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# Variational Methods for Two Resonant Problems on Time Scales*

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**Abstract**

Using variational methods we study the generalization of two classical second order periodic problems in the context of time scales. On the one hand, we study a forced pendulum-type equation. On the other hand, we obtain solutions for a bounded nonlinearity under Landesman–Lazer type conditions.

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## 1. Introduction

The area of "dynamic equations on time scales" [3, 4, 14] is a new and general field of mathematics that enables a more accurate mathematical description of discrete-continuous processes than the singular fields of differential or difference equations. Such hybrid processes appear in the population dynamics of certain species that feature nonoverlapping generations: the change in population from one generation to the next is discrete and so is modelled by a difference equation, while within-generation dynamics vary continuously (due to mortality rates, resource consumption, predation, interaction etc.) and thus are described by a differential equation [5, p. 619]. In particular, the area of dynamic equations on time scales is more general and flexible than either differential equations or difference equations and hence appears to be the way forward when modelling the above types of hybrid processes.

This work is concerned with the existence of solutions for the second order dynamic equation on time scales

$$y^{\Delta\Delta} + f(t, y^{\sigma}) = 0, \qquad t \in [0, T]_{\mathbb{T}} \tag{1.1}$$

under periodic conditions

$$y(0) = y(\sigma^2(T)), \qquad y^{\Delta}(0) = y^{\Delta}(\sigma(T)). \tag{1.2}$$

We shall study (1.1)–(1.2) by variational methods in two particular situations:

Firstly, when the nonlinearity $f$ is $\omega$-periodic in the second variable for some $\omega > 0$. More precisely, we shall consider as a model case the forced pendulum equation on time scales

$$y^{\Delta\Delta} + a\sin(y^{\sigma}) = p(t), \qquad t \in [0, T]_{\mathbb{T}}. \tag{1.3}$$

For the continuous case $\mathbb{T} = \mathbb{R}$, it is well known that if $\overline{p} := \dfrac{1}{T}\displaystyle\int_0^T p(t)dt = 0$, then (1.3) has at least two geometrically different $T$-periodic solutions, i.e., not differing in a multiple of $2\pi$. The first solution was obtained by Hamel [7] in 1922, and the second one by Mawhin and Willem [10] in 1984. In this work, we extend this result for a general time scale.

Secondly, we study the case of a nonperiodic but bounded nonlinearity $f$ under the so-called Landesman–Lazer conditions.

A vast literature exists on Landesman–Lazer type conditions for resonant problems in the continuous case, starting at the pioneering work [9] for a second order elliptic (scalar) differential equation under Dirichlet conditions. For a survey on Landesman–Lazer conditions see e.g., [11].

Existence results for both of the above situations on time scales have been obtained in [2] by topological methods. Here, the focus is put on the variational structure of the problem. This allows, in particular, to give a positive answer to the problem of finding a periodic solution of equation (1.3) when the forcing term $p$ has zero average. It is pertinent to note, however, that the variational setting for the periodic problem does not include the case $\sigma^2(T) \neq \sigma(T)$.

For completeness, let us recall the essential terminology of time scales.

**Definition 1.1.** A time scale $\mathbb{T}$ is a nonempty, closed subset of $\mathbb{R}$, equipped with the topology induced from the standard topology on $\mathbb{R}$.

**Definition 1.2.** The forward (backward) jump operator $\sigma$ at $t$ for $t < \sup \mathbb{T}$ (respectively $\rho$ at $t$ for $t > \inf \mathbb{T}$) is given by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}) \text{ for all } t \in \mathbb{T}.$$

Additionally $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$. Furthermore, denote $\sigma^2(t) = \sigma(\sigma(t))$ and $y^\sigma(t) = y(\sigma(t))$.

**Definition 1.3.** If $\sigma(t) > t$, then the point $t$ is called right-scattered, while if $\rho(t) < t$, then $t$ is termed left-scattered. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then the point $t$ is called right-dense, while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say that $t$ is left-dense.

