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Oscillation Criteria for Self-Adjoint Matrix Equations on Time Scales

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Abstract

We obtain oscillation criteria for a second-order self-adjoint matrix equation on a time scale in terms of the eigenvalues of the coefficient matrices and the graininess and backward graininess functions. We then illustrate these results with some non-trivial examples.

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1. Introduction

Dynamic equations on time scales have been introduced by Aulbach and Hilger [9, 18] to unify and extend the theory of ordinary differential equations, difference equations, quantum equations, and all other differential systems defined over nonempty closed subsets of the real line. Already several important problems concerning higher-order scalar dynamic equations on time scales, involving only delta differentiation, have been developed [11–13]. In [7] and [22] second-order self-adjoint linear dynamic equations on time scales were introduced and examined by making use of both delta and nabla derivatives. Quite recently [5], two classes of higher-order dynamic equations on time scales, involving both delta and nabla derivatives simultaneously, were shown to be (formally) self-adjoint in the classic sense provided certain self-adjoint boundary conditions were satisfied. In this present work we aim to extend these notions to systems by providing an analysis of the second-order (delta nabla) matrix dynamic equation

$$\left(PX^{\Delta}\right)^{\nabla}(t) + Q(t)X(t) = 0. \tag{1.1}$$

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More commonly authors [1–3], [12, Chapter 5], [17] focus on

$$\left(PX^{\Delta}\right)^{\Delta}(t) + Q(t)X^{\sigma}(t) = 0 \tag{1.2}$$

an equation they often dub the "self-adjoint" equation because it admits a Lagrange identity. The self-adjoint form (1.1), however, is an appropriate generalization and extension of the classic self-adjoint form from ordinary differential equations [15, 21, 23–26]

$$(PX')'(t) + Q(t)X(t) = 0,$$

and the discrete version [4, 20]

$$\Delta\left(P(t)\Delta X(t-1)\right) + Q(t)X(t) = 0,$$

to dynamic equations on time scales. An advantage of (1.1) over (1.2) is that the associated Green's function $G(t,s)$ for the boundary value problem for (1.1) is symmetric in the usual sense, i.e., $G(t,s) = G(s,t)$ [7]. The associated Green's function $G(t,s)$ for the boundary value problem for (1.2) does not satisfy this property. Instead it satisfies, $G(t,s) = G(\sigma(s), \rho(t))$. Davidson and Rynne [16] also discuss some advantages of studying (1.1) over (1.2) when considering the basic formulation of second-order two-point Sturm–Liouville type boundary value problems on time scales. Finally, in [8] Atici and Uysal show how the nabla derivative is more appropriate than the delta derivative for a specific business problem.

The paper is constructed as follows. In Section 2 we introduce the time scale calculus. In Section 3 we give some results concerning the self-adjoint matrix equation and its relationship to the Riccati equation. In Section 4 we give some oscillation results for this self-adjoint equation.

2. Time Scales Introduction

By a time scale $\mathbb{T}$ we just mean a nonempty closed subset of $\mathbb{R}$.

Definition 2.1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Finally, the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t$$

and the *backwards graininesss function* $\nu : \mathbb{T}_\kappa \to [0, \infty)$ is defined by

$$\nu(t) = t - \rho(t).$$

We also need the sets $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$ which are derived from the time scale as follows: If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, $\mathbb{T}_\kappa := \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}_k = \mathbb{T}$. Also, we will use the notation $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$, $\mathbb{T}_{\kappa^2} = (\mathbb{T}_\kappa)_\kappa$, and $\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

Definition 2.2. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$C_{\text{rd}} = C_{\text{rd}}(\mathbb{T}) = C_{\text{rd}}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{\text{rd}} = C^1_{\text{rd}}(\mathbb{T}) = C^1_{\text{rd}}(\mathbb{T}, \mathbb{R}).$$

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is called *ld-continuous* provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$C_{\text{ld}} = C_{\text{ld}}(\mathbb{T}) = C_{\text{ld}}(\mathbb{T}, \mathbb{R}).$$

Now we consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the so-called delta derivative of f at a point $t \in \mathbb{T}^\kappa$ and the nabla derivative of f at a point $t \in \mathbb{T}_\kappa$. By convention we will define $\lim_{s \to t} f(s) = f(t)$ if t is an isolated point.

Definition 2.4. [Delta Derivative] Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(s)) - f(t)}{\sigma(s) - t},$$

$(\sigma(s) \neq t)$, provided this limit exists. We call $f^\Delta(t)$ the *delta derivative* of f at t. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^\kappa$.

Definition 2.5. [Nabla Derivative] Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_\kappa$. Then we define

$$f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(\rho(s))}{t - \rho(s)},$$

$(\rho(s) \neq t)$, provided this limit exists. We call $f^\nabla(t)$ the *nabla derivative* of f at t. Moreover, we say that f is *nabla differentiable* on $\mathbb{T}_\kappa$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$. The function $f^\nabla : \mathbb{T}_\kappa \to \mathbb{R}$ is then called the nabla derivative of f on $\mathbb{T}_\kappa$.

Definition 2.6. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is ν-regressive provided

$$1 - \nu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}_\kappa.$$

For more on time scales definitions and notation see [12].

3. Self-Adjoint Matrix Equation and the Associated Riccati Equation

In this paper we are concerned with the self-adjoint second-order matrix dynamic equation

$$LX(t) = [P(t)X^\Delta(t)]^\nabla + Q(t)X(t) = 0 \tag{3.1}$$

on a time scale $\mathbb{T}$ and the associated Riccati equation

$$RZ(t) = Z^\nabla + Q(t) + F(t) = 0, \tag{3.2}$$

where

$$F(t) := Z^{\rho *}(t)[P^\rho(t) + \nu(t)Z^\rho(t)]^{-1}Z^\rho(t). \tag{3.3}$$

Throughout we will assume $P(t) > 0$ for $t \in \mathbb{T}^\kappa_\kappa$, and that $P(t)$ and $Q(t)$ are Hermitian ($P(t) = P^*(t)$, $Q(t) = Q^*(t)$ for $t \in \mathbb{T}^\kappa$ where $P^*(t)$ is the conjugate transpose of $P(t)$). We will also assume P and Q are continuous.

