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# Boundary Value Problems for First Order Impulsive Difference Equations*

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**Abstract**

In this paper, first order impulsive difference equations with linear boundary conditions are discussed. By using a new comparison theorem and the method of upper and lower solutions coupled with the monotone iterative technique, criteria on the existence of minimal and maximal solutions are obtained.

**AMS subject classification:** 34D20, 34A37.
**Keywords:** Impulsive difference equations, linear boundary conditions, upper and lower solutions, monotone iterative technique.

## 1. Introduction

In this paper, a boundary value problem (BVP) for impulsive difference equations with a family of linear two point boundary conditions is studied. The result is based on methods of upper and lower solutions and the monotone iterative technique. For constructing monotone sequences of a corresponding linear equation, the extremal solutions of the considered problem are given. The obtained result is an extension of the known one for impulsive difference equations with periodic boundary conditions [1], and also generalizes initial value problems as well as boundary value problems for ordinary difference equations and impulsive difference equations.

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The method of upper and lower solutions coupled with the monotone iterative technique provides an effective mechanism to prove constructive existence results for initial and boundary value problems for nonlinear differential equations and difference equations in recent years [2–6]. The basic idea of this method is that using the upper and lower solutions as an initial iteration, one can construct monotone sequences from corresponding linear equations, and these sequences converge monotonically to the maximal and minimal solutions of the nonlinear equations.

This paper is organized as follows. In Section 2, we establish a comparison principle. Then we discuss the existence and uniqueness of the solutions for linear BVP for impulsive difference equations. In Section 3, by use of the monotone iterative technique and the method of upper and lower solutions, we obtain existence theorems of extremal solutions for impulsive difference equations with linear boundary value conditions.

## 2. Preliminaries and Comparison Result

Consider the BVP for impulsive difference equations

$$\begin{cases} \Delta x(n) = f(n, x(n)), \quad n \neq n_k, n \in J, \\ \Delta x(n_k) = I_k(x(n_k)), \quad n = n_k, \\ Mx(0) - Nx(T) = C, \quad k = 1, 2, \dots, p, \end{cases} \tag{2.1}$$

where $\Delta$ denotes the forward difference operator, i.e., $\Delta x(n) = x(n+1) - x(n)$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $J = [0, T] = \{0, 1, \dots, T\}$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $(k = 1, 2, \dots, p)$, $M > 0$, $N \geq 0$ and $C$ are real constants, and $\{n_k\}_{k=1}^{p}$ are fixed points, such that $0 < n_1 < n_2 < \dots < n_p < T$, $T \in \mathbb{N}$.

Let $\Omega$ denote the set of real-valued functions defined on $J$, and let $\mathbb{N}$ denote the set of all natural numbers. For all $x \in \Omega$, put $\|x\| = \max_{n \in J} |x(n)|$.

**Definition 2.1.** The functions $\alpha$ and $\beta$ are called lower and upper solutions of (2.1), respectively, if

$$\begin{cases} \Delta \alpha(n) \leq f(n, \alpha(n)) - r_\alpha, \quad n \neq n_k, n \in J, \\ \Delta \alpha(n_k) \leq I_k(\alpha(n_k)) - d_{\alpha_k}, \quad n = n_k, \\ M\alpha(0) - N\alpha(T) = C, \quad k = 1, 2, \dots, p, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \Delta \beta(n) \geq f(n, \beta(n)) + r_\beta, \quad n \neq n_k, n \in J, \\ \Delta \beta(n_k) \geq I_k(\beta(n_k)) + d_{\beta_k}, \quad n = n_k, \\ M\beta(0) - N\beta(T) = C, \quad k = 1, 2, \dots, p, \end{cases} \tag{2.3}$$

where for $0 < L < 1$, $L_k < 1$, $r_\alpha$, $r_\beta$, $d_{\alpha_k}$ and $d_{\beta_k}$ $(k = 1, 2, \dots, p)$ are given by

$$r_\alpha = \begin{cases} \dfrac{(M-N)(Ln+1) + LNT}{T} [M\alpha(0) - N\alpha(T)] & \text{if } M\alpha(0) > N\alpha(T), \\ 0 & \text{if } M\alpha(0) \leq N\alpha(T), \end{cases}$$

$$r_\beta = \begin{cases} \dfrac{(M-N)(Ln+1)+LNT}{T}[M\beta(0)-N\beta(T)] & \text{if } M\beta(0) < N\beta(T), \\ 0 & \text{if } M\beta(0) \geq N\beta(T), \end{cases}$$

$$d_{\alpha_k} = \begin{cases} \dfrac{(M-N)(L_k n_k+1)+L_k NT}{T}[M\alpha(0)-N\alpha(T)] & \text{if } M\alpha(0) > N\alpha(T), \\ 0 & \text{if } M\alpha(0) \leq N\alpha(T), \end{cases}$$

$$d_{\beta_k} = \begin{cases} \dfrac{(M-N)(L_k n_k+1)+L_k NT}{T}[M\beta(0)-N\beta(T)] & \text{if } M\beta(0) < N\beta(T), \\ 0 & \text{if } M\beta(0) \geq N\beta(T), \end{cases}$$

and $\alpha, \beta \in \Omega$, $\alpha(n) \leq \beta(n)$ $(\forall n \in J)$.

To establish our main result, we also need the following lemmas.

**Lemma 2.2.** Assume that
(i) the sequence $n_k$ satisfies $0 \leq n_0 < n_1 < \ldots < n_k < \ldots$ with $\lim\limits_{k\to\infty} n_k = \infty$;
(ii) for $k \in \mathbb{N}$, $n \geq n_0$,

$$\begin{cases} \Delta m(n) \leq l_n m(n) + q_n & n \neq n_k, \\ m(n_k+1) \leq b_k m(n_k) + e_k, \end{cases}$$

where $\{l_n\}$ and $\{q_n\}$ are two real-valued sequences and $l_n > -1$, $b_k$ and $e_k$ are constants, and $b_k \geq 0$. Then

$$\begin{aligned} m(n) \leq m(n_0) & \prod_{n_0<n_k<n} b_k \prod_{n_0<i<n, i\neq n_k, k\in\mathbb{N}} (1+l_i) \\ & + \sum_{i=n_0, i\neq n_k}^{n-1} \prod_{i<n_k<n} b_k \prod_{i<s<n, s\neq n_k} (1+l_s) q_i \\ & + \sum_{n_0<n_k<n} e_k \prod_{n_k<n_j<n} b_j \prod_{n_k<i<n, i\neq n_j, j\in\mathbb{N}} (1+l_i). \end{aligned}$$

**Remark 2.3.** This lemma is a discrete version of [3, Theorem 1.4.1].

**Lemma 2.4.** Assume that $m \in \Omega$ satisfies

$$\begin{cases} \Delta m(n) \leq -Lm(n) - r_m, & n \neq n_k, \ n \in J, \\ \Delta m(n_k) \leq -L_k m(n_k) - d_{m_k}, & k = 1, 2, \ldots, p, \\ Mm(0) - Nm(T) = C, \end{cases} \tag{2.4}$$

where $0 < L < 1$, $L_k < 1$ for $k = 1, 2, \ldots, p$, $M > 0$, $N \geq 0$ are constants such that

$$\frac{N}{M}(1-L)^{N-p-1} \prod_{k=1}^{p} (1-L_k) < 1,$$

and

$$r_m = \begin{cases} \dfrac{(M-N)(Ln+1)+LNT}{T}[Mm(0)-Nm(T)] & \text{if } Mm(0) > Nm(T), \\ 0 & \text{if } Mm(0) \le Nm(T), \end{cases}$$

$$d_{m_k} = \begin{cases} \dfrac{(M-N)(L_k n_k+1)+L_k NT}{T}[Mm(0)-Nm(T)] & \text{if } Mm(0) > Nm(T), \\ 0 & \text{if } Mm(0) \le Nm(T). \end{cases}$$

Then $m(n) \le 0$ for $n \in J$.

*Proof.* **Case I:** $C \le 0$, i.e., $Mm(0) \le Nm(T)$. Consider the following inequalities:

$$\begin{cases} \Delta m(n) \le -Lm(n), & n \ne n_k,\ n \in J \\ \Delta m(n_k) \le -L_k m(n_k), & k = 1, 2, \ldots, p. \end{cases}$$

By Lemma 2.2, we get

$$m(n) \le m(0) \prod_{0<i<n, i \ne n_k} (1-L) \prod_{0<n_k<n} (1-L_k). \tag{2.5}$$

Let $n = T$. Then we have

$$m(T) \le m(0)(1-L)^{N-p-1} \prod_{k=1}^{p} (1-L_k).$$

For $C \le 0$, we have $m(0) \le \dfrac{N}{M} m(T)$ and

$$m(0)\left[1 - \frac{N}{M}(1-L)^{N-p-1} \prod_{k=1}^{p} (1-L_k)\right] \le 0.$$

Hence, $m(0) \le 0$. From (2.5) we have $m(n) \le 0$ for $n \in J$.

**Case II:** $C > 0$. Let

$$\overline{m}(n) = m(n) + g(n), \quad n \in J,$$

where $g(n) = \dfrac{Mn + N(T-n)}{T}[Mm(0) - Nm(T)]$, $n \in J$. Then

$$g(0) = N[Mm(0) - Nm(T)], \quad g(T) = M[Mm(0) - Nm(T)] \text{ and } g(n) \ge 0.$$

Hence, we have

$$\begin{aligned} M\overline{m}(0) - N\overline{m}(T) &= Mm(0) + Mg(0) - Nm(T) - Ng(T) \\ &= Mm(0) - Nm(T) = C, \end{aligned}$$

and

$$
\begin{aligned}
\Delta\overline{m}(n) &= \Delta m(n) + \Delta g(n) \\
&\leq -Lm(n) - \frac{(M-N)(Ln+1)+LNT}{T}[Mm(0) - Nm(T)] \\
&\quad + \frac{M-N}{T}[Mm(0) - Nm(T)] \\
&= -Lm(n) - L\frac{(M-N)n+NT}{T}[Mm(0) - Nm(T)] \\
&= -Lm(n) - Lg(n) = -L\overline{m}(n), \qquad n \neq n_k, \ \ n \in J,
\end{aligned}
$$

$$
\begin{aligned}
\Delta\overline{m}(n_k) &= \Delta m(n_k) + \Delta g(n_k) \\
&\leq -L_k m(n_k) - \frac{(M-N)(L_k n+1)+L_k NT}{T}[Mm(0) - Nm(T)] \\
&\quad + \frac{M-N}{T}[Mm(0) - Nm(T)] \\
&= -L_k m(n_k) - L_k\frac{(M-N)n+NT}{T}[Mm(0) - Nm(T)] \\
&= -L_k m(n_k) - L_k g(n_k) = -L_k\overline{m}(n_k), \qquad k = 1, 2, \ldots, p.
\end{aligned}
$$

In view of case I, we see that $\overline{m}(n) \leq 0$ for $n \in J$. Therefore $m(n) \leq 0$ for $n \in J$. The proof of Lemma 2.4 is complete. ■

Let us consider the linear impulsive difference equations with linear boundary conditions

$$
\begin{cases}
\Delta u(n) + Lu(n) = \sigma(n), \ \ n \neq n_k, \ n \in J, \\
\Delta u(n_k) = -L_k u(n_k) + I_k(\eta(n_k)) + L_k\eta(n_k), \ \ k = 1, 2, \ldots, p, \\
Mm(0) - Nm(T) = C,
\end{cases}
\tag{2.6}
$$

where $L$, $L_k$ ($k = 1, 2, \ldots, p$) are constants, $I_k \in C(J, \mathbb{R})$, $\eta \in \Omega$ and $\sigma \in C(J, \mathbb{R})$.