If $\mathbb{T}$ has a left-scattered maximum at $m$, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$. Otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

**Definition 1.4.** Fix $t \in \mathbb{T}^\kappa$ and let $y : \mathbb{T} \to \mathbb{R}$. Then $y^\Delta(t)$ is the number (if it exists) with the property that given $\epsilon > 0$ there is a neighborhood $U$ of $t$ such that, for all $s \in U$

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|.$$

Here $y^\Delta(t)$ is termed the (delta) derivative of $y(t)$ at $t$.

**Theorem 1.5. [8]** Assume that $y : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$.

(i) If $y$ is differentiable at $t$, then $y$ is continuous at $t$.

(ii) If $y$ is continuous at $t$ and $t$ is right-scattered, then $y$ is differentiable at $t$ and

$$y^\Delta(t) = \frac{y(\sigma(t)) - y(t)}{\sigma(t) - t}.$$

(iii) If $y$ is differentiable and $t$ is right-dense, then

$$y^\Delta(t) = \lim_{s \to t} \frac{y(t) - y(s)}{t - s}.$$

(iv) If $y$ is differentiable at $t$, then $y(\sigma(t)) = y(t) + \mu(t)y^\Delta(t)$.

**Definition 1.6.** The function $y$ is said to be right-dense continuous, that is $y \in C_{\mathrm{rd}}(\mathbb{T}; \mathbb{R})$ if:

(a) $y$ is continuous at every right-dense point $t \in \mathbb{T}$, and

(b) $\lim\limits_{s \to t^-} y(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

We shall use the standard notation for the different intervals in $\mathbb{T}$. For example, if $a, b \in \mathbb{R}$ with $a < b$, then the closed interval of numbers between $a$ and $b$ will be denoted by $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \le t \le b\}$.

The paper is organized as follows. In Section 2 we introduce some preliminary results concerning the Sobolev spaces in time scales.

In Section 3, we study the periodic problem for equation (1.3). Following the ideas in [10], we generalize a standard existence and multiplicity result to the context of time scales.

Finally, in Section 4 we obtain a Landesman–Lazer type result for (1.1) under periodic conditions.

## 2. Preliminary Results

Let us recall the Lebesgue measure in times scales, defined for example in [6], which can be constructed in the following way.

For $a < b \in \mathbb{T}$, consider $\mathcal{A} \subset \mathcal{P}([a, b)_{\mathbb{T}})$, the completion of the Borel $\sigma$-algebra generated by the family

$$\{[x, y)_{\mathbb{T}} : a \le x < y \le b, x, y \in \mathbb{T}\}.$$

Hence, there is a unique $\sigma$-additive measure $\mu_\Delta : \mathcal{A} \to \mathbb{R}^+$ defined over this basis as: $\mu_\Delta([x, y)_{\mathbb{T}}) = y - x$. As mentioned in [1], it is easy to see that $\mu_\Delta$ can be characterized as

$$\mu_\Delta = \lambda + \sum_{i \in I} (\sigma(t_i) - t_i)\delta_{t_i},$$

where $\lambda$ is the Lebesgue measure on $\mathbb{R}$, and $\{t_i\}_{i \in I}$ is the (at most countable) set of all right-scattered points of $\mathbb{T}$. A function $f$ which is measurable with respect to $\mu_\Delta$ is called $\Delta$-measurable, and the Lebesgue integral over $[a, b)_{\mathbb{T}}$ is denoted by

$$\int_a^b f(t)\ \Delta t := \int_{[a,b)_{\mathbb{T}}} f(t)\ d\mu_\Delta.$$

Thus, for $1 \le p < \infty$ the Banach spaces $L^p$ may be defined in the standard way, namely

$$L^p_\Delta([a, b)_{\mathbb{T}}) := \left\{ \hat{f} : f \text{ is } \Delta\text{-measurable and } \int_a^b |f(t)|^p\ \Delta t < \infty \right\},$$

where $\hat{f}$ denotes the equivalence class of $f$ of all the $\Delta$-measurable functions that coincide with $f$ almost everywhere for the $\Delta$-measure. The norm of this space will be denoted by

$$\|f\|_{L^p_\Delta} := \left( \int_a^b |f(t)|^p\ \Delta t \right)^{1/p}.$$

Next, we shall introduce as in [1] the idea of *weak time scale derivative* (for brevity, *weak derivative*):