Definition 3.1. Let $\mathbb{D}$ denote the set of all $n \times n$ matrix functions X defined on $\mathbb{T}$ such that X^Δ is continuous on $\mathbb{T}^\kappa$ and $(PX^\Delta)^\nabla$ is left-dense continuous on $\mathbb{T}^\kappa_\kappa$. We say that X is a solution of the matrix equation (3.1) on $\mathbb{T}$ provided $X \in \mathbb{D}$ and $LX(t) = 0$ for all $t \in \mathbb{T}^\kappa_\kappa$.

Definition 3.2. If $X, Y \in \mathbb{D}$, then we define the *Wronskian matrix* of $X(t)$ and $Y(t)$ by

$$W[X(t), Y(t)] = X^*(t)P(t)Y^\Delta(t) - [P(t)X^\Delta(t)]^*Y(t)$$

for $t \in \mathbb{T}^\kappa$.

For the sake of convenience to the reader we omit the proofs of the following results. They can be found in [6, 12, 14].

Theorem 3.3. [Lagrange Identity] If $X, Y \in \mathbb{D}$, then

$$(W[X(t), Y(t)])^\nabla = X^*(t)LY(t) - (LX(t))^*Y(t)$$

for $t \in \mathbb{T}^\kappa_\kappa$.

An immediate corollary of the Lagrange identity is Abel's formula.

Corollary 3.4. [Abel's Formula] If X and Y are solutions of (3.1) on $\mathbb{T}$, then

$$W[X(t), Y(t)] = C$$

for $t \in \mathbb{T}^\kappa$, where C is a constant matrix.

Definition 3.5. Let $X, Y \in \mathbb{D}$ and W be given as in (3.2). We say X is a prepared (conjoined, isotropic) solution of (3.1) iff X is a solution of (3.1) and

$$W(X, X)(t) \equiv 0, \quad t \in \mathbb{T}^\kappa.$$

Theorem 3.6. Assume that X is a solution of (3.1) on $\mathbb{T}$. Then the following are equivalent:

(i) X is a prepared solution;

(ii) $X^*(t)P(t)X^\Delta(t)$ is Hermitian for all $t \in \mathbb{T}^\kappa$;

(iii) $X^*(t_0)P(t_0)X^\Delta(t_0)$ is Hermitian for some $t_0 \in \mathbb{T}^\kappa$.

Lemma 3.7. Let X be a solution of (3.1). If X is prepared, then

$$X^*(\sigma(t))P(t)X(t)$$

is Hermitian for all $t \in \mathbb{T}^\kappa$. Conversely, if there is a $t_0 \in \mathbb{T}^\kappa$ such that $\mu(t_0) > 0$ and $X^*(\sigma(t_0))P(t_0)X(t_0)$ is Hermitian, then X is a prepared solution of (3.1). Also, if X is a nonsingular prepared solution of (3.1), then $P(t)X(\sigma(t))X^{-1}(t)$, $P(t)X(t)X^{-1}(\sigma(t))$ and $Z(t) := P(t)X^\Delta(t)X^{-1}(t)$ are Hermitian for all $t \in \mathbb{T}^\kappa$.

Lemma 3.8. Assume that X is a prepared solution of (3.1) on $\mathbb{T}$. Then the following are equivalent:

(a) $X^*(\sigma(t))P(t)X(t) > 0$ on $\mathbb{T}^\kappa$;

(b) $X(t)$ is nonsingular and $P(t)X(\sigma(t))X^{-1}(t) > 0$ on $\mathbb{T}^\kappa$;

(c) $X(t)$ is nonsingular and $P(t)X(t)X^{-1}(\sigma(t)) > 0$ on $\mathbb{T}^\kappa$.

We shall now consider the associated Riccati equation (3.2).

Definition 3.9. Define the set $\mathbb{D}_R$ to be the set of all $n \times n$ matrix functions $Z(t)$ defined on $\mathbb{T}^\kappa$ such that Z^∇ is left-dense continuous on $\mathbb{T}^\kappa_\kappa$ and such that $P^\rho(t) + \nu(t)Z^\rho(t)$ is invertible for all $t \in \mathbb{T}^\kappa_\kappa$. We say that Z is a solution of (3.2) on $\mathbb{T}_\kappa$ provided $Z \in \mathbb{D}_R$ and $RZ(t) = 0$ for all $t \in \mathbb{T}^\kappa_\kappa$.

Theorem 3.10. If (3.1) has a prepared solution X such that $X(t)$ is invertible for all $t \in \mathbb{T}$, then Z defined by

$$Z(t) := P(t)X^\Delta(t)X^{-1}(t)$$

for $t \in \mathbb{T}^\kappa$ is a Hermitian solution of (3.2) on $\mathbb{T}_\kappa$. Conversely, if (3.2) has a Hermitian solution Z on $\mathbb{T}_\kappa$, then there exists a prepared solution X of (3.1) such that $X(t)$ is invertible for all $t \in \mathbb{T}$ and $Z(t) = P(t)X^\Delta(t)X^{-1}(t)$.

Theorem 3.11. The self-adjoint matrix equation (3.1) has a prepared solution X on $\mathbb{T}$ with $X^*(\rho(t))P(\rho(t))X(t) > 0$ on $\mathbb{T}_\kappa$ if and only if the Riccati equation (3.2) has a Hermitian solution Z on $\mathbb{T}_\kappa$ satisfying $P^\rho + \nu Z^\rho > 0$ on $\mathbb{T}_\kappa$.

4. Oscillation Results

Definition 4.1. Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. We say that (3.1) is *nonoscillatory* on $[a, \infty)$ provided there is a prepared solution X of $LX = 0$ and a $t_0 \in [a, \infty)$ such that

$$X^*(\rho(t))P(\rho(t))X(t) > 0$$

on $[t_0, \infty)$. Otherwise we say (3.1) is *oscillatory* on $[a, \infty)$.

