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Positive Solutions for First Order Difference Equations with Impulses*

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Abstract

In this paper, we prove the existence of positive solutions for a class of first-order impulsive difference equations with periodic boundary value conditions.

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1. Introduction

The study of difference equations has caused a greater interest, for example, see [2, 4, 8–10, 13]. In [13], the authors studied the oscillation and stability of first order difference equations with impulses. In [8–10], the author investigated the existence of positive solutions of second-order difference equations without impulses. In [2], the authors obtained the existence of positive periodic solutions for second-order difference equations without impulses. On the other hand, the periodic boundary value problem (PBVP for short) of impulsive differential equations has been the subject of recent research, see [3, 6, 7, 11, 12, 14]. However, there are only a few publications about periodic solutions of impulsive difference equations. In this paper, we will make a contribution to

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the theory of PBVP of difference equations with impulses. To this end, we consider the PBVP

$$\begin{gathered} -\Delta y(n) + p(n)y(n) = f(n, y(n)),\ n \in [0, N-1],\ n \neq n_j, \\ \Delta y(n_j) = b_j y(n_j),\ j = 1, \dots, k, \\ y(0) = y(N), \end{gathered} \tag{1.1}$$

where $N \in \mathbb{N}$, $n_j \in [0, N-1]$ are fixed impulsive points and $0 = n_0 < n_1 < \cdots < n_k < n_{k+1} = N-1$, $\{p(n)\}_{n=0}^{N-1}, b_j$, $(j = 1, 2, \dots, k)$ are real number sequences, $\{y(n)\}_{n=0}^{N}$ is a desired solution, $\Delta y(n) = y(n+1) - y(n)$, and for two integers $a \leq b$, $[a, b]$ denotes the discrete segment being the set $\{a, a+1, \dots, b\}$. We will assume that

(H_1) $p(n) > -1$ and $\prod_{\substack{s=0 \\ s \neq n_j}}^{N-1} [p(s)+1] \geq 1, b_j > -1, (j = 1, \dots, k), \prod_{j=1}^{k} (1+b_j) > 1.$

Subject to PBVP (1.1), the linear equation

$$\begin{gathered} -\Delta y(n) + p(n)y(n) = h(n),\ n \in [0, N-1],\ n \neq n_j, \\ \Delta y(n_j) = b_j y(n_j),\ j = 1, \dots, k, \end{gathered} \tag{1.2}$$

together with

$$y(0) = y(N) \tag{1.3}$$

is called the corresponding linear problem of PBVP (1.1).

This paper is concerned with the existence of positive solutions of (1.1). Our tool in this paper will be the well-known Krasnoselskii fixed point theorem [2, 10] and the Leggett–Williams fixed point theorem [1, 5].

2. Green's Function for (1.2)–(1.3)

Consider the corresponding linear homogeneous equation of (1.2),

$$\begin{gathered} -\Delta u(n) + p(n)u(n) = 0,\ n \in [0, N-1], n \neq n_j, \\ \Delta u(n_j) = b_j u(n_j),\ j = 1, \dots, k, \end{gathered} \tag{2.1}$$

under the initial condition $u(0) = 1$. Denote the solution of (2.1) by $\{u(n)\}_{n=0}^{N}$. We get the following lemma.

Lemma 2.1. If $\{u(n)\}_{n=0}^{N}$ is a solution of (2.1), then

$$u(n) = \prod_{0 < n_j < n} (1+b_j)[p(n_j)+1]^{-1} \prod_{s=0}^{n-1} [p(s)+1],\ n \in [0, N].$$

Proof. From $\Delta u(n) - p(n)u(n) = 0$, we have

$$u(n+1) = [p(n)+1]u(n).$$

For $n \in [0, n_1]$, we easily see that

$$u(n) = \prod_{s=0}^{n-1} [p(s) + 1].$$

Hence

$$u(n_1) = \prod_{s=0}^{n_1-1} [p(s) + 1],$$

$$u(n_1 + 1) = (1 + b_1)u(n_1) = (1 + b_1) \prod_{s=0}^{n_1-1} [p(s) + 1]. \tag{2.2}$$

For $n \in [n_1 + 1, n_2]$, we have

$$u(n) = \prod_{s=n_1+1}^{n-1} [p(s) + 1]u(n_1 + 1).$$

Substituting (2.2) into the above equation, we find

$$u(n) = (1 + b_1)[p(n_1) + 1]^{-1} \prod_{s=0}^{n-1} [p(s) + 1].$$

If for $n \in [n_j + 1, n_{j+1}]$, we have

$$u(n) = \prod_{i=1}^{j} (1 + b_i)[p(n_i) + 1]^{-1} \prod_{s=0}^{n-1} [p(s) + 1],$$

then

$$u(n_{j+1}) = \prod_{i=1}^{j} (1 + b_i)[p(n_i) + 1]^{-1} \prod_{s=0}^{n-1} [p(s) + 1],$$
$$u(n_{j+1} + 1) = (1 + b_{j+1})u(n_{j+1}).$$

Hence, for $n \in [n_{j+1} + 1, n_{j+2}]$, we can get

$$\begin{aligned} u(n) &= \prod_{s=n_{j+1}+1}^{n-1} [p(s) + 1]u(n_{j+1} + 1) \\ &= (1 + b_{j+1}) \prod_{i=1}^{j} (1 + b_i)[p(n_i) + 1]^{-1} \prod_{s=0}^{n_{j+i}-1} [p(s) + 1] \prod_{s=n_{j+1}+1}^{n-1} [p(s) + 1] \\ &= \prod_{i=1}^{j+1} (1 + b_i)[p(n_i) + 1]^{-1} \prod_{s=0}^{n-1} [p(s) + 1]. \end{aligned}$$

Therefore, by induction, for all $n \in [0, N]$, we have

$$u(n) = \prod_{0<n_j<n}(1+b_j)[p(n_j)+1]^{-1}\prod_{s=0}^{n-1}[p(s)+1].$$

It is easy to see that $u(n) > 0$ for $n \in [0, N]$, $u(N) - 1 > 0$. ■

Theorem 2.2. Let $\{y(n)\}_{n=0}^{N}$ be a solution of PBVP (1.2)–(1.3). Then

$$y(n) = \sum_{\substack{s=0 \\ s\neq n_j}}^{N-1} G(n,s)h(s),\ n \in [0, N],$$

where

$$G(n,s) = \frac{1}{u(N)-1}\begin{cases} \dfrac{u(n)}{u(s+1)},\ 0 \le s < n \le N-1, \\ \dfrac{u(N)u(n)}{u(s+1)},\ 0 \le n \le s \le N-1. \end{cases}$$

Proof. First, we claim that

$$z(n) = -u(n)\left[\sum_{s=0}^{n-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_i\le n-1}\frac{1}{u(n_i+1)}h(n_i)\right]$$

is a particular solution of (1.2). Indeed, for $n \neq n_j$, we have

$$z(n+1) = -u(n+1)\left[\sum_{s=0}^{n-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_j\le n}\frac{1}{u(n_j+1)}h(n_j)\right] - h(n),$$

$$z(n) = -u(n)\left[\sum_{s=0}^{n-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_j\le n-1}\frac{1}{u(n_j+1)}h(n_j)\right],$$

hence

$$\begin{aligned}\Delta z(n) &= -\Delta u(n)\left[\sum_{s=0}^{n-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_j\le n-1}\frac{1}{u(n_j+1)}h(n_j)\right] - h(n) \\ &= -p(n)u(n)\left[\sum_{s=0}^{n-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_j\le n-1}\frac{1}{u(n_j+1)}h(n_j)\right] - h(n) \\ &= p(n)z(n) - h(n).\end{aligned}$$