**Lemma 2.5.** Let $0 < L < 1$. Then $u \in \Omega$ is a solution of (2.6) if and only if $u$ is a solution of the following impulsive summation equation:

$$
\begin{aligned}
u(n) = {} & \frac{C}{M - N(1-L)^T} + \sum_{j=0, j\neq n_k}^{T-1} G(n, j)\sigma(j) \\
& + \sum_{0<n_k\leq T-1} G(n, n_k)\left[(L - L_k)u(n_k) + I_k(\eta(n_k)) + L_k(\eta(n_k))\right],
\end{aligned}
\tag{2.7}
$$

where

$$
G(n, j) = \frac{N}{M - N(1-L)^T}
\begin{cases}
\dfrac{(1-L)^n}{(1-L)^{j+1}} & 0 \leq j \leq n-1, \\[2ex]
\dfrac{(1-L)^{T+n}}{(1-L)^{j+1}} & n \leq j \leq T-1.
\end{cases}
$$

*Proof.* Set $y(n) = \dfrac{u(n)}{(1-L)^n}$, $n \in J$. From (2.6), we see that $y(n)$ satisfies

$$\begin{cases} y(n+1) = y(n) + \dfrac{\sigma(n)}{(1-L)^{n+1}}, \ \ n \neq n_k, \ n \in J, \\ \Delta y(n_k) = \dfrac{L-L_k}{1-L} y(n_k) + \dfrac{I_k(\eta(n_k)) + L_k \eta(n_k)}{(1-L)^{n_k+1}}, \ \ k = 1, 2, \dots, p, \\ My(0) - Ny(T)(1-L)^T = C. \end{cases} \tag{2.8}$$

From (2.8), we have

$$\begin{aligned} y(n) = y(0) + \sum_{j=0,\, j\neq n_k}^{n-1} \frac{\sigma(j)}{(1-L)^{j+1}} \\ + \sum_{0<n_k\leq n-1} \left( \frac{L-L_k}{1-L} y(n_k) + \frac{I_k(\eta(n_k)) + L_k(\eta(n_k))}{(1-L)^{n_k+1}} \right). \end{aligned} \tag{2.9}$$

Let $n = T$ in (2.9). Then we get

$$\begin{aligned} y(T) = y(0) + \sum_{j=0,\, j\neq n_k}^{T-1} \frac{\sigma(j)}{(1-L)^{j+1}} \\ + \sum_{0<n_k\leq T-1} \left( \frac{L-L_k}{1-L} y(n_k) + \frac{I_k(\eta(n_k)) + L_k(\eta(n_k))}{(1-L)^{n_k+1}} \right). \end{aligned} \tag{2.10}$$

From the boundary conditions $y(T) = \dfrac{My(0) - C}{N(1-L)^T}$, we obtain

$$\begin{aligned} y(0) = \frac{C}{M - N(1-L)^T} + \frac{N(1-L)^T}{M - N(1-L)^T} \Bigg[ \sum_{j=0,\, j\neq n_k}^{T-1} \frac{\sigma(j)}{(1-L)^{j+1}} \\ + \sum_{0<n_k\leq T-1} \left( \frac{L-L_k}{1-L} y(n_k) + \frac{I_k(\eta(n_k)) + L_k(\eta(n_k))}{(1-L)^{n_k+1}} \right) \Bigg]. \end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.9) and using $y(n) = \dfrac{u(n)}{(1-L)^n}$, $n \in J$, we see that $u$ satisfies (2.7). If $u$ is a solution of (2.7), then $u$ satisfies (2.6). ■

**Lemma 2.6.** Assume that $0 < L < 1$, $L_k < 1$ for $k = 1, 2, \dots, p$, $I_k \in C(J, \mathbb{R})$, $\eta \in \Omega$, and

$$\frac{N}{M - N(1-L)^T} \sum_{k=1}^{p} |L - L_k| < 1.$$

Then the BVP (2.6) has a unique solution in $\Omega$.

*Proof.* For any $u \in \Omega$, consider the operator $F$ defined by

$$Fu(n) = \frac{C}{M - N(1-L)^T} + \sum_{j=0, j\neq n_k}^{T-1} G(n, j)\sigma(j) + \sum_{0<n_k\le T-1} G(n, n_k)\left[(L - L_k)u(n_k) + I_k(\eta(n_k)) + L_k(\eta(n_k))\right],$$

where

$$G(n, j) = \frac{N}{M - N(1-L)^T} \begin{cases} \dfrac{(1-L)^n}{(1-L)^{j+1}} & 0 \le j \le n-1, \\ \dfrac{(1-L)^{T+n}}{(1-L)^{j+1}} & n \le j \le T-1. \end{cases}$$

