International Journal of Difference Equations.
ISSN 0973-6069 Volume 1 Number 2 (2006), pp. 205–218

http://www.ripublication.com/ijde.htm

# Oscillation of Second Order Half-Linear Dynamic Equations on Discrete Time Scales

**M. Bohner**
*University of Missouri–Rolla,
Department of Mathematics and Statistics,
Rolla, Missouri 65401, U.S.A.
E-mail: bohner@umr.edu*

**S.H. Saker**
*Mansoura University, