International Journal of Difference Equations (IJDE).
ISSN 0973-6069 Volume 2 Number 1 (2007), pp. 13–23

http://www.ripublication.com/ijde.htm

Function Spaces and their Dual Spaces on Time Scales

Ozlem Batit
Department of Mathematics, Ege University,
35100 Bornova, Izmir, Turkey
E-mail: ozlem.batit@ege.edu.tr

Abstract

In this paper we introduce some function spaces on time scales along with their dual spaces and obtain some relations between them. As an application, the infinite matrix transformation on times scales is studied.

AMS subject classification: 40C05, 46A20, 46E15.
Keywords: Time scales, function spaces, duals, matrix transformations.

1. Introduction

Calculus on time scales has been introduced by Bernd Aulbach and Stefan Hilger [1,9] to unify discrete and continuous analysis. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers, so that it is a complete metric space with the metric $d(t, s) = |t - s|$. The books by Bohner and Peterson [4, 5] are excellent references for calculus on time scales.

In the paper by Kizmaz [10], the following sequence spaces are defined:

$$\begin{aligned} l_\infty(\Delta) &= \{x = (x_k) : \Delta x \in l_\infty\}, \\ c(\Delta) &= \{x = (x_k) : \Delta x \in c\}, \\ c_0(\Delta) &= \{x = (x_k) : \Delta x \in c_0\}, \end{aligned}$$

where $\Delta x_k = x_{k+1} - x_k$, $k \in \mathbb{Z}^+$. These sequence spaces are Banach spaces with norm

$$\|x\| = |x_1| + \|\Delta x\|_\infty,$$

Received January 31, 2007; Accepted February 20, 2007

where $\|x\|_\infty = \sup_{1\le k<\infty} |x_k|$. Also some properties of these spaces and of their α, β, γ-dual spaces have been given. Finally, matrix classes related to these sequence spaces were studied. Later Et [6], Et and Çolak [7] generalized the results of [10] considering the n-th power of the operator Δ.

In [2], the results on dual spaces in which methods of [10] were adapted, had been devoted to the function space

$$L_\infty(D) = \left\{ f \mid f : [0,\infty) \to \mathbb{R},\ Df = f' \in L_\infty \right\}.$$

This function space is a Banach space with norm

$$\|f\| = |f(0)| + \|f'\|_\infty,$$

where

$$\|f\|_\infty = \sup_{0\le x<\infty} |f(x)|.$$

In this paper our aim is to unify and generalize the above mentioned results by defining the function spaces and their dual spaces on time scales.

The paper is organized as follows. In Section 2, we give some basic concepts of the time scale calculus and also introduce some function spaces and define α, β, γ-dual spaces. In Section 3, we obtain a relation between the α-dual of some function spaces and construct the $\alpha\alpha$-dual of $L_\infty(\Delta)$. In Section 4, we study infinite matrix transformations on special time scales as an application of the results in the earlier sections.

2. Preliminaries

First, we shall briefly mention some basic definitions of time scale calculus for the reader's convenience. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} :\ s > t\}.$$

If $\sigma(t) > t$, we say that t is right-scattered, and if $\sigma(t) = t$, then t is called right-dense. The graininess $\mu : \mathbb{T} \to [0,\infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

For $a, b \in \mathbb{T}$ with $a \le b$ we define the closed interval $[a,b]$ in $\mathbb{T}$ by $[a,b] = \{t \in \mathbb{T} : a \le t \le b\}$. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}\setminus\{t_0\}$ if $\mathbb{T}$ has a left-scattered maximum t_0, otherwise $\mathbb{T} = \mathbb{T}^\kappa$.

Now, let f be a function defined on $\mathbb{T}$ and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \le \epsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta derivative of f at t. Moreover, we say that f is delta differentiable on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

Note that in the case $\mathbb{T} = \mathbb{R}$ we have $f^{\Delta}(t) = f'(t)$ and in the case $\mathbb{T} = \mathbb{Z}$ we have $f^{\Delta}(t) = f(t+1) - f(t)$.

Here, F is called an antiderivative of a function f defined on $\mathbb{T}$ if $F^{\Delta} = f$ holds on $\mathbb{T}^{\kappa}$. In this case we define a Cauchy integral by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s),$$

where $s, t \in \mathbb{T}$.