For every $u$ and $v$, we have

$$\begin{aligned} |Fu(n) - Fv(n)| &\le \sum_{0<j_k\le T-1} |G(n, n_k)[(L - L_k)(u(n_k) - v(n_k))]| \\ &\le \frac{N}{M - N(1-L)^T} \sum_{k=1}^{p} |L - L_k||u(n_k) - v(n_k)| \\ &\le \left\{ \frac{N}{M - N(1-L)^T} \sum_{k=1}^{p} |L - L_k| \right\} \|u - v\|. \end{aligned}$$

Hence, $\|Fu - Fv\| = \max_{n\in J} |Fu(n) - Fv(n)| = \alpha\|u - v\|$, where

$$\alpha = \frac{N}{M - N(1-L)^T} \sum_{k=1}^{p} |L - L_k| < 1.$$

Thus the operator $F$ is a contraction on $\Omega$. That is, there is a unique element $u \in \Omega$ such that $u = Fu$. This $u$ is the unique solution of (2.6). The proof of Lemma 2.6 is complete. ■

## 3. Main Result

**Theorem 3.1.** Assume that

$(A_0)$: The function $f \in C(J \times \mathbb{R}, \mathbb{R})$ satisfies

$$f(n, x) - f(n, y) \ge -L(x - y),$$

whenever $\alpha(n) \le y \le x \le \beta(n)$, $n \in J$, where $0 < L < 1$.

$(A_1)$: The functions $I_k \in C(\mathbb{R}, \mathbb{R})$ satisfy

$$I_k(x) - I_k(y) \geq -L_k(x - y),$$

whenever $\alpha(n_k) \leq y \leq x \leq \beta(n_k)$, $L_k < 1$, $k = 1, 2, \ldots, p$.

$(A_2)$:

$$\frac{N}{M}(1 - L)^{N-p-1} \prod_{k=1}^{p}(1 - L_k) < 1.$$

$(A_3)$:

$$\frac{N}{M - N(1 - L)^T} \sum_{k=1}^{p} |L - L_k| < 1.$$

Then there exist monotone sequences $\{\alpha_j(n)\}$, $\{\beta_j(n)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, such that $\lim_{j\to\infty} \alpha_j(n) = \rho(n)$, $\lim_{j\to\infty} \beta_j(n) = r(n)$ uniformly on $J$, in which $\rho(n)$ and $r(n)$ are the minimal and the maximal solutions of the BVP (2.1), respectively, such that

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_j \leq \rho \leq x \leq r \leq \beta_j \leq \ldots \leq \beta_2 \leq \beta_1 \leq \beta_0 \quad \text{on } J,$$

where $x$ is any solution of the BVP (2.1) such that $\alpha(n) \leq x \leq \beta(n)$ on $J$.

*Proof.* Let $[\alpha, \beta] = \{x \in \Omega : \alpha(n) \leq x(n) \leq \beta(n),\ n \in J\}$. For any $\eta \in C[\alpha, \beta]$, we consider the linear impulsive difference equations (2.6), where

$$\sigma(n) = f(n, \eta(n)) + L\eta(n), \quad n \in J.$$

By Lemma 2.5, it clear that for every such $\eta$, there exists a unique solution $u \in \Omega$ of (2.6). Define a mapping $A$ by $A\eta = u$. This mapping will be used to define the sequences $\{\alpha_j(n)\}$, $\{\beta_j(n)\}$. We prove that

(a) $\alpha \leq A\alpha$, $\beta \geq A\beta$;

(b) $A$ is a monotone nondecreasing operator on the segment $[\alpha, \beta]$. To prove (a), set $A\alpha_0 = \alpha_1$, where $\alpha_1$ is the unique solution of (2.6) with $\eta = \alpha_0$. Setting $m = \alpha_0 - \alpha_1$, we have $Mm(0) - Nm(T) = [M\alpha_0(0) - N\alpha_0(T)] - [M\alpha_1(0) - N\alpha_1(T)]$. We see that