We now introduce some notation that we will use in the remainder of this paper. If A is an $n \times n$ Hermitian matrix, then it is well known that all the eigenvalues of A are real. We let $\lambda_i(A)$ denote the ith eigenvalue of A so that

$$\lambda_{\max}(A) = \lambda_1(A) \geq \cdots \geq \lambda_n(A) = \lambda_{\min}(A).$$

The *trace* of a matrix A is denoted by

$$\operatorname{tr}(A) := \sum_{i=1}^n a_{ii}.$$

We shall make frequent use of Weyl's theorem [19, Theorem 4.3.1], which says that if A and B are Hermitian matrices, then

$$\lambda_i(A) + \lambda_{\max}(B) \geq \lambda_i(A+B) \geq \lambda_i(A) + \lambda_{\min}(B).$$

We now state and prove an oscillation theorem for the matrix equation

$$X^{\Delta\nabla}(t) + Q(t)X(t) = 0. \tag{4.1}$$

Theorem 4.2. Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. Assume for any $t_0 \in [a, \infty)$ there exists $t_0 \leq a_0 < b_0$ such that $\mu(a_0) > 0$, $\mu(b_0) > 0$ and

$$\lambda_{\max}\left(\int_{a_0}^{b_0} Q(t)\nabla t\right) \geq \frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)}. \tag{4.2}$$

Then equation (4.1) is oscillatory on $[a, \infty)$.

Proof. Assume (4.1) is nonoscillatory on $[a, \infty)$. Then there exists a prepared solution X of (4.1) and a $t_0 \in [a, \infty)$ such that

$$X^*(\rho(t))X(t) > 0$$

on $[t_0, \infty)$. We make the Riccati substitution

$$Z(t) = X^{\Delta}(t)X^{-1}(t)$$

for $t \in [t_0, \infty)$. Then by Theorem 3.11,

$$I + \nu(t)Z^{\rho}(t) > 0$$

on $[t_0, \infty)$ and $Z(t)$ satisfies the Riccati equation (3.2) on $[t_0, \infty)$. By hypothesis, there exist $t_0 \leq a_0 < b_0$ such that $\mu(a_0) > 0$, $\mu(b_0) > 0$ and inequality (4.2) holds. Note that since $\mu(a_0) > 0$, we have that $\sigma(a_0) > a_0$ and hence $\rho(\sigma(a_0)) = a_0$. Similarly, $\rho(\sigma(b_0)) = b_0$.

So now

$$RZ = 0 \Longrightarrow Z^{\nabla}(t) = -Q(t) - Z^{*\rho}(t)[I + \nu(t)Z^{\rho}(t)]^{-1}Z^{\rho}(t).$$

Now nabla integrate both sides from a_0 to $t > a_0$.

$$\Longrightarrow Z(t) - Z(a_0) = -\int_{a_0}^{t} Q(s)\nabla s - \int_{a_0}^{t} Z^{\rho}(s)[I + \nu(s)Z^{\rho}(s)]^{-1}Z^{\rho}(s)\nabla s$$

$$\Longrightarrow Z(t) = Z(a_0) - \int_{a_0}^{t} Q(s)\nabla s - \int_{a_0}^{\sigma(a_0)} Z^{\rho}(s)[I + \nu(s)Z^{\rho}(s)]^{-1}Z^{\rho}(s)\nabla s$$

$$- \int_{\sigma(a_0)}^{t} Z^{\rho}(s)[I + \nu(s)Z^{\rho}(s)]^{-1}Z^{\rho}(s)\nabla s$$

$$\Longrightarrow Z(t) = Z(a_0) - \int_{a_0}^{t} Q(s)\nabla s - \mu(a_0)Z(a_0)[I + \nu(\sigma(a_0))Z(a_0)]^{-1}Z(a_0)$$

$$- \int_{\sigma(a_0)}^{t} Z^{\rho}(s)[I + \nu(s)Z^{\rho}(s)]^{-1}Z^{\rho}(s)\nabla s$$

$$\Longrightarrow Z(t) + \int_{a_0}^{t} Q(s)\nabla s = Z(a_0) - \mu(a_0)Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}Z(a_0)$$

$$- \int_{\sigma(a_0)}^{t} Z^{\rho}(s)[I + \nu(s)Z^{\rho}(s)]^{-1}Z^{\rho}(s)\nabla s$$

$$\Longrightarrow Z(t) + \int_{a_0}^{t} Q(s)\nabla s \leq Z(a_0) - \mu(a_0)Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}Z(a_0)$$

$$\Longrightarrow Z(t) + \int_{a_0}^{t} Q(s)\nabla s$$

$$\leq Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}(I + \mu(a_0)Z(a_0) - \mu(a_0)Z(a_0)).$$

Thus,

$$Z(t) + \int_{a_0}^{t} Q(s)\nabla s \le Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}. \tag{4.3}$$

Now let U be a unitary matrix ($U^*U = I$) such that

$$Z(a_0) = U^* D U,$$

where

$$D := \operatorname{diag}(d_1, \dots, d_n),$$

where $d_i = \lambda_i(Z(a_0))$ is the ith eigenvalue of $Z(a_0)$, $i = 1, 2, \dots$. Consider

$$\begin{aligned} Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1} &= U^*DU[I + \mu(a_0)U^*DU]^{-1} \\ &= U^*DU[U^*(I + \mu(a_0)D)U]^{-1} \\ &= U^*DUU^{-1}[I + \mu(a_0)D]^{-1}(U^*)^{-1} \\ &= U^*D[I + \mu(a_0)D]^{-1}U. \end{aligned}$$

Thus $Z(a_0)[I+\mu(a_0)Z(a_0)]^{-1}$ is similar to $D[I+\mu(a_0)D]^{-1}$. Hence they have the same eigenvalues. Since $I+\nu(t)Z^\rho(t) > 0$ on $[t_0, \infty)$, we have that $I+\nu(\sigma(a_0))Z^\rho(\sigma(a_0)) > 0$ and hence $I + \mu(a_0)Z(a_0) > 0$. So then $1 + \mu(a_0)d_i > 0$. Also $h(x) := \dfrac{x}{1 + \mu(a_0)x}$ is increasing when $1 + \mu(a_0)x > 0$. Thus we have that

$$\begin{aligned} \lambda_i\left(Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}\right) &= \lambda_i\left(D[I + \mu(a_0)D]^{-1}\right) \\ &= \frac{d_i}{1 + \mu(a_0)d_i}. \end{aligned}$$