Throughout this paper, we assume that $\mathbb{T}$ is unbounded above. Now let us suppose that a real-valued function f is defined on $[a, \infty) = \{t \in \mathbb{T} : t \geq a\}$ and is integrable from a to any point $A \in \mathbb{T}$ with $A \geq a$. If the integral

$$F(A) = \int_a^A f(t)\Delta t$$

approaches a finite limit as $A \to \infty$, then we call that limit the improper integral of first kind of f from a to ∞ and write

$$\int_a^{\infty} f(t)\Delta t = \lim_{A\to\infty} \left\{ \int_a^A f(t)\Delta t \right\}.$$

In the papers by Guseinov [8] and by Bohner and Guseinov [3], many properties of the Δ-integral on time scales are given.

Let $\mathbb{T}$ be a time scale such that $\mathbb{T} \subset [0, \infty)$ and there exists a subset $\{t_k : k \in \mathbb{N}_0\} \subset \mathbb{T}$ with $0 = t_0 < t_1 < t_2 < \ldots$ and $\lim\limits_{k\to\infty} t_k = \infty$. We define the spaces of continuous functions

$$L_{\infty} = \{f | f : \mathbb{T} \to \mathbb{R},\ \sup_{t\in\mathbb{T}} |f(t)| < \infty\},$$
$$C = \{f | f : \mathbb{T} \to \mathbb{R},\ \lim_{t\to\infty} f(t) < \infty\},$$
$$C_0 = \{f | f : \mathbb{T} \to \mathbb{R},\ \lim_{t\to\infty} f(t) = 0\},$$

with the norm

$$\|f\|_{\infty} = \sup_{t\in\mathbb{T}} |f(t)|.$$

One can easily see that these are normed linear spaces and $C_0 \subset C \subset L_{\infty}$. Next we define

$$L_{\infty}(\Delta) = \{f | f \in K,\ f^{\Delta} \in L_{\infty}\},$$
$$C(\Delta) = \{f | f \in K,\ f^{\Delta} \in C\},$$
$$C_0(\Delta) = \left\{f | f \in K,\ f^{\Delta} \in C_0\right\},$$

where $K = \{f | f : \mathbb{T} \to \mathbb{R}$ and f is Δ-differentiable on $\mathbb{T}^\kappa\}$. It is also easy to show that these function spaces are Banach spaces with the norm

$$\|f\|_\Delta = |f(0)| + \|f^\Delta\|_\infty$$

and $C_0(\Delta) \subset C(\Delta) \subset L_\infty(\Delta)$.

Next we define the operator

$$\phi : L_\infty(\Delta) \to L_\infty(\Delta)$$

with $\phi(f(t)) = f(t) - f(0)$. It is clear that ϕ is a bounded linear operator on $L_\infty(\Delta)$. Also $\phi\,[L_\infty(\Delta)] = \{g \in L_\infty(\Delta) : g(0) = 0\}$ is a subspace of $L_\infty(\Delta)$ and is a space with norm $\|f\|_\Delta = \|f^\Delta\|_\infty$.

On the other hand we define the operator

$$D : \phi L_\infty(\Delta) \to L_\infty$$

with $D(f) = f^\Delta$. D is a linear isometry so that the spaces $\phi L_\infty(\Delta)$ and L_∞ are equivalent normed spaces.

Definition 2.1. Let X be a function space and $a \in \mathbb{T}$. We define the dual spaces of $F \subset X$ in the following way:

(i) $F^\alpha = \left\{ f : \int_0^a |f(t)|\Delta t < \infty, \int_{\mathbb{T}} |f(t)g(t)|\Delta t < \infty \text{ for all } g \in F \right\}$,

(ii) $F^\beta = \left\{ f : \int_0^a |f(t)|\Delta t < \infty, \int_{\mathbb{T}} f(t)g(t)\Delta t \text{ is convergent for all } g \in F \right\}$,

(iii) $F^\gamma = \left\{ f : \int_0^a |f(t)|\Delta t < \infty, \sup_{s\in\mathbb{T}} \left| \int_0^s f(t)g(t)\Delta t \right| < \infty \text{ for all } g \in F \right\}$.