**Definition 2.1.** Let $f \in L^p_\Delta([a,b)_\mathbb{T})$. A weak derivative of $f$ (if it exists) is a $\Delta$-measurable function $g$ such that

$$\int_a^b f(t)\varphi^\Delta(t)\Delta t = -\int_a^b g(t)\varphi^\sigma(t)\Delta t$$

for any $\varphi \in [C^1_{\rm rd}]_0([a,b]) := \{\varphi \in C^1_{\rm rd}([a,b]) : \varphi(a) = \varphi(b) = 0\}$.

**Remark 2.2.** If $f \in C^1_{\rm rd}([a,b])$, then by the product rule it follows that $f^\Delta$ is also a weak derivative of $f$.

**Remark 2.3.** Let $g \in C_{\rm rd}([a,b])$, and define $f(t) = \displaystyle\int_0^t g(s)\Delta s$. Then, by the fundamental theorem (see [6]) it follows that $g$ is a weak derivative of $f$.

**Remark 2.4.** It is easy to see that if $f$ has zero weak derivative, then $f \equiv c$ for some constant $c$. In view of the previous remark, we deduce that if $f$ has a right-dense continuous weak derivative, then it belongs to $C^1_{\rm rd}$.

Thus, the Sobolev spaces $W^{1,p}_\Delta([a,b))$ may be defined as in the continuous case:

$$W^{1,p}_\Delta([a,b)) := \{f \in L^p_\Delta([a,b)) : f \text{ has a weak derivative } f^\Delta \in L^p_\Delta([a,b))\},$$

equipped with the norm

$$\|f\|_{W^{1,p}_\Delta} := \left(\|f\|^p_{L^{1,p}_\Delta} + \|f^\Delta\|^p_{L^{1,p}_\Delta}\right)^{1/p}.$$

In particular, for $p = 2$ we shall denote $H^1_\Delta([a,b)) := W^{1,2}_\Delta([a,b))$, and its norm is induced by the inner product given by

$$\langle f, g\rangle_{H^1_\Delta} := \int_a^b [f(t)g(t) + f^\Delta(t)g^\Delta(t)]\,\Delta t.$$

Basic properties of Sobolev spaces in time scales can be found in [1].

## 3. The Forced Pendulum Equation

In this section we prove the existence of periodic solutions for the pendulum equation on a time scale $\mathbb{T}$ when the forcing term has zero average. Note that if $y$ is a solution of (1.3)–(1.2), then $y + 2k\pi$ is also a solution for any integer $k$. So, in order to establish an appropriate multiplicity result we shall say that two solutions $y_1$, $y_2$ are geometrically different if $y_1 - y_2 \neq 2k\pi$ for all $k \in \mathbb{Z}$.

**Theorem 3.1.** Assume that $\sigma^2(T) = \sigma(T)$, and that $p \in L^1([0, \sigma(T))$ satisfies $\overline{p} = 0$, where $\overline{p} := \dfrac{1}{\sigma(T)} \displaystyle\int_0^{\sigma(T)} p(t)\Delta t$. Then problem (1.3)–(1.2) has at least two geometrically different solutions, i.e., not differing in a multiple of $2\pi$.