Using $h(x) := \dfrac{x}{1 + \mu(a_0)x} < \dfrac{1}{\mu(a_0)}$ when $1 + \mu(a_0)x > 0$, we get that

$$\lambda_i\left(Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}\right) = \frac{d_i}{1 + \mu(a_0)d_i} < \frac{1}{\mu(a_0)}.$$

So from (4.3), we have

$$\lambda_i\left(Z(t) + \int_{a_0}^{t} Q(s)\nabla s\right) \le \lambda_i\left(Z(a_0)[I + \mu(a_0)Z(a_0)]^{-1}\right) < \frac{1}{\mu(a_0)}.$$

So

$$\frac{1}{\mu(a_0)} > \lambda_{\max}\left(Z(t) + \int_{a_0}^{t} Q(s)\nabla s\right) \ge \lambda_{\max}\left(\int_{a_0}^{t} Q(s)\nabla s\right) + \lambda_{\min}(Z(t))$$

by Weyl's inequality. Now let $t = b_0$ to get that

$$\lambda_{\max}\left(\int_{a_0}^{b_0} Q(s)\nabla s\right) < \frac{1}{\mu(a_0)} - \lambda_{\min}(Z(b_0)).$$

Since $I + \nu(t)Z^{\rho}(t) > 0$ on $[t_0, \infty)$, we have that $I + \nu(\sigma(b_0))Z^{\rho}(\sigma(b_0)) > 0$ and hence $I + \mu(b_0)Z(b_0) > 0$. Thus $\lambda_{\min}(Z(b_0)) > \dfrac{-1}{\mu(b_0)}$. Hence

$$\lambda_{\max}\left(\int_{a_0}^{b_0} Q(t)\nabla t\right) < \frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)}$$

which is a contradiction to our assumption. Thus equation (4.1) is oscillatory on $[a, \infty)$. ∎

We now give the following examples which are similar to the examples given in [12]. Since the calculations are similar, we will omit the proofs.

Example 4.3. Assume the scalar functions $q_i : \mathbb{T}_\kappa \to \mathbb{R}$, $1 \le i \le n$ are ld-continuous on $\mathbb{T} := \bigcup_{k=0}^{\infty}\left[k, k+\frac{1}{2}\right]$ with $q_1 > q_i$ on $\mathbb{T}$ for all i and assume for each $t_0 \in [0, \infty)$ there is a $k_0 \in \mathbb{N}$, and a $l_0 \in \mathbb{N}$, such that $k_0 \ge t_0$ and

$$\sum_{j=1}^{l_0}\int_{k_0+j}^{k_0+j+\frac{1}{2}} q_1(t)\nabla t + \frac{1}{2}\sum_{j=0}^{l_0-1} q_1^{\sigma}\left(k_0 + j + \frac{1}{2}\right) \ge 4. \tag{4.4}$$

Then, if

$$Q(t) := \operatorname{diag}(q_1(t), q_2(t), \ldots, q_n(t)),$$

it follows that the matrix dynamic equation

$$X^{\Delta\nabla} + Q(t)X = 0$$

is oscillatory on $\mathbb{T}$.

Example 4.4. Let $\mathbb{T} := q^{\mathbb{N}_0} = \{q^k : k = 0, 1, 2, \ldots\}$ where $q > 1$. Then the q-difference equation

$$x^{\Delta\nabla}(t) + \frac{cq}{(q-1)t^2}x(t) = 0,$$

where $c > 1$ is a constant, is oscillatory on $\mathbb{T}$.

Corollary 4.5. Assume $a \in \mathbb{T}$ and $\sup\mathbb{T} = \infty$. If there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset [a, \infty)$ such that $\lim_{k\to\infty} t_k = \infty$ with $\mu(t_k) \ge K > 0$ for some $K > 0$ and such that

$$\limsup_{t_k\to\infty} \lambda_{\max}\left(\int_a^{t_k} Q(s)\nabla s\right) = +\infty, \tag{4.5}$$

then equation (4.1) is oscillatory.

Proof. Let $t_0 \in [a, \infty)$. Choose k_0 sufficiently large so that $a_0 := t_{k_0} \in [t_0, \infty)$. Using (4.5), we can pick $k_1 > k_0$ sufficiently large so that with $b_0 := t_{k_1}$ we have

$$\lambda_{\max}\left(\int_{a_0}^{b_0} Q(s)\nabla s\right) \geq \frac{2}{K} \geq \frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)}.$$

So by Theorem 4.2, (4.1) is oscillatory. ■

Theorem 4.6. Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$ and suppose there is a strictly increasing sequence $\{t_k\}_{k=1}^{\infty} \subset [a, \infty)$ such that $\mu(t_k) > 0$ for $k = 1, 2, \ldots$, with $\lim_{k\to\infty} t_k = \infty$. Further, assume that there is a sequence $\{\tau_k\}_{k=1}^{\infty} \subset [a, \infty)$ such that $\sigma(\tau_k) > \tau_k \geq \sigma(t_k)$, for $k = 1, 2, \ldots$ such that

$$\lambda_{\min}\left(\frac{P(t_k)}{\mu(t_k)} + \frac{P(\tau_k)}{\mu(\tau_k)} - \int_{t_k}^{\tau_k} Q(s)\nabla s\right) \leq 0. \tag{4.6}$$

Then (3.1) is oscillatory on $[a, \infty)$.