F^α, F^β and F^γ are called α, β and γ-dual spaces of F, respectively.

The proof of the following theorem easily follows from the above definition.

Theorem 2.2. Let F and G be function spaces. Then

(i) $F^\alpha \subseteq F^\beta \subseteq F^\gamma$,

(ii) $F \subseteq G$ implies $G^* \subseteq F^*$, $* = \alpha, \beta, \gamma$.

3. Main Results

Lemma 3.1. If $f \in \phi L_\infty(\Delta)$, then

$$\sup_{t\in\mathbb{T}\setminus\{0\}} \frac{|f(t)|}{t} < \infty.$$

Proof. If $f \in \phi L_\infty(\Delta)$, then $\left|f^\Delta(t)\right| \le M$ for all $t \in \mathbb{T} \setminus \{0\}$, where M is a positive constant. Let $t \in \mathbb{T} \setminus \{0\}$. Then we use the properties of the Δ-integral on time scales to reach the desired result

$$|f(t)| = |f(t) - f(0)| = \left|\int_0^t f^\Delta(s)\Delta s\right| \le \int_0^t |f^\Delta(s)|\Delta s \le \int_0^t M\Delta s = Mt.$$

This completes the proof. ■

Theorem 3.2. We have

$$[\phi L_\infty(\Delta)]^\alpha = \left\{f : \int_0^a |f(t)|\,\Delta t < \infty, \int_{\mathbb{T}} t|f(t)|\Delta t < \infty\right\}.$$

Proof. Define

$$D_1 = \left\{f : \int_0^a |f(t)|\Delta t < \infty, \int_{\mathbb{T}} t|f(t)|\Delta t < \infty\right\}.$$

One can easily see that $[\phi L_\infty(\Delta)]^\alpha \subset D_1$. If $f \in D_1$, then for an $t_0 \in \mathbb{T} \setminus \{0\}$,

$$\begin{aligned} \int_{\mathbb{T}} |f(t)g(t)|\Delta t &= \int_0^{t_0} |f(t)g(t)|\Delta t + \int_{t_0}^{\infty} t|f(t)|\frac{|g(t)|}{t}\Delta t \\ &\le \int_0^{t_0} |f(t)g(t)|\Delta t + \sup_{t\in[t_0,\infty)} \frac{|g(t)|}{t} \int_{t_0}^{\infty} t|f(t)|\Delta t \\ &< \infty \end{aligned}$$

for all $g \in \phi L_\infty(\Delta)$. This implies that $f \in [\phi L_\infty(\Delta)]^\alpha$. ■

Remark 3.3. It is easy to see that the set D_1 is nonempty since one can consider f as zero function. The Dirichlet–Abel test is a main tool for presenting a nontrivial example.

Theorem 3.4. (Dirichlet–Abel Test [3]) Let the following conditions be satisfied.

(i) f is integrable from a to any point $A \in \mathbb{T}$ with $A \ge a$, and the integral $F(A) = \int_a^A f(t)\Delta t$ is bounded for all $A \ge a$.

(ii) g is monotone on $[a, \infty)$ and $\lim_{t\to\infty} g(t) = 0$.

Then the improper integral of first kind of the form $\int_a^\infty f(t)g(t)\Delta t$ is convergent.

Example 3.5. Let $f(t) = \dfrac{t + \sigma(t) + 2}{(t+1)^2(\sigma(t)+1)^2(t^2+1)}$, $t \in \mathbb{T}$. We now use the Dirichlet–Abel test to see the convergence of the integral

$$\int_0^a \left|\frac{t + \sigma(t) + 2}{(t+1)^2(\sigma(t)+1)^2(t^2+1)}\right| \Delta t.$$