For a proof of Theorem 3.1, let us consider the space

$$H := \mathbb{R} + H_0^1([0, \sigma(T))_{\mathbb{T}}) = \{y : H^1([0, \sigma(T)) : y(0) = y(\sigma(T))\}$$

with the induced norm $\|y\| := \|y\|_{H_\Delta^1}$, and the functional $J_p : H \to \mathbb{R}$ given by

$$J_p(y) = \int_0^{\sigma(T)} \left( \frac{y^\Delta(t)^2}{2} + a\cos(y^\sigma(t)) + p(t)y^\sigma(t) \right) \Delta t.$$

It is clear that if $y$ is a critical point of $J_p$, then $y$ is a weak solution of the problem (and then, from Remark 2.4, classical). Indeed, by simple computation it follows that $J_p \in C^1(H, \mathbb{R})$, and its derivative $DJ_p : H \to H^*$ is given by:

$$DJ_p(y)(\varphi) = \int_0^{\sigma(T)} \left( y^\Delta(t)\varphi^\Delta(t) - a\sin(y^\sigma(t))\varphi^\sigma(t) + p(t)\varphi^\sigma(t) \right) \Delta t.$$

Thus, if we consider $\varphi \in H_0^1([0, \sigma(T))_{\mathbb{T}})$ we deduce that $y$ is a weak solution of equation (1.3), and hence classical. Moreover, taking $\varphi \equiv 1$ we obtain:

$$a \int_0^{\sigma(T)} \sin(y^\sigma(t))\ \Delta t = \int_0^{\sigma(T)} p(t)\ \Delta t = 0.$$

Integrating (1.3) we conclude that

$$y^\Delta(\sigma(T)) - y^\Delta(0) = \int_0^{\sigma(T)} y^{\Delta\Delta}(t)\ \Delta t = 0.$$

In order to obtain solutions of (1.3)–(1.2) as critical points of $J_p$, we need a compactness condition. It is worthy to note, however, that $J_p$ does not satisfy the so-called *Palais–Smale condition*.

**Definition 3.2.** Let $E$ be a Banach space and $J \in C^1(E, \mathbb{R})$. It is said that $J$ satisfies (PS) if any sequence $\{y_n\} \subset E$ such that $|J(y_n)| \leq c$ for some constant $c$ and $DJ(y_n) \to 0$, has a convergent subsequence in $E$.

This "lack of compactness" is due to the fact that the functional $J_p$ is $2\pi$-periodic, i.e., $J_p(y) = J_p(y + 2\pi)$. Indeed, from the computations below it follows that $J_p$ admits a Palais–Smale sequence, i.e., a sequence $\{y_n\}$ such that $J_p(y_n)$ is bounded and $DJ_p(y_n) \to 0$. If $\{y_n\}$ has a convergent subsequence, still denoted $\{y_n\}$, then $\{y_n + 2n\pi\}$ is a Palais–Smale sequence with no convergent subsequences.

The following version of the mountain pass lemma, proved by Pucci and Serrin in [12], assumes a weaker compactness condition which is appropriate for the present case:

**Theorem 3.3.** Let $E$ be a Banach space and let $J \in C^1(E, \mathbb{R})$, $e \in E$ and $R > r > 0$ such that $\|e\| > R$, and

$$\max\{J(e), J(0)\} \leq b := \inf_{r \leq \|x\| \leq R} J(x).$$

Let

$$\Gamma = \{\gamma \in C([0,1]), E) : \gamma(0) = 0, \gamma(1) = e\},$$

and

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)).$$

Moreover, assume that the following conditions hold:

1. If $\{y_n\} \subset H$ satisfies $J(y_n) \to c$ and $DJ(y_n) \to 0$, then $c$ is a critical value of $J$.
2. (BPS) If the sequence $\{y_n\} \subset H$ is bounded, with $J(y_n)$ bounded and $DJ(y_n) \to 0$, then $\{y_n\}$ has a convergent subsequence in $H$.

Then $c$ is a critical value of $J$. Moreover, if $c = b$, then there is a critical point $y$ such that $J(y) = b$ and $\|y\| = \dfrac{r+R}{2}$.

