{a}{\rho}, \quad t_0 > 0, \tag{2.57}$$

$$\operatorname{Tr} A = -\bar{p} = -\frac{a}{\rho}, \quad |A| = \bar{q} = \frac{b}{t\rho}.$$

From (1.15) it follows that if there exist two different phase functions such that the generalized characteristic equation (see (1.13))

$$\operatorname{Hov}(t) = \theta^2 - \theta \operatorname{Tr}(A) + |A| - a_{12}(1 - \nu\theta)\left(\frac{a_{11} - \theta}{a_{12}}\right)^{\nabla} = 0,$$

is satisfied, then $M_j \equiv 0$, $j = 1, 2$ and condition (1.17) of Theorem 1.1 disappears.

For the Euler system (1.1) with the matrix $A(t)$ given by (2.57) this equation becomes

$$\mathrm{Hov}(t) = \theta^2(t) + \frac{a\theta(t)}{\rho(t)} + \frac{b}{t\rho(t)} + (1 - \nu(t)\theta(t))\theta^{\nabla}(t) = 0. \tag{2.58}$$

We seek a solution of this nonlinear Riccati equation in the form

$$\theta(t) = \frac{\lambda}{t}.$$

In view of

$$\frac{\theta^{\nabla}(t)}{\theta(t)} = -\frac{1}{\rho(t)}, \quad \nu(t) = t - \rho(t), \quad \theta_j^2 - \nu\theta_j^{\nabla}\theta_j = \frac{t\theta_j^2}{\rho}$$

the characteristic equation (2.58) becomes

$$\mathrm{Hov}(t) = \frac{\lambda^2 - (1-a)\lambda + b}{t\rho(t)} = 0$$

or

$$\lambda^2 - (1-a)\lambda + b = 0,$$

which is the usual characteristic quadratic equation with solutions

$$\lambda_{1,2} = \frac{1-a}{2} \pm \lambda, \qquad \lambda = \sqrt{\frac{(1-a)^2}{4} - b}. \tag{2.59}$$

Choosing the phase functions

$$\theta_j(t) = \frac{\lambda_j}{t}, \quad j = 1, 2, \tag{2.60}$$

we have

$$\mathrm{Hov}_j(t) = M_j(t) \equiv 0.$$

Well-known exact solutions of the Euler system can be constructed by using the phase functions (2.60).

So condition (1.17) is satisfied and from Theorem 1.1 it follows that the system (1.1) with matrix $A(t)$ defined by (2.57) is asymptotically stable if and only if the condition

$$\lim_{t\to\infty} \theta_j^{k-1}\widehat{e}_{\theta_j}(t, t_0) = 0, \quad k, j = 1, 2 \tag{2.61}$$

is satisfied.

Example 2.7. Consider system (1.1) with

$$A = \begin{bmatrix} 0 & 1 \\ -\dfrac{tb}{\rho(t)(t^2+1)} & -\dfrac{a}{\rho(t)} \end{bmatrix}, \quad t_0 > 0, \tag{2.62}$$

$$\operatorname{Tr} A = -\frac{a}{\rho}, \quad |A| = \frac{tb}{\rho(t)(1+t^2)}.$$

For this system we cannot solve the generalized characteristic equation

$$\operatorname{Hov}(t) = \theta^2(t) + \frac{a\theta(t)}{\rho(t)} + \frac{tb}{\rho(t)(t^2+1)} + (1 - \nu(t)\theta(t))\theta^{\nabla}(t) = 0. \tag{2.63}$$

Choosing the phase functions by formula (2.60), the same way as in Example 2.6, from (1.13) we get

$$\operatorname{Hov}_j(t) = \frac{\lambda_j^2 + (a-1)\lambda_j}{t\rho} + |A| = -\frac{b}{t\rho(t)} + \frac{tb}{\rho(t)(1+t^2)} = -\frac{b}{t\rho(t)(1+t^2)},$$

$$Q_0 = \frac{\operatorname{Hov}_1 - \operatorname{Hov}_2}{2\theta} \equiv 0, \quad K(t) \le \left| \frac{C \operatorname{Hov}_1}{\theta} \right| \le \frac{C}{t\rho(t)(1+t^2)}. \tag{2.64}$$

Condition (1.21) becomes the condition

$$\left| 1 + \frac{at\nu(t) + b\nu^2(t)}{t\rho(t)} \right| \ge \beta > 0, \quad \text{for all} \quad t \in \mathbb{T}_\infty. \tag{2.65}$$

Condition (1.22) is satisfied and conditions (1.30), (1.31) turn to

$$2\operatorname{Re}[\lambda_j] < \frac{\nu(t)}{t}|\lambda_j|^2, \quad \int_{t_0}^{\infty} \frac{\nabla s}{\nu(s)} = \infty, \quad j = 1, 2. \tag{2.66}$$

Using the estimate (2.64) one can simplify condition (1.19):

$$\lim_{t\to\infty} \widehat{e}_{K_0}(t, t_0) < \infty, \quad K_0 = \frac{C}{t^3\rho(t)}. \tag{2.67}$$

So from conditions (2.65), (2.66), (2.67) it follows from Theorem 1.3 that the system (1.1) with matrix $A(t)$ given by (2.62) is asymptotically stable. Note that if $\nu(t) \equiv 0$, then conditions (2.65), (2.67) are satisfied and (2.66) becomes $\operatorname{Re}[\lambda_j] < 0$.

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