If we take $g_1(t) = \dfrac{t+\sigma(t)+2}{(t+1)^2(\sigma(t)+1)^2}$, $h_1(t) = \dfrac{1}{t^2+1}$, $t \in \mathbb{T}$, then we have that

$$\int_0^A |g_1(t)|\Delta t = -\frac{1}{(A+1)^2} + 1$$

for all $A \in \mathbb{T}$ $(A \geq 0)$ and $\lim_{t\to\infty} h_1(t) = 0$, $h_1^\Delta(t) \leq 0$ so that h_1 is monotone decreasing on $\mathbb{T}^\kappa$. Hence

$$\int_0^a g_1(t)h_1(t)\Delta t = \int_0^a |f(t)|\Delta t < \infty$$

for all $a \in \mathbb{T}$. To verify that the second condition is valid, we have

$$\int_0^\infty t\left|\frac{t+\sigma(t)+2}{(t+1)^2(\sigma(t)+1)^2(t^2+1)}\right|\Delta t = \int_0^{s_0} t|f(t)|\Delta t + \int_{s_0}^\infty t|f(t)|\Delta t < \infty,$$

where $s_0 > 0$, $s_0 \in \mathbb{T}$. We take

$$g_2(t) = \frac{t+\sigma(t)+2}{(t+1)^2(\sigma(t)+1)^2},\ h_2(t) = \frac{t}{t^2+1},\ t \in [s_0,\infty).$$

We have that

$$\int_{s_0}^A |g_2(t)|\,\Delta t = -\frac{1}{(A+1)^2} + \frac{1}{(s_0+1)^2}$$

is bounded for all $A \in \mathbb{T}$ $(A \geq s_0)$ and $\lim_{t\to\infty} h_2(t) = 0$, $h_2^\Delta(t) < 0$ so that h_2 is decreasing on $[s_0,\infty)$. This implies

$$\int_{\mathbb{T}} t|f(t)|\Delta t = \int_{\mathbb{T}} t\left|\frac{t+\sigma(t)+2}{(t+1)^2(\sigma(t)+1)^2(t^2+1)}\right|\Delta t < \infty.$$

Hence we obtain $f \in D_1$.

Lemma 3.6. If $f \in L_\infty(\Delta)$, then

$$\sup_{t\in\mathbb{T}} \frac{|f(t)|}{t+1} < \infty.$$

Proof. If $f \in L_\infty(\Delta)$, then $|f^\Delta(t)| \leq N$ for all $t \in \mathbb{T}$, where N is a positive constant. Let $t \in \mathbb{T}$. Then we use the properties of the Δ-integral on time scales to reach the desired result

$$|f(t)| - |f(0)| \leq |f(t) - f(0)| = \left|\int_0^t f^\Delta(s)\Delta s\right| \leq \int_0^t |f^\Delta(s)|\Delta s \leq \int_0^t N\Delta s = Nt.$$

Then it follows that

$$|f(t)| \leq Nt + |f(0)| < A(t+1),\ A = \max\{N, |f(0)|\}.$$

This completes the proof. ∎

Theorem 3.7. We have

$$[\phi C(\Delta)]^{\alpha} = [\phi L_{\infty}(\Delta)]^{\alpha} .$$

Proof. Let $f \in [\phi C(\Delta)]^{\alpha}$. Then $\int_{\mathbb{T}} |f(t)g(t)|\Delta t$ is convergent for all $g \in \phi C(\Delta)$. We can take $g(t) = t, t \in \mathbb{T}$. Therefore $f \in [\phi L_{\infty}(\Delta)]^{\alpha}$ by Theorem 3.2. It can easily seen that $[\phi L_{\infty}(\Delta)]^{\alpha} \subseteq [\phi C(\Delta)]^{\alpha}$ by Theorem 2.2. ■

Theorem 3.8. We have

$$[C(\Delta)]^{\alpha} = [L_{\infty}(\Delta)]^{\alpha} .$$

Proof. Let f be an element of the space $[C(\Delta)]^{\alpha}$. Then $f \in [\phi L_{\infty}(\Delta)]^{\alpha}$ by Theorem 3.2. If we use Lemma 3.6, then

$$\begin{aligned} \int_{\mathbb{T}} |f(t)g(t)|\Delta t &= \int_{\mathbb{T}} (t+1)|f(t)|\frac{|g(t)|}{t+1}\Delta t \\ &\leq \sup_{t\in\mathbb{T}} \frac{|g(t)|}{t+1}\left(\int_{\mathbb{T}} t|f(t)|\Delta t + \int_{\mathbb{T}} |f(t)|\Delta t\right) \\ &< \infty \end{aligned}$$

for all $g \in L_{\infty}(\Delta)$. This implies that $f \in [L_{\infty}(\Delta)]^{\alpha}$. The other side of the inclusion follows from Theorem 2.2. ■