*Proof of Theorem 3.1.* As $\overline{p} = 0$, it is immediate that $J_p$ is bounded from below. On the other hand, from the $2\pi$-periodicity of $J_p$ we may assume, for a minimizing sequence $\{y_n\}$, that $y_n(0) \in [0, 2\pi]$, then writing $y_n(t) - y_n(0) = \int_0^t y_n^\Delta(s)\, \Delta s$ we deduce that

$$\|y_n\| \leq \|y_n - y_n(0)\| + D \leq C\|y_n^\Delta\|_{L^2_\Delta} + D$$

for some constants $C$ and $D$. Furthermore,

$$\|y_n^\Delta\|^2_{L^2_\Delta} \leq 2J_p(y_n) + C$$

for some $C$, and as $J_p(y_n)$ is bounded we deduce that $\{y_n\}$ is bounded. As $H$ is reflexive, from standard results we conclude that $J_p$ achieves a minimum $y_0$.

Next, consider the functional $I(y) := J_p(y_0 + y)$. Then

$$I(0) = I(2\pi) = \min_{y \in H} I(y),$$

and if we set $0 < r < R < 2\pi$, all the assumptions of Theorem 3.3 hold for $e = 2\pi$. Indeed, if $I(y_n) \to c$ and $DI(y_n) \to 0$, as before we may assume that $y_n(0) \in [0, 2\pi]$. Then $\{y_n\}$ is bounded, and from the compactness of the imbeddings $H \hookrightarrow C([0, \sigma(T)])$

and $H \hookrightarrow (H, w)$ (where $w$ denotes the weak topology), we may assume also that $y_n \to y$ uniformly, and weakly in $H$. Then $DI(y_n)(y - y_n) \to 0$, that is to say:

$$\int_0^{\sigma(T)} \left( \left(y_n^\Delta + y_0^\Delta\right)\left(y_n^\Delta - y^\Delta\right) - a\sin(y_n^\sigma + y_0^\sigma)(y_n^\sigma - y^\sigma) + p(y_n^\sigma - y^\sigma) \right)(t)\Delta t \to 0.$$

It follows that $\int_0^{\sigma(T)} y_n^\Delta(t)(y_n^\Delta(t) - y^\Delta(t))\ \Delta t \to 0$, and as $\int_0^{\sigma(T)} y^\Delta(t)(y_n^\Delta(t) - y^\Delta(t))\ \Delta t \to 0$ we conclude that

$$\int_0^{\sigma(T)} (y_n^\Delta - y^\Delta)^2(t)\Delta t \to 0,$$

i.e., $y_n \to y$ in $H$ (and $DI(y) = 0$). In a similar way, it is seen that $I$ satisfies (BPS).

Now, if $c > \min_{y\in H} I(y)$, then there exists a critical point $z$ of $I$ with $I(z) = c \neq I(y_0)$, and $y = y_0 + z$ is a critical point of $J_p$ such that $y - y_0 \notin 2\pi\mathbb{Z}$. On the other hand, if $c = \min_{y\in H} I(y)$, then there exists a critical point $z$ of $I$ with $\|z\| = \dfrac{R+r}{2}$, and hence $y = y_0 + z$ is a critical point of $J_p$, with $0 < \|y - y_0\| < 2\pi$. ■

## 4. Landesman–Lazer Conditions

In this section we study problem (1.1)–(1.2) under Landesman–Lazer type conditions. As before, for a variational formulation of the problem we shall assume that the time scale satisfies the condition $\sigma^2(T) = \sigma(T)$.

Let $F(t, u) = \int_0^u f(t, s)ds$, and consider the functional

$$J(y) = \int_0^{\sigma(T)} \left( \frac{y^\Delta(t)^2}{2} - F(t, y^\sigma(t)) \right) \Delta t,$$

defined over the space $H$ given in the previous section. Then

$$DJ(y)(\varphi) = \int_0^{\sigma(T)} \left( y^\Delta(t)\varphi^\Delta(t) - f(t, y^\sigma(t))\varphi^\sigma(t) \right) \Delta t,$$

and as before we deduce that any critical point of $J$ is a classical solution of (1.1)–(1.2).