For $n = n_j$, $j = 1, \dots, k$, we have

$$
\begin{aligned}
z(n_j) &= -u(n_j)\left[\sum_{s=0}^{n_j-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_i\le n_j-1}\frac{1}{u(n_i+1)}h(n_i)\right],\\
z(n_j+1) &= -u(n_j+1)\left[\sum_{s=0}^{n_j}\frac{1}{u(s+1)}h(s) - \sum_{0<n_i\le n_j}\frac{1}{u(n_i+1)}h(n_i)\right]\\
&= -u(n_j+1)\left[\sum_{s=0}^{n_j-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_i\le n_j-1}\frac{1}{u(n_i+1)}h(n_i)\right],
\end{aligned}
$$

hence

$$
\begin{aligned}
\Delta z(n_j) &= -\Delta u(n_j)\left[\sum_{s=0}^{n_j-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_i\le n_j-1}\frac{1}{u(n_i+1)}h(n_i)\right]\\
&= -b_j u(n_j)\left[\sum_{s=0}^{n_j-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_i\le n_j-1}\frac{1}{u(n_i+1)}h(n_i)\right]\\
&= b_j z(n_j).
\end{aligned}
$$

Therefore z is a particular solution of (1.2). It follows that the general solution of (1.2) has the form

$$
y(n) = cu(n) - u(n)\left[\sum_{s=0}^{n-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_i\le n-1}\frac{1}{u(n_i+1)}h(n_i)\right], \tag{2.3}
$$

where c will be determined. Taking into account $y(0) = y(N)$, we get

$$
c = y(0) = y(N) = cu(N) - u(N)\left[\sum_{s=0}^{N-1}\frac{1}{u(s+1)}h(s) - \sum_{j=1}^{k}\frac{1}{u(n_j+1)}h(n_i)\right].
$$

Hence

$$
c = \frac{u(N)}{u(N)-1}\left[\sum_{s=0}^{N-1}\frac{1}{u(s+1)}h(s) - \sum_{j=1}^{k}\frac{1}{u(n_j+1)}h(n_j)\right]. \tag{2.4}
$$

Substituting (2.4) into (2.3), we have

$$\begin{aligned} y(n) &= \frac{u(N)u(n)}{u(N)-1}\left[\sum_{s=0}^{N-1}\frac{1}{u(s+1)}h(s) - \sum_{j=1}^{k}\frac{1}{u(n_j+1)}h(n_j)\right] \\ &\quad - u(n)\left[\sum_{s=0}^{n-1}\frac{1}{u(s+1)}h(s) - \sum_{0<n_i\le n-1}\frac{1}{u(n_j+1)}h(n_j)\right] \\ &= \left[\sum_{s=0}^{N-1} G(n,s)h(s) - \sum_{0<n_j\le N-1} G(n,n_j)h(n_j)\right] \\ &= \sum_{\substack{s=0\\ s\neq n_j}}^{N-1} G(n,s)h(s). \end{aligned}$$

This concludes the proof. ■

Since $G(n,s) > 0$ for $n, s \in [0,N]$, let us set

$$m = \min_{n,s\in[0,N]} G(n,s),\ M = \max_{n,s\in[0,N]} G(n,s).$$

3. Existence of one Positive Solution of (1.1)

In this section, we consider the nonlinear problem (1.1). We assume the function $f(n,\xi)$ satisfies the following condition:

(H_2) $f : [0, N-1] \times \mathbb{R} \to \mathbb{R}$ is continuous in ξ and $f(n,\xi) \geq 0$ for $\xi \in \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

Define an N-dimensional Banach space

$$E = \{y :\ [0,N] \to R :\ y(0) = y(N)\}$$

with the norm

$$||y|| = \max_{n\in[0,N]} |y(n)|,$$

and a cone

$$K = \left\{ y \in E :\ y(n) \geq 0, n \in [0,N], \min_{n\in[0,N]} y(n) \geq \frac{m}{M}||y|| \right\}.$$

By Theorem 2.2, solving PBVP (1.1) is equivalent to solving the summation equation

$$y(n) = \sum_{\substack{s=0\\ s\neq n_j}}^{N-1} G(n,s)f(s,y(s)),\ n \in [0,N],$$

and consequently, it is equivalent to finding fixed points of the operator $A : E \to E$ defined by

$$Ay(n) = \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} G(n,s) f(s, y(s)). \tag{3.1}$$