Corollary 3.9. Let F stand for L_{∞} or C. Then

$$[F(\Delta)]^{\alpha} = \left\{ f : \int_0^a |f(t)|\Delta t < \infty,\ \int_{\mathbb{T}} t|f(t)|\Delta t < \infty \right\}.$$

Theorem 3.10. Let F stand for L_{∞} or C. Then

$$[F(\Delta)]^{\alpha\alpha} = \left\{ f : \int_0^a |f(t)|\Delta t < \infty,\ \sup_{t\in\mathbb{T}} \frac{|f(t)|}{t+1} < \infty \right\}.$$

Proof. From Definition 2.1, we can write that

$$[L_{\infty}(\Delta)]^{\alpha\alpha} = \left\{ f : \int_0^a |f(t)|\Delta t < \infty,\ \int_{\mathbb{T}} |f(t)g(t)|\Delta t < \infty \text{ for all } g \in [L_{\infty}(\Delta)]^{\alpha} \right\}.$$

Let

$$D_2 = \left\{ f : \int_0^a |f(t)|\Delta t < \infty,\ \sup_{t\in\mathbb{T}} \frac{|f(t)|}{t+1} < \infty \right\}.$$

If $f \in D_2$, then for all $g \in [L_{\infty}(\Delta)]^{\alpha}$,

$$\begin{aligned} \int_{\mathbb{T}} |f(t)g(t)|\Delta t &= \int_{\mathbb{T}} (t+1)\frac{|f(t)|}{t+1}|g(t)|\Delta t \\ &\leq \sup_{t\in\mathbb{T}} \frac{|f(t)|}{t+1}\left(\int_{\mathbb{T}} (t+1)|g(t)|\Delta t\right) \\ &< \infty. \end{aligned}$$

This implies that $f \in [L_\infty(\Delta)]^{\alpha\alpha}$.

Now suppose that $f \in [L_\infty(\Delta)]^{\alpha\alpha}$ and $f \notin D_2$. Then we have $\sup_{t\in\mathbb{T}} \frac{|f(t)|}{t+1} = \infty$. So there is a strictly increasing sequence $\langle t_n \rangle$ such that $t_n \in \mathbb{T}$, with $0 < t_1 < t_2 < t_3 < \dots$ and

$$\frac{t_k^{-1}\,|f(t_k)|}{t_{k+1}-t_k} > \frac{(t_k+1)^{-1}|f(t_k)|}{t_{k+1}-t_k} > k^2.$$

Without loss of generality, $f(t) \neq 0$, $t \in \langle t_n \rangle$. We define the function g by

$$g(t) = \begin{cases} |f(t_k)|^{-1}, & t = t_k \\ 0, & t \neq t_k. \end{cases}$$

Then we have

$$\begin{aligned} \int_{\mathbb{T}} t|g(t)|\Delta t &= \int_{\bigcup_{k=1}^{\infty}\{t_k\}} t_k |f(t_k)|^{-1}\Delta t_k \\ &\le \int_{\bigcup_{k=1}^{\infty}\{t_k\}} \frac{1}{k^2(t_{k+1}-t_k)}\Delta t_k \\ &= \sum_{k=1}^{\infty}\frac{1}{k^2} \\ &< \infty. \end{aligned}$$

It is easy to see $\int_0^a |g(t)|\Delta t < \infty$. Hence $g \in [L_\infty(\Delta)]^{\alpha}$ and

$$\begin{aligned} \int_{\mathbb{T}} |f(t)g(t)|\Delta t &= \int_{\bigcup_{k=1}^{\infty}\{t_k\}} |f(t_k)g(t_k)|\Delta t_k \\ &= \int_{\bigcup_{k=1}^{\infty}\{t_k\}} |f(t_k)||f(t_k)|^{-1}\Delta t_k \\ &= \int_{\bigcup_{k=1}^{\infty}\{t_k\}} 1\Delta t_k \\ &= \infty. \end{aligned}$$

This is a contradiction to our assumption that $f \in [L_\infty(\Delta)]^{\alpha\alpha}$. Hence $f \in D_2$. ■