Proof. Assume (3.1) is nonoscillatory on $[a, \infty)$. Then there is a prepared solution X of (3.1) and a $t_0 \in [a, \infty)$ such that

$$X^*(\rho(t))P(\rho(t))X(t) > 0$$

on $[t_0, \infty)$. We make the Riccati substitution

$$Z(t) := P(t)X^{\Delta}(t)X^{-1}(t)$$

for $t \in [a, \infty)$. Then by Theorem 3.11, we get that

$$P^{\rho}(t) + \nu(t)Z^{\rho}(t) > 0$$

on $[t_0, \infty)$ and $Z(t)$ is a Hermitian solution of the Riccati equation (3.2) on $[t_0, \infty)$. Let $\{t_k\}$, $\{\tau_k\}$ be the sequences given in the statement of this theorem. Pick a fixed integer k so that $t_k \geq t_0$. Nabla integrating both sides of the Riccati equation from t_k to τ_k we obtain

$$\begin{aligned}
Z(\tau_k) &= Z(t_k) - \int_{t_k}^{\tau_k} Q(s)\nabla s - \int_{t_k}^{\tau_k} F(s)\nabla s \\
&= Z(t_k) - \int_{t_k}^{\tau_k} Q(s)\nabla s - \int_{t_k}^{\sigma(t_k)} F(s)\nabla s - \int_{\sigma(t_k)}^{\tau_k} F(s)\nabla s \\
&= Z(t_k) - \mu(t_k)F^{\sigma}(t_k) - \int_{t_k}^{\tau_k} Q(s)\nabla s - \int_{\sigma(t_k)}^{\tau_k} F(s)\nabla s \\
&= Z(t_k) - \mu(t_k)Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}Z(t_k) \\
&\quad - \int_{t_k}^{\tau_k} Q(s)\nabla s - \int_{\sigma(t_k)}^{\tau_k} F(s)\nabla s
\end{aligned}$$

$$
\begin{aligned}
&= Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}(P(t_k) + \mu(t_k)Z(t_k) - \mu(t_k)Z(t_k)) \\
&\quad - \int_{t_k}^{\tau_k} Q(s)\nabla s - \int_{\sigma(t_k)}^{\tau_k} F(s)\nabla s \\
&= Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}P(t_k) - \int_{t_k}^{\tau_k} Q(s)\nabla s - \int_{\sigma(t_k)}^{\tau_k} F(s)\nabla s \\
&\le Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}P(t_k) - \int_{t_k}^{\tau_k} Q(s)\nabla s.
\end{aligned}
$$

So we have that

$$
Z(\tau_k) \le Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}P(t_k) - \int_{t_k}^{\tau_k} Q(s)\nabla s. \tag{4.7}
$$

Now note that

$$
\begin{aligned}
Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}P(t_k) &= Z(t_k)[P(t_k) + \mu(t_k)P(t_k)X^{\Delta}(t_k)X^{-1}(t_k)]^{-1}P(t_k) \\
&= Z(t_k)[P(t_k)(X(t_k) + \mu(t_k)X^{\Delta}(t_k))X^{-1}(t_k)]^{-1}P(t_k) \\
&= Z(t_k)[P(t_k)X^{\sigma}(t_k)X^{-1}(t_k)]^{-1}P(t_k) \\
&= Z(t_k)X(t_k)(X^{\sigma})^{-1}(t_k)P^{-1}(t_k)P(t_k) \\
&= Z(t_k)X(t_k)(X^{\sigma})^{-1}(t_k) \\
&= P(t_k)X^{\Delta}(t_k)X^{-1}(t_k)X(t_k)(X^{\sigma})^{-1}(t_k) \\
&= P(t_k)X^{\Delta}(t_k)(X^{\sigma})^{-1}(t_k) \\
&= P(t_k)\left[\frac{X^{\sigma}(t_k) - X(t_k)}{\mu(t_k)}\right](X^{\sigma})^{-1}(t_k) \\
&= \frac{P(t_k)}{\mu(t_k)}\left[I - X(t_k)(X^{\sigma})^{-1}(t_k)\right] \\
&= \frac{P(t_k)}{\mu(t_k)} - \frac{P(t_k)}{\mu(t_k)}X(t_k)X^{-1}(\sigma(t_k)).
\end{aligned}
$$

So we have that

$$
Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}P(t_k) = \frac{P(t_k)}{\mu(t_k)} - \frac{P(t_k)}{\mu(t_k)}X(t_k)X^{-1}(\sigma(t_k)). \tag{4.8}
$$

We know $X^*(\rho(t))P(\rho(t))X(t) > 0$ on $[t_0, \infty)$. So $X^*(\rho(\sigma(t_k)))P(\rho(\sigma(t_k)))\,X(\sigma(t_k))$ > 0 for all t_k. Thus $X^*(t_k)P(t_k)\,X(\sigma(t_k)) > 0$. From Lemma 3.7, we know that $X^{*\sigma}PX$ is Hermitian. Thus we get that $X^*(\sigma(t_k))P(t_k)X(t_k) > 0$. Now from Lemma 3.8, we get that $P(t_k)X(t_k)X^{-1}(\sigma(t_k)) > 0$. So from (4.8) we have that

$$
Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}P(t_k) < \frac{P(t_k)}{\mu(t_k)}.
$$

Thus from (4.7) we get that

$$
Z(\tau_k) < \frac{P(t_k)}{\mu(t_k)} - \int_{t_k}^{\tau_k} Q(s)\nabla s. \tag{4.9}
$$

Now $P^\rho(t) + \nu(t)Z^\rho(t) > 0$ on $[t_0, \infty)$ implies $P^\rho(\sigma(\tau_k)) + \nu(\sigma(\tau_k))Z^\rho(\sigma(\tau_k)) > 0$ which implies that $P(\tau_k) + \mu(\tau_k)Z(\tau_k) > 0$ and hence $\dfrac{-P(\tau_k)}{\mu(\tau_k)} < Z(\tau_k)$.