Lemma 3.1. [2] Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1,\ \bar{\Omega}_1 \subset \Omega_2$, and let

$$A : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $||Au|| \leq ||u||,\ u \in K \cap \partial\Omega_1$ and $||Au|| \geq ||u||,\ u \in K \cap \partial\Omega_2$; or

(ii) $||Au| \geq ||u||, u \in K \cap \partial\Omega_1$ and $||Au|| \leq ||u||,\ u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Lemma 3.2. A is a completely continuous operator.

Proof. Let $y_m(n), y_0(n) \in E$ with $y_m(n) \to y_0(n)$ as $m \to \infty$. From (3.1) and since $f(n, \xi)$ is continuous in ξ, as $m \to \infty$, we have

$$|Ay_m(n) - Ay_0(n)| \leq M \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} |f(s, y_m(s) - f(s, y_0(s))| \to 0.$$

Hence $||Ay_m - Ay_0|| \to 0$. It follows that the operator A is continuous.

Further, if $Y \subset E$ is a bounded set, then $||y|| \leq C_1 = const$ for all $y \in Y$. Set $C_2 = \max_{n \in [0, N-1]} f(n, y), y \in Y$. Then from (3.1) we get, for all $y \in Y$

$$||Ay|| \leq M \sum_{s=0}^{N-1} |f(s, y(s))| \leq MNC_2.$$

This shows that $A(Y)$ is a bounded set in E. Since E is N-dimensional, $A(Y)$ is relatively compact in E.

Therefore A is a completely continuous operator. ∎

Lemma 3.3. $A(K) \subset K$.

Proof. For any $y \in K$, by (3.1), for all $n \in [0, N]$, we have

$$||Ay|| = \max_{n \in [0,N]} |Ay(n)| \leq M \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} |f(s, y(s))|,$$

$$Ay(n) \geq m \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} |f(s, y(s))| \geq \frac{m}{M} ||Ay||,$$

hence

$$\min_{n\in[0,N]} Ay(n) \geq \frac{m}{M}||Ay||.$$

This implies $Ay \in K$, that is $A(K) \subset K$. ■

In the next theorem we also assume the following condition on $f(n, \xi)$:

(H_3) There exist numbers $0 < r < R < \infty$, such that for all $n \in [0, N-1]$,

$$f(n,\xi) \leq \frac{1}{M(N-k)}\xi, \text{if } 0 \leq \xi \leq r; f(n,\xi) \geq \frac{M}{m^2(N-k)}\xi, \text{ if } R \leq \xi < \infty.$$

Theorem 3.4. Assume that conditions $(H_1) - (H_3)$ are satisfied. Then PBVP (1.1) has at least one solution $y = \{y(n)\}_{n=0}^{N}$ such that

$$\frac{m}{M}r \leq y(n) \leq \frac{M}{m}R, \ n \in [0, N]. \tag{3.2}$$

Proof. Let $\Omega_1 = \{y \in E : \|y\| < r\}$. Thus for $y \in K \cap \partial\Omega_1$, we have $0 \leq y(s) \leq r$, $s \in [0, N]$, and

$$\begin{aligned} Ay(n) &\leq M \sum_{\substack{s=0\\ s\neq n_j}}^{N-1} f(s, y(s)) \leq M \frac{1}{(N-k)M} \sum_{\substack{s=0\\ s\neq n_j}}^{N-1} y(s) \\ &\leq \frac{1}{N-k}\|y\|(N-k) = \|y\|. \end{aligned}$$