$$\begin{aligned}
\Delta m(n) &= \Delta\alpha_0(n) - \Delta\alpha_1(n) \\
&\leq f(n, \alpha_0(n)) - r_{\alpha_0} - f(n, \alpha_1(n)) + r_{\alpha_1} \\
&\leq f(n, \alpha_0(n)) - r_{\alpha_0} - [-L\alpha_1(n) + f(n, \alpha_0(n)) + L\alpha_0(n)] + r_{\alpha_1} \\
&= -Lm(n) - r_{\alpha_0} + r_{\alpha_1} \\
&= -Lm(n) - r_m, \quad n \neq n_k,\ \ n \in J,
\end{aligned}$$

$$\begin{aligned}
\Delta m(n_k) &= \Delta\alpha_0(n_k) - \Delta\alpha_1(n_k) \\
&\leq I_k(\alpha_0(n_k)) - d_{\alpha_{0k}} - I_k(\alpha_1(n_k)) + d_{\alpha_{1k}} \\
&\leq I_k(\alpha_0(n_k)) - d_{\alpha_{0k}} - [-L_k\alpha_1(n_k) + I_k(\alpha_0(n_k)) + L_k\alpha_0(n_k)] + d_{\alpha_{1k}} \\
&= -L_k m(n_k) - d_{\alpha_{0k}} + d_{\alpha_{1k}} \\
&= -L_k m(n) - d_{m_k}, \quad k = 1, 2, \ldots, p,
\end{aligned}$$

where $r_m, d_{m_k}$ $k = 1, 2, \ldots, p$ are given by

$$r_{\alpha_0} - r_{\alpha_1} = r_m$$
$$= \begin{cases} \dfrac{(M-N)(Ln+1)+LNT}{T}[Mm(0) - Nm(T)] & \text{if } Mm(0) > Nm(T), \\ 0 & \text{if } Mm(0) \le Nm(T), \end{cases}$$

$$d_{\alpha_{0k}} - d_{\alpha_{1k}} = d_{m_k}$$
$$= \begin{cases} \dfrac{(M-N)(L_k n_k+1)+L_k NT}{T}[Mm(0) - Nm(T)] & \text{if } Mm(0) > Nm(T), \\ 0 & \text{if } Mm(0) \le Nm(T). \end{cases}$$

By Lemma 2.4, we get $m(n) \le 0$ on $J$, i.e., $\alpha \le A\alpha$. Similarly, we can prove that $\beta \ge A\beta$.

To prove (b), let $\eta_1,\ \eta_2 \in [\alpha, \beta]$ such that $\eta_1 \le \eta_2$. Suppose that $u_1 = A\eta_1$ and $u_2 = A\eta_2$. Set $m = u_1 - u_2$. Using $(A_0)$, $(A_1)$ and (2.6), we get

$$\begin{aligned} \Delta m(n) &= \Delta u_1(n) - \Delta u_2(n) \\ &= [-Lu_1(n) + f(n, \eta_1(n)) + L\eta_1(n)] - [-Lu_2(n) + f(n, \eta_2(n)) + L\eta_2(n)] \\ &\le -L(u_1(n) - u_2(n)) \\ &= -Lm(n), \quad n \ne n_k,\ n \in J, \end{aligned}$$

$$\begin{aligned} \Delta m(n_k) &= \Delta u_1(n_k) - \Delta u_2(n_k) \\ &= [-L_k u_1(n) + I_k(\eta_1(n_k)) + L_k\eta_1(n_k)] \\ &\quad -[-L_k u_2(n_k) + I_k(\eta_2(n_k)) + L_k\eta_2(n_k)] \\ &\le -L_k(u_1(n_k) - u_2(n_k)) \\ &= -L_k m(n_k), \quad k = 1, 2, \ldots, p, \end{aligned}$$

$$Mm(0) - Nm(T) = Mu_1(0) - Nu_1(T) - (Mu_2(0) - Nu_2(T)) = C - C = 0.$$

In view of Lemma 2.4, we have $m(n) \le 0$ on $J$, i.e., $u_1 \le u_2$.