Using this and (4.9) we have that

$$-\frac{P(\tau_k)}{\mu(\tau_k)} < \frac{P(t_k)}{\mu(t_k)} - \int_{t_k}^{\tau_k} Q(s)\nabla s \Rightarrow \frac{P(t_k)}{\mu(t_k)} + \frac{P(\tau_k)}{\mu(\tau_k)} - \int_{t_k}^{\tau_k} Q(s)\nabla s > 0$$
$$\Rightarrow \lambda_{\min}\left(\frac{P(t_k)}{\mu(t_k)} + \frac{P(\tau_k)}{\mu(\tau_k)} - \int_{t_k}^{\tau_k} Q(s)\nabla s > 0\right) > 0$$

which is a contradiction. Hence (3.1) is oscillatory on $[a, \infty)$. ■

Corollary 4.7. Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. A necessary condition for the self-adjoint equation (3.1) to be nonoscillatory on $[a, \infty)$ is that for any strictly increasing sequence $\{t_k\}_{k=1}^\infty \subset [a, \infty)$ such that $\mu(t_k) > 0$ for $k = 1, 2, \ldots$, with $\lim_{k\to\infty} t_k = \infty$ there is a subsequence $\{t_j\} \subset \{t_k\}$ such that

$$D_j := \frac{P(t_j)}{\mu(t_j)} + \frac{P(t_{j+1})}{\mu(t_{j+1})} - \int_{t_j}^{t_{j+1}} Q(s)\nabla s > 0$$

for $j = 1, 2, \ldots$.

Proof. We obtain this result by taking $\tau_j = t_{j+1}$ in the previous theorem. ■

Corollary 4.8. Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. Assume that there is a strictly increasing sequence $\{t_k\}_{k=1}^\infty \subset [a, \infty)$ such that $\mu(t_k) > 0$ for $k = 1, 2, \ldots$, with $\lim_{k\to\infty} t_k = \infty$. Further, assume that there are sequences $\{s_k\}_{k=1}^\infty \subset [a, \infty)$ and $\{\tau_k\}_{k=1}^\infty \subset [a, \infty)$ such that $\sigma(s_k) > s_k \geq \sigma(\tau_k) > \tau_k \geq \sigma(t_k)$, for $k = 1, 2, \ldots$ such that

$$\int_{t_k}^{\tau_k} Q(t)\nabla t \geq \frac{P(t_k)}{\mu(t_k)}$$

and

$$\lambda_{\min}\left(\frac{P(s_k)}{\mu(s_k)} - \int_{\tau_k}^{s_k} Q(t)\nabla t\right) \leq 0$$

for $k = 1, 2, \ldots$. Then (3.1) is oscillatory on $[a, \infty)$.

Proof. It follows from Weyl's inequality that if A and B are Hermitian matrices, then

$$\lambda_{\min}(A - B) \leq \lambda_{\min}(A) - \lambda_{\min}(B).$$

We will use this fact in the following chain of inequalities. Consider

$$\begin{aligned}
&\lambda_{\min}\left(\frac{P(s_k)}{\mu(s_k)}+\frac{P(t_k)}{\mu(t_k)}-\int_{t_k}^{s_k} Q(t)\nabla t\right)\\
&=\lambda_{\min}\left(\left[\frac{P(s_k)}{\mu(s_k)}-\int_{\tau_k}^{s_k} Q(t)\nabla t\right]-\left[\int_{t_k}^{\tau_k} Q(t)\nabla t-\frac{P(t_k)}{\mu(t_k)}\right]\right)\\
&\le \lambda_{\min}\left(\frac{P(s_k)}{\mu(s_k)}-\int_{\tau_k}^{s_k} Q(t)\nabla t\right)-\lambda_{\min}\left(\int_{t_k}^{\tau_k} Q(t)\nabla t-\frac{P(t_k)}{\mu(t_k)}\right)\\
&\le 0.
\end{aligned}$$

Hence the result follows from Theorem 4.6. ■

Corollary 4.9. Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. Suppose also that

$$\lim_{t\to\infty} \lambda_{\min}\left(\int_a^t Q(s)\nabla s\right)=\infty \tag{4.10}$$

and for each $T \in [a, \infty)$ there is a $t \in [T, \infty)$ such that $\mu(t) > 0$ and

$$\lambda_{\min}\left(\frac{P(t)}{\mu(t)}-\int_T^t Q(s)\nabla s\right)\le 0. \tag{4.11}$$

Then (3.1) is oscillatory on $[a, \infty)$.

Proof. Let $\{t_k\} \subset [a, \infty)$ be a strictly increasing sequence with $\mu(t_k) > 0$ for $k = 1, 2, \ldots$, and $\lim_{k\to\infty} t_k = \infty$. Using (4.10) we get for each k there is a $\tau_k > \sigma(t_k)$ so that

$$\int_{t_k}^{\tau_k} Q(t)\nabla t \ge \frac{P(t_k)}{\mu(t_k)},$$

$k = 1, 2, \ldots$. Using (4.11) we get that for each $k = 1, 2, \ldots$ there is an $s_k \ge \sigma(\tau_k)$ so that

$$\lambda_{\min}\left(\frac{P(s_k)}{\mu(s_k)}-\int_{\tau_k}^{s_k} Q(t)\nabla t\right)\le 0.$$

It follows from Corollary 4.8 that (3.1) is oscillatory on $[a, \infty)$. ■

Theorem 4.10. Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. Assume for each $t_0 \in [a, \infty)$ there is a strictly increasing sequence $\{t_k\}_{k=1}^{\infty} \subset [t_0, \infty)$ with $\mu(t_k) > 0$ and $\lim_{k\to\infty} t_k = \infty$, and there are constants K_1, K_2 such that $0 < K_1 \le \mu(t_k) \le K_2$ for $k = 1, 2, \ldots$ such that

$$\lim_{k\to\infty} \lambda_{\max}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)\ge \frac{1}{\mu(t_1)}.$$

Further assume that there is a constant M such that

$$\operatorname{tr}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)\ge M$$

for $k \geq 1$. Then equation (4.1) is oscillatory on $[a, \infty)$.