Hence

$$\|Ay\| \leq \|y\|, \text{ for } y \in K \cap \partial\Omega_1. \tag{3.3}$$

Further, let

$$R_1 = \frac{M}{m}R \text{ and } \Omega_2 = \{y \in E : \ \|y\| < R_1\}.$$

Then $y \in K$ and $\|y\| = R_1$ imply

$$\min_{n\in[0,N]} y(n) \geq \frac{m}{M}\|y\| = \frac{m}{M}R_1 = R,$$

thus $y(s) \geq R$ for all $s \in [0, N]$, and

$$\begin{aligned} Ay(n) &\geq m \sum_{\substack{s=0\\ s\neq n_j}}^{N-1} f(s, y(s)) \geq \frac{mM}{m^2(N-k)} \sum_{\substack{s=0\\ s\neq n_j}}^{N-1} y(s) \\ &\geq \frac{M}{m(N-k)} \sum_{\substack{s=0\\ s\neq n_j}}^{N-1} \frac{m}{M}\|y\| = \|y\|. \end{aligned}$$

Hence

$$\|Ay\| \geq \|y\|, \text{ for all } y \in K \cap \partial\Omega_2. \tag{3.4}$$

In view of (3.3) and (3.4), the condition (i) of Lemma 3.1 is satisfied. It follows that A has a fixed point y in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, and we have $r \leq \|y\| \leq R_1$. Because of $y \in K$, we have $y(n) \geq \frac{m}{M}\|y\|,\ n \in [0, N]$. It follows that (3.2) holds. ■

Using Theorem 3.4, we can obtain the following corollary.

Corollary 3.5. If conditions $(H_1) - (H_2)$ hold, and

(H_3') $\lim\limits_{\xi\to 0^+} \dfrac{f(n,\xi)}{\xi} = 0$ and $\lim\limits_{\xi\to\infty} \dfrac{f(n,\xi)}{\xi} = \infty$ for all $n \in [0, N-1]$,

then the PBVP (1.1) has at least one positive solution.

Proof. From $\lim\limits_{\xi\to 0^+} \dfrac{f(n,\xi)}{\xi} = 0$ and $\lim\limits_{\xi\to\infty} \dfrac{f(n,\xi)}{\xi} = \infty$, we can conclude that there exist $r > 0$ sufficiently small and $R > 0$ sufficiently large such that

$$f(n,\xi) \leq \frac{1}{M(N-k)}\xi, \text{if } 0 \leq \xi \leq r; f(n,\xi) \geq \frac{M}{m^2(N-k)}\xi, \text{ if } R \leq \xi < \infty.$$

By Theorem 3.4, we obtain that PBVP (1.1) has at least one solution. ■

We assume the following condition on $f(n,\xi)$:

(H_4) There exist numbers $0 < r < R < \infty$ such that for all $n \in [0, N-1]$,

$$f(n,\xi) \geq \frac{M}{m^2(N-k)}\xi, \text{ if } 0 \leq \xi \leq r; f(n,\xi) \leq \frac{1}{M(N-k)}, \text{ if } R \leq \xi < \infty.$$

Theorem 3.6. Assume that conditions (H_1), (H_2) and (H_4) are satisfied. Then the PBVP (1.1) has at least one solution such that

$$\frac{m}{M}r \leq y(n) \leq \frac{M}{m}R,\ n \in [0, N].$$

Proof. Let $\Omega_3 = \{y \in E :\ \|y\| < r\}$. Then for $y \in K$ with $\|y\| = r$, we have

$$\begin{aligned} Ay(n) &\geq m \sum_{\substack{s=0\\ s\neq n_j}}^{N-1} f(s, y(s)) \geq \frac{mM}{m^2(N-k)} \sum_{\substack{s=0\\ s\neq n_j}}^{N-1} y(s) \\ &\geq \frac{M}{m(N-k)}(N-k)\frac{m}{M}\|y\| = \|y\|, \end{aligned}$$

hence

$$\|Ay\| \geq \|y\|, \text{ for } y \in K \cap \partial\Omega_3. \tag{3.5}$$

On the other hand, set $\Omega_4 = \{y \in E : \ \|y\| < R_1\}$, $R_1 = \dfrac{M}{m}R$. Then $y \in K$ and $\|y\| = R_1$ imply

$$\min_{n\in[0,N]} y(n) \geq \frac{m}{M}\|y\| = \frac{m}{M}R_1 = R,$$

hence $y(s) \geq R$ for all $s \in [0, N]$, and

$$\begin{aligned} Ay(n) &\leq M \sum_{\substack{s=0 \\ s\neq n_j}}^{N-1} f(s, y(s)) \leq \frac{M}{M(N-k)} \sum_{\substack{s=0 \\ s\neq n_j}}^{N-1} y(s) \\ &\leq M \frac{1}{M(N-k)}(N-k)\|y\| = \|y\|. \end{aligned}$$