We shall prove the existence of critical points of $J$ under the following conditions: Let $f : [0, T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ be bounded and continuous, with limits at infinity

$$\lim_{s\to\pm\infty} f(t, s) := f^\pm(t).$$

In this context, the Landesman–Lazer conditions for problem (1.1)–(1.2) read as follows:

$$\int_0^{\sigma(T)} f^+(t)\ \Delta t < 0 < \int_0^{\sigma(T)} f^-(t)\ \Delta t \tag{4.1}$$

or

$$\int_0^{\sigma(T)} f^-(t)\ \Delta t < 0 < \int_0^{\sigma(T)} f^+(t)\ \Delta t. \tag{4.2}$$

**Theorem 4.1.** Assume that (4.1) or (4.2) holds. Then (1.1)–(1.2) admits at least one solution.

For a proof of Theorem 4.1 we shall apply two standard results.

**Theorem 4.2.** Let $J$ satisfy (PS), and assume that $J$ is coercive. Then there exists $y_0 \in H$ such that $J(y_0) = \inf\limits_{y\in H} J(y)$.

**Theorem 4.3. (Rabinowitz, [13])** Let $E$ be a Banach space and $J : E \to \mathbb{R}$ a $C^1$ functional satisfying (PS). Furthermore, assume that $E = E_1 \oplus E_2$, with $\dim(E_1) < \infty$, and

$$\max_{x\in E_1:\|x\|=R} J(x) < \inf_{y\in E_2} J(y)$$

for some $R > 0$. Then $J$ has at least one critical point.

*Proof of Theorem 4.1.* In the first place, let us prove that $J$ satisfies the Palais–Smale condition given in Definition 3.2. Assume that

$$|J(y_n)| \le c, \qquad \|DJ(y_n)\|_{H^*} := \varepsilon_n \to 0, \tag{4.3}$$

where $H^*$ denotes the dual space of $H$. We claim that $y_n$ is bounded: Indeed, otherwise we may suppose that $\|y_n\| \to \infty$. Set $v_n = \dfrac{y_n}{\|y_n\|}$, then as in the previous section taking a subsequence we may assume that $v_n \to v$ weakly and $v_n \to v$ uniformly. Moreover, from the inequality

$$\int_0^{\sigma(T)} \left(\frac{y_n^\Delta(t)^2}{2} - F(t, y_n^\sigma(t))\right) \Delta t \le c$$

and the fact that

$$|F(t,u)| = \left|\int_0^u f(t,s)ds\right| \le C|u|$$

for some constant $C$, we deduce that

$$\int_0^{\sigma(T)} \frac{y_n^\Delta(t)^2}{\|y_n\|^2}\ \Delta t \to 0.$$

As before, we have that

$$\|y_n - y_n(0)\| \le C\|y_n^\Delta\|_{L^2_\Delta}$$

for some constant $C$. Thus, if we write

$$v_n = \frac{y_n - y_n(0)}{\|y_n\|} + \frac{y_n(0)}{\|y_n\|},$$

the first term goes to 0. Then, taking a subsequence we may assume that $v_n \to c_0$ uniformly for some constant $c_0 \neq 0$. Furthermore,

$$v_n^\Delta = \frac{y_n^\Delta}{\|y_n\|} \to 0,$$

and we conclude that $v_n \to c_0$ in $H$.