Proof. Assume (4.1) is nonoscillatory on $[a, \infty)$. Then there is a prepared solution X of (4.1) and a $t_0 \in [a, \infty)$ such that

$$X^*(\rho(t))X(t) > 0$$

on $[t_0, \infty)$. We make the Riccati substitution

$$Z(t) := X^{\Delta}(t)X^{-1}(t)$$

for $t \in [a, \infty)$. Then by Theorem 3.11, we get that

$$I + \nu(t)Z^{\rho}(t) > 0$$

on $[t_0, \infty)$ and $Z(t)$ is a Hermitian solution of the Riccati equation $Z^{\Delta}(t)+Q(t)+F(t) = 0$ on $[t_0, \infty)$, where $F := Z^{\rho}[I + \nu Z^{\rho}]^{-1}Z^{\rho}$. Corresponding to t_0, let $\{t_k\}_{k=1}^{\infty} \subset [t_0, \infty)$ be the sequence guaranteed in the statement of the theorem. Nabla integrating both sides of the Riccati equation from t_1 to t_k, where $k > 1$ gives

$$\begin{aligned}
&Z(t_k) + \int_{t_1}^{t_k} Q(s)\nabla s + \int_{t_1}^{t_k} F(s)\nabla s = Z(t_1)\\
&\Rightarrow Z(t_k) + \int_{t_1}^{t_k} Q(s)\nabla s + \int_{t_1}^{\sigma(t_1)} F(s)\nabla s + \int_{\sigma(t_1)}^{t_k} F(s)\nabla s = Z(t_1)\\
&\Rightarrow Z(t_k) + \int_{t_1}^{t_k} Q(s)\nabla s + \mu(t_1)F^{\sigma}(t_1) + \int_{\sigma(t_1)}^{t_k} F(s)\nabla s = Z(t_1)\\
&\Rightarrow Z(t_k) + \int_{t_1}^{t_k} Q(s)\nabla s + \int_{\sigma(t_1)}^{t_k} F(s)\nabla s\\
&\qquad = Z(t_1) - \mu(t_1)Z(t_1)[I + \mu(t_1)Z(t_1)]^{-1}Z(t_1)\\
&\Rightarrow Z(t_k) + \int_{t_1}^{t_k} Q(s)\nabla s + \int_{\sigma(t_1)}^{t_k} F(s)\nabla s\\
&\qquad = Z(t_1)[I + \mu(t_1)Z(t_1)]^{-1}(I + \mu(t_1)Z(t_1) - \mu(t_1)Z(t_1))\\
&\Rightarrow Z(t_k) + \int_{t_1}^{t_k} Q(s)\nabla s + \int_{\sigma(t_1)}^{t_k} F(s)\nabla s = Z(t_1)[I + \mu(t_1)Z(t_1)]^{-1}.
\end{aligned}$$

So

$$\lambda_{\max}\left(Z(t_k) + \int_{t_1}^{t_k} Q(s)\nabla s + \int_{\sigma(t_1)}^{t_k} F(s)\nabla s\right) = \lambda_{\max}\left(Z(t_1)[I + \mu(t_1)Z(t_1)]^{-1}\right). \tag{4.12}$$

By Weyl's inequality, we get that

$$\begin{aligned}
&\lambda_{\max}\left(Z(t_1)[I+\mu(t_1)Z(t_1)]^{-1}\right)\\
&\quad\geq \lambda_{\max}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)+\lambda_{\min}\left(Z(t_k)+\int_{\sigma(t_1)}^{t_k} F(s)\nabla s\right)\\
&\quad\geq \lambda_{\max}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)+\lambda_{\min}\left(Z(t_k)\right)+\lambda_{\min}\left(\int_{\sigma(t_1)}^{t_k} F(s)\nabla s\right)\\
&\quad\geq \lambda_{\max}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)+\lambda_{\min}\left(Z(t_k)\right).
\end{aligned}$$

So

$$\lambda_{\max}\left(Z(t_1)[I+\mu(t_1)Z(t_1)]^{-1}\right)\geq \lambda_{\max}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)+\lambda_{\min}\left(Z(t_k)\right). \tag{4.13}$$

Taking the limit of both sides as $k\to\infty$ and using the fact that $\lim_{k\to\infty}\lambda_{\min}(Z(t_k))=0$, which we will prove later, we get

$$\begin{aligned}
\lambda_{\max}\left(Z(t_1)[I+\mu(t_1)Z(t_1)]^{-1}\right) &\geq \lim_{k\to\infty}\lambda_{\max}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)\\
&\geq \frac{1}{\mu(t_1)}
\end{aligned}$$

by assumption. But in Theorem 4.2 we showed that $I+\mu(t_1)Z(t_1)>0$ implies that

$$\lambda_{\max}\left(Z(t_1)[I+\mu(t_1)Z(t_1)]^{-1}\right)<\frac{1}{\mu(t_1)} \tag{4.14}$$

and this gives us a contradiction. Hence to complete the proof of this theorem, it remains to prove that $\lim_{k\to\infty}\lambda_{\min}(Z(t_k))=0$ holds.

Since $I+\nu(t)Z^{\rho}(t)>0$ on $[t_0,\infty)$ we get that $F(t)\geq 0$ on $[t_0,\infty)$ and hence $F^{\sigma}(t)\geq 0$ on $[t_0,\infty)$. Thus $\operatorname{tr}(F^{\sigma}(t))\geq 0$. Hence,

$$\begin{aligned}
\sum_{j=1}^{k}\mu(t_j)\lambda_i(F^{\sigma}(t_j)) &\leq \sum_{j=1}^{k}\mu(t_j)\operatorname{tr}(F^{\sigma}(t_j))\\
&= \sum_{j=1}^{k}\int_{t_j}^{\sigma(t_j)}\operatorname{tr}(F(t))\nabla t\\
&\leq \int_{t_1}^{\sigma(t_k)}\operatorname{tr}(F(t))\nabla t\\
&\leq \operatorname{tr}\left(\int_{t_1}^{\sigma(t_k)} F(t)\nabla t\right)\\
&\leq n\lambda_{\max}\left(\int_{t_1}^{\sigma(t_k)} F(t)\nabla t\right)
\end{aligned}$$

for all $k > 1$. From (4.13) and (4.14) we get

$$\frac{1}{\mu(t_1)} > \lambda_{\max}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right) + \lambda_{\min}\left(Z(t_k)\right).$$