Hence

$$\|Ay\| \leq \|y\|, \text{ for all } y \in K \cap \partial\Omega_4. \tag{3.6}$$

(3.5) and (3.6) show that the condition (ii) of the Lemma 3.1 is satisfied. It follows that by Lemma 3.1, A has a fixed point y in $K \cap (\bar{\Omega}_4 \setminus \Omega_3)$. Hence we have $r \leq \|y\| \leq R_1$, further $\dfrac{m}{M}r \leq y(n) \leq \dfrac{M}{m}R$. ■

Corollary 3.7. If $(H_1) - (H_2)$ hold, and

(H_4') $\displaystyle\lim_{\xi\to 0^+} \frac{f(n,\xi)}{\xi} = \infty$ and $\displaystyle\lim_{\xi\to\infty} \frac{f(n,\xi)}{\xi} = 0$ for all $n \in [0, N-1]$,

then the PBVP (1.1) has at least one positive solution.

The proof of Corollary 3.7 is similar to that of Corollary 3.5, so we omit it here.

We state two results corresponding to Corollary 3.5 and Corollary 3.7. There are some differences but the ideas and techniques are the same. Thus, we present these results without proofs.

Theorem 3.8. Assume $(H_1), (H_2)$ and one of the following conditions holds:

(H_5) $\displaystyle\lim_{\xi\to 0^+} \frac{f(n,\xi)}{\xi} = 0,$ for $n \in [0, N-1]$ and $f(n,\xi) \geq \dfrac{M}{m^2(N-k)}\xi,$ if $R \leq \xi < \infty,\ n \in [0, N-1]$;

(H_6) $\displaystyle\lim_{\xi\to\infty} \frac{f(n,\xi)}{\xi} = \infty,$ for $n \in [0, N-1]$ and $f(n,\xi) \leq \dfrac{1}{M(N-k)}\xi,$ if $0 \leq \xi \leq r, n \in [0, N-1]$,

where r and R are real numbers. Then the PBVP (1.1) has at least one positive solution.

Theorem 3.9. Assume $(H_1), (H_2)$ and one of the following conditions holds:

(H_7) $\lim\limits_{\xi\to 0^+}\dfrac{f(n,\xi)}{\xi}=\infty,$ for $n\in[0,N-1]$ and $f(n,\xi)\le\dfrac{1}{M(N-k)}\xi,$ if $R\le\xi<\infty,\ n\in[0,N-1]$;

(H_8) $\lim\limits_{\xi\to\infty}\dfrac{f(n,\xi)}{\xi}=0,$ for $n\in[0,N-1]$ and $f(n,\xi)\ge\dfrac{M}{m^2(N-k)}\xi$ if $0\le\xi\le r, n\in[0,N-1]$.

Then the PBVP (1.1) has at least one positive solution.

Example 3.10. Consider the problem

$$\begin{aligned}-\Delta y(n)&=y^2(n),\ n\in[0,2],n\neq 1,\\ \Delta y(1)&=b_1y(1),\\ y(0)&=y(3),\end{aligned}\tag{3.7}$$

where $b_1>0$ is a constant, $p(n)\equiv 0$. Let $f(n,\xi)=\xi^2$. Then we easily check that conditions (H_1), (H_2) and (H_3') hold. Hence by Corollary 3.5, equation (3.7) has at least one positive solution.