On the other hand, from (4.3) we have

$$-2c \le \int_0^{\sigma(T)} \left(y_n^\Delta(t)^2 - 2F(t, y_n^\sigma(t))\right) \Delta t \le 2c$$

and $\|DJ(y_n)(y_n)\| \le \varepsilon_n \|y_n\|$, or equivalently

$$-\varepsilon_n \|y_n\| \le \int_0^{\sigma(T)} \left(-y_n^\Delta(t)^2 + f(t, y_n^\sigma(t)) y_n^\sigma(t)\right) \Delta t \le \varepsilon_n \|y_n\|.$$

Hence

$$\left| \int_0^{\sigma(T)} [f(t, y_n^\sigma(t)) - 2\tilde{F}(t, y_n^\sigma(t))] v_n^\sigma(t)\ \Delta t \right| \le \frac{2c}{\|y_n\|} + \varepsilon_n, \tag{4.4}$$

where

$$\tilde{F}(t,u) = \begin{cases} \dfrac{F(t,u)}{u} & \text{if } u \neq 0 \\ f(t,0) & \text{if } u = 0. \end{cases}$$

Suppose for example that $c_0 > 0$. Then $y_n(t) \to +\infty$ uniformly. Moreover, for any $\varepsilon > 0$ and $t \in [0, \sigma(T)]_{\mathbb{T}}$ we may fix $u_0$ such that if $u \ge u_0$, then $|f(t,u) - f^+(t)| < \dfrac{\varepsilon}{2}$. Then

$$\left| \frac{F(t,u)}{u} - f^+(t) \right| = \left| \frac{F(t,u_0)}{u} + \frac{1}{u} \int_{u_0}^{u} [f(t,s) - f^+(t)]\, ds - \frac{u_0}{u} f^+(t) \right| < \varepsilon$$

for $u \gg 0$. It follows that $\tilde{F}(t,u) \to f^+(t)$ for $u \to +\infty$, and by dominated convergence and (4.4) we conclude that

$$\int_0^{\sigma(T)} f^+(t) c_0\ \Delta t = 2 \int_0^{\sigma(T)} f^+(t) c_0\ \Delta t,$$

which contradicts (4.1) and (4.2). Thus, $y_n$ is bounded (the proof is analogous if $c_0 < 0$).

Next, taking a subsequence we may assume that $y_n \to y$ weakly in $H$ and uniformly for some $y$. As $\|y_n\|$ is bounded,

$$|DJ(y_n)(y_n - y)| \le \varepsilon_n \|y_n - y\| \to 0.$$

Moreover, as $f$ is bounded, the uniform convergence of $y_n \to y$ implies that the second term of $DJ(y_n)(y_n - y)$ tends to 0, and hence

$$\int_0^{\sigma(T)} y_n^\Delta(t)(y_n^\Delta(t) - y^\Delta(t))\ \Delta t \to 0.$$

Then, as $y_n \to y$ weakly in $H$, we conclude that

$$\int_0^{\sigma(T)} (y_n^\Delta - y^\Delta)^2(t)\Delta t \to 0,$$

and thus $y_n \to y$ in $H$.

Next, we shall prove that if (4.1) holds, then $J$ is coercive. Indeed, suppose that $J(y_n) \le c$ and $\|y_n\| \to \infty$. In the same way as before, we deduce that $\dfrac{y_n}{\|y_n\|} \to c_0 \in \mathbb{R} \setminus \{0\}$. If for example $c_0 > 0$ then

$$\lim_{n\to\infty} \int_0^{\sigma(T)} \frac{F(t, y_n^\sigma(t))}{\|y_n\|}\ \Delta t = c_0 \int_0^{\sigma(T)} f^+(t)\ \Delta t < 0,$$

and the same conclusion holds if $c_0 < 0$. As $J(y_n) \le c$, for $n$ large we obtain that

$$\frac{\|y_n^\Delta\|^2_{L^2_\Delta}}{\|y_n\|} < 0,$$

a contradiction.