Now using (4.12) and (4.14) we get

$$\begin{aligned}\frac{1}{\mu(t_1)} &> \lambda_{\max}\left(Z(t_k) + \int_{t_1}^{t_k} Q(s)\nabla s + \int_{\sigma(t_1)}^{t_k} F(s)\nabla s\right)\\ &\geq \lambda_{\min}(Z(t_k)) + \lambda_{\min}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right) + \lambda_{\max}\left(\int_{\sigma(t_1)}^{t_k} F(s)\nabla s\right),\end{aligned}$$

where we used Weyl's inequality in the second step. Thus

$$\frac{1}{\mu(t_1)} - \lambda_{\min}(Z(t_k)) \geq \lambda_{\min}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right) + \lambda_{\max}\left(\int_{\sigma(t_1)}^{t_k} F(s)\nabla s\right).$$

Now $I + \nu(t)Z^{\rho}(t) > 0$ on $[t_0, \infty)$ implies that $I + \mu(t_k)Z(t_k) > 0$, which implies $1 + \mu(t_k)\lambda_i(Z(t_k)) > 0$. Hence $\frac{1}{\mu(t_k)} > -\lambda_i(Z(t_k))$ for all i.

So then using this and the above inequality, we get that

$$\frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)} > \lambda_{\min}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right) + \lambda_{\max}\left(\int_{\sigma(t_1)}^{t_k} F(s)\nabla s\right). \tag{4.15}$$

Since we are assuming that (4.1) is nonoscillatory, from Theorem 4.2, we can, without loss of generality, assume that

$$\lambda_{\max}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right) < \frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)}$$

for all $k > 1$. Therefore,

$$\begin{aligned}M &\leq \operatorname{tr}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)\\ &= \sum_{i=1}^{n}\lambda_i\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)\\ &= \lambda_{\min}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right) + \sum_{i=1}^{n-1}\lambda_i\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)\\ &< \lambda_{\min}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right) + (n-1)\left(\frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)}\right)\end{aligned}$$

which gives us

$$\frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)} - \lambda_{\min}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right) < n\left(\frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)}\right) - M.$$

From (4.15) we then get that

$$\begin{aligned}\lambda_{\max}\left(\int_{\sigma(t_1)}^{t_k} F(s)\nabla s\right) &< \left(\frac{1}{\mu(t_1)}+\frac{1}{\mu(t_k)}\right)-\lambda_{\min}\left(\int_{t_1}^{t_k} Q(s)\nabla s\right)\\ &< n\left(\frac{1}{\mu(t_1)}+\frac{1}{\mu(t_k)}\right)-M.\end{aligned}$$

Therefore,

$$\sum_{j=1}^{k}\mu(t_j)\,\mathrm{tr}(F^{\sigma}(t_j)) \le n\lambda_{\max}\left(\int_{t_1}^{\sigma(t_k)} F(t)\nabla t\right)$$

implies that

$$\begin{aligned}\sum_{j=1}^{\infty}\mu(t_j)\,\mathrm{tr}(F^{\sigma}(t_j)) &\le \lim_{k\to\infty} n\lambda_{\max}\left(\int_{\sigma(t_1)}^{t_k} F(t)\nabla t\right)\\ &\le \lim_{k\to\infty} n\left[n\left(\frac{1}{\mu(t_1)}+\frac{1}{\mu(t_k)}\right)-M\right].\\ &< \infty.\end{aligned}$$

Since $\lambda_i(F^{\sigma}(t)) \le \mathrm{tr}(F^{\sigma}(t))$ and $0 < K_1 \le \mu(t_k) \le K_2$ for $k = 1, 2, \ldots$ we have that

$$\sum_{j=1}^{\infty} K_1\lambda_i(F^{\sigma}(t_j)) \le \sum_{j=1}^{\infty}\mu(t_j)\,\mathrm{tr}(F^{\sigma}(t_j)) < \infty.$$

Thus

$$\lim_{k\to\infty}\lambda_i(F^{\sigma}(t_k)) = 0$$

and hence

$$\lim_{k\to\infty}\frac{[\lambda_i(Z(t_k))]^2}{1+\mu(t_k)\lambda_i(Z(t_k))} = 0$$

for $i = 1, 2, \ldots, n$.

Similar to the argument we used in Theorem 4.2, we can show

$$\lambda_i(F^{\sigma}(t_k)) = \frac{d_i^2}{1+\mu(t_k)d_i}.$$

Now when $\lambda_i(Z(t_k)) \ge 0$ we have that

$$0 \le \frac{[\lambda_i(Z(t_k))]^2}{1+K_2\lambda_i(Z(t_k))} \le \frac{[\lambda_i(Z(t_k))]^2}{1+\mu(t_k)\lambda_i(Z(t_k))}$$

and when $\lambda_i(Z(t_k)) \le 0$ we have that

$$0 \le \frac{[\lambda_i(Z(t_k))]^2}{1+K_1\lambda_i(Z(t_k))} \le \frac{[\lambda_i(Z(t_k))]^2}{1+\mu(t_k)\lambda_i(Z(t_k))}.$$

So we either get that

$$\lim_{k\to\infty} \frac{[\lambda_i(Z(t_k))]^2}{1 + K_2\lambda_i(Z(t_k))} = 0$$

or that

$$\lim_{k\to\infty} \frac{[\lambda_i(Z(t_k))]^2}{1 + K_1\lambda_i(Z(t_k))} = 0.$$

In either case we have that

$$\lim_{k\to\infty} \lambda_i(Z(t_k)) = 0$$

for $i = 1, 2, \ldots, n$. This completes the proof. ■

Example 4.11. If $\mathbb{T} = h\mathbb{N}_0$, where $h > 0$, $c \geq \frac{1}{h^2}$, and the function q has the values

$$c, -\frac{c}{2}, -\frac{c}{2}, \frac{c}{3}, \frac{c}{3}, \frac{c}{3}, -\frac{c}{4}, -\frac{c}{4}, -\frac{c}{4}, -\frac{c}{4}, \frac{c}{5}, \ldots$$

at $0, h, 2h, 3h, \ldots$, respectively, then $x^{\Delta\nabla} + q(t)x = 0$ is oscillatory on $\mathbb{T}$.

Finally, we note that similar results for the nabla delta matrix equation can be obtained. For details see [14].

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