Remark 3.11. We note that when (3.7) has no impulse effect, it is reduced to

$$-\Delta y(n)=y^2(n),\ n\in[0,2],\tag{3.8}$$

$$y(0)=y(3).\tag{3.9}$$

From (3.8), we obtain

$$y(1)=y(0)[1-y(0)],\ y(2)=y(1)[1-y(1)],\ y(3)=y(2)[1-y(2)].$$

If $0<y(0)<1$, then

$$y(3)<y(2)<y(1)<y(0).$$

This shows that the periodic boundary value condition (3.9) is not satisfied.

If $y(0)\ge 1$, then $y(n)\le 0$, $n=1,2,3$. This also shows that (3.9) is not satisfied. Hence the PBVP (3.8)–(3.9) has no positive solution.

4. Existence of Three Positive Solutions of (1.1)

Let E be a real Banach space with cone K. A map $\beta: K\to[0,+\infty)$ is said to be a nonnegative continuous concave functional on K if β is continuous and

$$\beta(tx+(1-t)y)\ge t\beta(x)+(1-t)\beta(y)$$

for all $x,y\in K$ and $t\in[0,1]$. Let a,b be two numbers such that $0<a<b$ and β a nonnegative continuous concave functional on K. We define the following convex sets:

$$K_a=\{x\in K:\ \|x\|<a\},$$

$$K(\beta, a, b) = \{x \in K : \ a \leq \beta(x), \|x\| \leq b\}.$$

Lemma 4.1. (Leggett–Williams fixed point theorem [5]). Let A: $\overline{K}_c \to \overline{K}_c$ be completely continuous and β be a nonnegative continuous concave functional on K such that $\beta(x) \leq \|x\|$ for all $x \in \overline{K}_c$. Suppose there exist $0 < d < a < b \leq c$ such that

(i) $\{x \in K(\beta, a, b) : \ \beta(x) > a\} \neq \emptyset$ and $\beta(Ax) > a$ for $x \in K(\beta, a, b)$,

(ii) $\|Ax\| < d$ for $\|x\| \leq d$,

(iii) $\beta(Ax) > a$ for $x \in K(\beta, a, c)$ with $\|Ax\| > b$.

Then A has at least three fixed points x_1, x_2, x_3 in $\overline{P}_c$ such that

$$\|x_1\| < d, \ a < \beta(x_2) \text{ and } \|x_3\| > d \text{ with } \beta(x_3) < a.$$

Now, we establish existence conditions of three positive solutions for PBVP (1.1). In this section, we assume (H_1) holds and let $f(n, \xi) = q(n) f_1(\xi)$, $q(n)$ and $f_1(y)$ is continuous, $q(n) > 0$ for $n \in [0, N-1]$, $f_1(y) \geq 0$ for $y \in \mathbb{R}^+$.

Theorem 4.2. Suppose that there exist numbers a and d with $0 < d < a$ such that

$$f_1(y) < \frac{d}{D}, \ \ y \in [0, d] \tag{4.1}$$

and

$$f_1(y) > \frac{a}{C}, \ \ y \in [a, a/\gamma] \tag{4.2}$$

where

$$D = \max_{n \in [0, N-1]} \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} G(n, s) q(s),$$

$$C = \min_{n \in [0, N-1]} \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} G(n, s) q(s), \ \gamma = \frac{m}{M}.$$

Suppose further that one of the following conditions holds:

(P_1) $\lim\limits_{y \to \infty} \dfrac{f_1(y)}{y} < 1/D$,

(P_2) there exists a number c such that $c > a/\gamma$ and if $y \in [0, c]$, then $f_1(y) < c/D$.

Then the boundary value problem (1.1) has at least three positive solutions.

Proof. For $y \in K$, let

$$\beta(y) = \min_{n \in [0, N]} y(n).$$

Then it is easy to check that β is a nonnegative continuous concave functional on K with $\beta(x) \leq \|x\|$ for $x \in K$ and by Lemma 3.2 and Lemma 3.3, $A : K \to K$ is completely continuous.