Finally, we observe that if (4.2) holds, then $J$ satisfies the assumptions of Theorem 4.3. Indeed, we may decompose $H$ into a direct sum $H = \mathbb{R} \oplus H_1$, with $H_1 := \{y \in H : y(0) = 0\}$. Then

$$\|y\| \le C\|y^\Delta\|_{L^2_\Delta} \qquad \text{for all } y \in H_1.$$

Hence

$$J(y) = \frac{1}{2}\|y^\Delta\|^2_{L^2_\Delta} - \int_0^{\sigma(T)} F(t, y^\sigma(t))\ \Delta t \ge \frac{1}{2}\|y^\Delta\|^2_{L^2_\Delta} - C\|y\|_{L^2_\Delta},$$

which proves that $J$ is coercive on $H_1$. Then $\inf\limits_{y\in H_1} J(y) > -\infty$. On the other hand, if $y \in \mathbb{R}$, then

$$J(y) = -\int_0^{\sigma(T)} F(t, y(t))\ \Delta t = -y \int_0^{\sigma(T)} \tilde{F}(t, y(t))\ \Delta t.$$

As $\displaystyle\int_0^{\sigma(T)} \tilde{F}(t, y(t))\ \Delta t \to \int_0^{\sigma(T)} f^\pm(t)\ \Delta t$ for $y \to \pm\infty$, we conclude that $J(y) \to -\infty$ as $y \to \pm\infty$. Then, if $|y| = R \gg 0$, it follows that $|J(y)| < \inf\limits_{y_1\in H_1} J(y_1)$, and this completes the proof. ■

**Example 4.4.** Let $f(t, y) = \xi(t)\arctan(y)$, where $\xi : [0, T]_\mathbb{T} \to \mathbb{R}$ is continuous and satisfies

$$\int_0^{\sigma(T)} \xi(t)\ \Delta t \ne 0.$$

Then (1.1)–(1.2) admits at least one solution.

Indeed, in this case

$$f^{\pm}(t) = \pm\frac{\pi}{2}\xi(t),$$

and clearly one of the conditions (4.1) or (4.2) is fulfilled.

## References


[1] Ravi P. Agarwal, Victoria Otero-Espinar, Kanishka Perera, and Dolores R. Vivero, Basic properties of Sobolev's spaces on time scales, *Adv. Difference Equ.*, pages Art. ID 38121, 14, 2006.

[2] P. Amster and C.C. Tisdell, Two classical periodic problems on time scales, 2007, Submitted.

[3] Martin Bohner, Gusein Guseinov, and Allan Peterson, Introduction to the time scales calculus, In *Advances in dynamic equations on time scales*, pages 1–15. Birkhäuser Boston, Boston, MA, 2003.

[4] Martin Bohner and Allan Peterson, *Dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2001. An introduction with applications.

[5] Javier G. P. Gamarra and Ricard V. Solé, Complex discrete dynamics from simple continuous population models, *Bull. Math. Biol.*, 64:611–620, 2002.

[6] Gusein Sh. Guseinov, Integration on time scales, *J. Math. Anal. Appl.*, 285(1):107–127, 2003.

[7] Georg Hamel, Über erzwungene Schwingungen bei endlichen Amplituden, *Math. Ann.*, 86(1-2):1–13, 1922.

[8] Stefan Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.*, 18(1-2):18–56, 1990.

[9] E.M. Landesman and A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.*, 19:609–623, 1969/1970.

[10] J. Mawhin and M. Willem, Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, *J. Differential Equations*, 52(2):264–287, 1984.

[11] Jean Mawhin, Landesman–Lazer conditions for boundary value problems: A nonlinear version of resonance, *Bol. Soc. Esp. Mat. Apl.*, 16:45–65, 2000.

[12] Patrizia Pucci and James Serrin, Extensions of the mountain pass theorem, *J. Funct. Anal.*, 59(2):185–210, 1984.

[13] Paul H. Rabinowitz, Some minimax theorems and applications to nonlinear partial differential equations, In *Nonlinear analysis (collection of papers in honor of Erich H. Rothe)*, pages 161–177. Academic Press, New York, 1978.

[14] Vanessa Spedding, Taming nature's numbers, *New Scientist*, 179(2404):28–31, 19 July 2003.