Remark 3.11. It is easy to see D_2 is nonempty. Let $f(t) = \frac{t+\sigma(t)+2}{(t+1)^2(\sigma(t)+1)^2}$. Then we have

$$\int_0^a |f(t)|\Delta t = \int_0^a \frac{t+\sigma(t)+2}{(t+1)^2(\sigma(t)+1)^2}\Delta t = -\frac{1}{(a+1)^2} + 1$$

and

$$\sup_{t\in\mathbb{T}} \frac{|f(t)|}{t+1} = \sup_{t\in\mathbb{T}} \frac{t+\sigma(t)+2}{(t+1)^3(\sigma(t)+1)^2} < \infty.$$

Hence $f \in D_2$.

4. Matrix Transformations

In this section, as an application, we shall study infinite matrix transformations on special time scales of the form $\mathbb{T} = \{t_k : t_1 = 0, t_k < t_{k+1}, k \in \mathbb{N}\}$. Let X and Y be function spaces defined on $\mathbb{T}$. We denote the set of all infinite matrices from space X to space Y by (X, Y).

Let $A = (g_n(t_k)\mu(t_k))$ be an infinite matrix of real valued functions g_i $(i \in \mathbb{N})$ which are continuous functions on $\mathbb{T}$ and $\mu(t_k) = t_{k+1} - t_k$. Define

$$\begin{aligned} A(f) &= \begin{pmatrix} g_1(t_1)(t_2-t_1) & g_1(t_2)(t_3-t_2) & \cdot & \cdot & \cdot \\ g_2(t_1)(t_2-t_1) & g_2(t_2)(t_3-t_2) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} f(t_1) \\ f(t_2) \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \\ &= \begin{pmatrix} g_1(t_1)f(t_1)(t_2-t_1) + g_1(t_2)f(t_2)(t_3-t_2) + \dots \\ g_2(t_1)f(t_1)(t_2-t_1) + g_2(t_2)f(t_2)(t_3-t_2) + \dots \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=1}^{\infty} g_1(t_k)f(t_k)\mu(t_k) \\ \sum_{k=1}^{\infty} g_2(t_k)f(t_k)\mu(t_k) \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}. \end{aligned}$$

We write formally, provided the series converges for each n and $A_n(f) \in Y$ whenever $f \in X$,

$$A_n(f) = \sum_{k=1}^{\infty} g_n(t_k)f(t_k)\mu(t_k), \ n \in \mathbb{N}$$

such that

$$\Delta A_n(f) = \sum_{k=1}^{\infty} \Delta g_n(t_k)f(t_k)\mu(t_k), \ n \in \mathbb{N}.$$

Lemma 4.1. The following is valid:

$$L_\infty^\beta = L_1 = \left\{ f \mid f : \mathbb{T} \to \mathbb{R}, \int_{\mathbb{T}} |f(t)|\Delta t < \infty \right\}.$$

Proof. Let

$$f \in L_\infty^\beta = \left\{ f \mid \int_0^a |f(t)|\Delta t < \infty, \int_{\mathbb{T}} f(t)g(t)\Delta t \text{ is convergent for all } g \in L_\infty \right\}.$$

Then we let g be

$$g(t) = \operatorname{sgn} f(t) = \begin{cases} 1, & f(t) > 0 \\ -1, & f(t) < 0 \\ 0, & f(t) = 0\,. \end{cases}$$

Hence $\int_{\mathbb{T}} |f(t)|\Delta t < \infty$ and $f \in L_1$. If f is an element of L_1, then for all $g \in L_\infty$

$$\left|\int_{\mathbb{T}} f(t)g(t)\Delta t\right| \leq \int_{\mathbb{T}} |f(t)g(t)|\Delta t \leq \sup_{t\in\mathbb{T}} |g(t)| \int_{\mathbb{T}} |f(t)|\Delta t < \infty.$$

Hence $f \in L_\infty^\beta$. ■

Theorem 4.2. *$A \in (L_\infty, C(\Delta))$ if and only if*

(i) $\sum_{k=1}^{\infty} |g_n(t_k)|\, \mu(t_k) < \infty$ *for each* $n \in \mathbb{N}$,

(ii) $B \in (L_\infty, C)$, *where* $B = (h_n(t_k)\mu(t_k)) = ((g_{n+1}(t_k) - g_n(t_k))\, \mu(t_k))$.