First, we prove that if (P_1) holds, then there exists a number c such that $c > a/\gamma$ and $A : \overline{K}_c \to \overline{K}_c$. To do this, by (P_1), there exist $T > 0$ and $\sigma < 1/D$ such that

$$f_1(y) \le \sigma y, \quad \text{for } y > T.$$

Set

$$e = \max_{u \in [0,T]} f_1(y).$$

It follows that $f_1(y) \le \sigma y + e$ for all $y \in [0, +\infty)$. Take

$$c > \max\left\{\frac{eD}{1 - \sigma D}, a/\gamma\right\}.$$

If $y \in \overline{P}_c$, then

$$\begin{aligned} Ay(n) &\le \max_{n \in [0,N]} \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} G(n,s) q(s) f_1(y(s)) \\ &\le \max_{n \in [0,N]} \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} G(n,s) q(s) (\sigma \|y\| + e) \\ &\le (\sigma c + e) D < c, \end{aligned}$$

that is

$$\|Au\| < c. \tag{4.3}$$

Next, we assert that if there exists a positive number r such that $f_1(y) < r/D$ for $y \in [0, r]$, then $A : \overline{K}_r \to K_r$. Indeed, if $y \in \overline{K}_r$, then

$$\|Au\| \le \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} G(n,s) q(s) f_1(y(s)) < \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} G(n,s) q(s) \cdot \frac{r}{D} = r. \tag{4.4}$$

Hence (4.3) and (4.4) show that if either (P_1) or (P_2) holds, then there exists a number $c > a/\gamma$ such that A maps $\overline{P}_c$ into P_c.

Note that if $r = d$, then we may assert further that A maps $\overline{K}_d$ into K_d by (4.1).

Now we show that $\{y \in K(\beta, a, a/\gamma) : \beta(y) > a\} \neq \emptyset$ and $\beta(Ay) > a$ for all $y \in K(\beta, a, a/\gamma)$. In fact, take $x(t) \equiv \dfrac{a + a/\gamma}{2} > a$, so $x \in \{y \in K(\beta, a, a/\gamma) : \beta(y) > a\}$. Moreover, for $y \in K(\beta, a, a/\gamma)$, we have $\beta(y) \ge a$, and then

$$a/\gamma \ge \|y\| \ge \min_{n \in [0,N]} y(n) = \beta(y) \ge a.$$

Thus, in view of (4.2), we obtain

$$\beta(Ay) = \min_{n \in [0,N]} \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} G(n,s) q(s) f_1(y) \ge \min_{n \in [0,N]} \sum_{\substack{s=0 \\ s \neq n_j}}^{N-1} G(n,s) q(s) \cdot \frac{a}{C} = a.$$

Finally, we assert that if $y \in K(\beta, a, c)$ and $\|Ay\| > a/\gamma$, then $\beta(Ay) > a$. To see this, if $y \in K(\beta, a, c)$ and $\|Ay\| > a/\gamma$, then we have

$$\beta(Ay) = \min_{n\in[0,N]} \sum_{\substack{s=0 \\ s\neq n_j}}^{N-1} G(n,s)q(s)f_1(y) \geq m \sum_{\substack{s=0 \\ s\neq n_j}}^{N-1} q(s)f_1(y). \tag{4.5}$$

On the other hand

$$\|Ay\| = \max_{n\in[0,N]} |Ay(n)| \leq M \sum_{\substack{s=0 \\ s\neq n_j}}^{N-1} q(s)f_1(y). \tag{4.6}$$

So we get from (4.5) and (4.6) that

$$\beta(Ay) \geq \gamma\|Ay\| > \gamma \cdot a/\gamma = a.$$

To sum up, all the hypotheses of Lemma 4.1 are satisfied by taking $b = a/\gamma$. Hence, A has at least three fixed points, that is, PBVP (1.1) has at least three positive solutions y_1, y_2 and y_3 such that

$$\|y_1\| < d,\ a < \beta(y_2) \text{ and } \|y_3\| > d \text{ with } \beta(y_3) < a.$$

This concludes the proof. ■

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