We can now define the sequences $\alpha_j = A\alpha_{j-1}$, $\beta_j = A\beta_{j-1}$ and conclude from the previous arguments that

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_j \le \ldots \le \beta_j \le \ldots \le \beta_2 \le \beta_1 \le \beta_0 \quad \text{on } J.$$

It then follows employing standard arguments that $\lim_{j\to\infty} \alpha_j(n) = \rho(n)$ and $\lim_{j\to\infty} \beta_j(n) = r(n)$ uniformly and monotonically on $J$. It is easy to show that $\rho$ and $r$ are solutions of (2.1) in view of the fact $\alpha_j(n)$, $\beta_j(n)$ $(j \in \mathbb{N})$ satisfy

$$\begin{aligned} \alpha_j(n) &= \frac{C}{M - N(1-L)^T} + \sum_{i=0, i\ne n_k}^{T-1} G(n,i)\sigma_{j-1}(i) \\ &\quad + \sum_{0<n_k\le T-1} G(n, n_k)[(L - L_k)\alpha_j(n_k) + I_k(\alpha_{j-1}(n_k)) + L_k(\alpha_{j-1}(n_k))], \end{aligned}$$

$$\beta_j(n) = \frac{C}{M - N(1-L)^T} + \sum_{i=0, i\neq n_k}^{T-1} G(n,i)\overline{\sigma}_{j-1}(i)$$
$$+ \sum_{0<n_k\leq T-1} G(n,n_k)[(L-L_k)\beta_j(n_k) + I_k(\beta_{j-1}(n_k)) + L_k(\beta_{j-1}(n_k))],$$

where

$$\sigma_{j-1}(n) = f(n,\alpha_{j-1}(n)) + L\alpha_{j-1}(n), \quad n \in J,$$

$$\overline{\sigma}_{j-1}(n) = f(n,\beta_{j-1}(n)) + L\beta_{j-1}(n), \quad n \in J.$$

To prove that $\rho$, $r$ are minimal and maximal solutions of (2.1), we have to show that if $x(n)$ is any solution of (2.1) such that $\alpha(n) \leq x(n) \leq \beta(n)$ on $J$, then $\alpha(n) \leq \rho(n) \leq x(n) \leq r(n) \leq \beta(n)$ on $J$. To do this, let $j \in \mathbb{N}$ such that $\alpha_j(n) \leq x(n) \leq \beta_j(n)$ on $J$, set $m = \alpha_{j+1} - x$ so that

$$\begin{aligned}\Delta m(n) &= \Delta\alpha_{j+1}(n) - \Delta x(n)\\ &= [-L\alpha_{j+1}(n) + f(n,\alpha_j(n)) + L\alpha_j(n)] - f(n,x(n))\\ &= -Lm(n), \quad n \neq n_k, \ n \in J,\end{aligned}$$

$$\begin{aligned}\Delta m(n_k) &= \Delta\alpha_{j+1}(n_k) - \Delta x(n_k)\\ &= [-L_k\alpha_{j+1}(n_k) + I_k(\alpha_j(n_k)) + L_k\alpha_j(n_k)] - I_k(x(n_k))\\ &= -L_k m(n_k), \quad k = 1,2,\ldots,p,\end{aligned}$$

$$Mm(0) - Nm(T) = M\alpha_{j+1}(0) - N\alpha_{j+1}(T) - (Mx(0) - Nx(T)) = C - C = 0.$$

By Lemma 2.4, $m(n) \leq 0$, and hence $\alpha_{j+1}(n) \leq x(n)$ on $J$. Similarly $x(n) \leq \beta_{j+1}(n)$ on $J$, and hence $\alpha_{j+1}(n) \leq x(n) \leq \beta_{j+1}(n)$ on $J$. Since $\alpha_0(n) \leq x(n) \leq \beta_0(n)$ on $J$, this proves by induction that $\alpha_j(n) \leq x(n) \leq \beta_j(n)$ on $J$ for every $j$. Taking the limit as $j \to \infty$, we conclude $\rho(n) \leq x(n) \leq r(n)$ on $J$ and the proof is complete. ■

**Remark 3.2.** When $M = 1$, $N = 1$, $C = 0$ the BVP (2.1) is reduced to a periodic boundary value problem of impulsive difference equations in [1].

**Remark 3.3.** When $M = 1$, $N = 1$, $C = 0$ and $I_k(x) \equiv 0$, $k = 1,2,\ldots,p$, the BVP (2.1) without impulses is reduced to a periodic problem of ordinary difference equations.

**Remark 3.4.** When $N = 0$, the BVP (2.1) is reduced to an initial value problem of impulsive difference equations.

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