Proof. ($\Rightarrow$) Let $A \in (L_\infty, C(\Delta))$. Then for all $n \in \mathbb{N}$, $\sum_{k=1}^{\infty} g_n(t_k) f(t_k)\mu(t_k)$ is convergent and $A_n(f) \in C(\Delta)$ for all $f \in L_\infty$.

(i) For all $f \in L_\infty$,

$$\int_{\mathbb{T}} g_n(t) f(t)\Delta t = \sum_{k=1}^{\infty} g_n(t_k) f(t_k)(t_{k+1} - t_k) = \sum_{k=1}^{\infty} g_n(t_k) f(t_k)\mu(t_k) < \infty.$$

It is easy to see that $\int_0^a |g_n(t_k)|\, \Delta t_k < \infty$. Hence $g_n \in L_\infty^\beta$ and $g_n \in L_1$ by Lemma 4.1. This implies that

$$\int_{\mathbb{T}} |g_n(t)|\, \Delta t = \sum_{k=1}^{\infty} |g_n(t_k)|\, (t_{k+1} - t_k) = \sum_{k=1}^{\infty} |g_n(t_k)|\, \mu(t_k) < \infty.$$

(ii) We have

$$\Delta A_n(f) = \sum_{k=1}^{\infty} \Delta g_n(t_k) f(t_k)\mu(t_k) = \sum_{k=1}^{\infty} (g_{n+1}(t_k) - g_n(t_k))\, f(t_k)\mu(t_k) = B_n(f),$$

for all $f \in L_\infty$. Therefore $B_n(f) \in C$. Hence $B \in (L_\infty, C)$.

($\Leftarrow$) Let (i) and (ii) be true. Then for all $n \in \mathbb{N}$ and $f \in L_\infty$,

$$|g_n(t_k) f(t_k) \mu(t_k)| \leq M \, |g_n(t_k)| \, \mu(t_k),$$

where $\sup_{t \in \mathbb{T}} |f(t)| = M$ with $M \in \mathbb{R}$. We get that $\sum_{k=1}^{\infty} g_n(t_k) f(t_k) \mu(t_k)$ is convergent for all $n \in \mathbb{N}$. Also we have $\Delta A_n(f) = B_n(f)$ and $B_n(f) \in C$. Hence $A_n(f) \in C(\Delta)$. This completes the proof. ■

Acknowledgments

I would like to thank Professor Ferhan Atici for her advise, valuable experience and help.

References

[1] Bernd Aulbach and Stefan Hilger, Linear dynamic processes with inhomogeneous time scale, In *Nonlinear dynamics and quantum dynamical systems (Gaussig, 1990)*, volume 59 of *Math. Res.*, pages 9–20. Akademie-Verlag, Berlin, 1990.

[2] Ozlem Batit, Some function spaces and their dual spaces, Master's thesis, Ege University, Izmir, Turkey, 2001.

[3] Martin Bohner and Gusein Sh. Guseinov, Improper integrals on time scales, *Dynam. Systems Appl.*, 12(1-2):45–65, 2003. Special issue: dynamic equations on time scales.

[4] Martin Bohner and Allan Peterson, *Dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2001. An introduction with applications.

[5] Martin Bohner and Allan Peterson, *Advances in dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2003.

[6] Mikail Et, On some difference sequence spaces, *Doğa Mat.*, 17(1):18–24, 1993.

[7] Mikail Et and Rıfat Çolak, On some generalized difference sequence spaces, *Soochow J. Math.*, 21(4):377–386, 1995.

[8] Gusein Sh. Guseinov, Integration on time scales, *J. Math. Anal. Appl.*, 285(1):107–127, 2003.

[9] Stefan Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.*, 18(1-2):18–56, 1990.

[10] H. Kızmaz, On certain sequence spaces, *Canad. Math. Bull.*, 24(2):169–176, 1981.

[11] Erwin Kreyszig, *Introductory functional analysis with applications*, John Wiley & Sons, New York-London-Sydney, 1978.

[12] I.J. Maddox, *Elements of functional analysis*, Cambridge University Press, Cambridge, second edition, 1988.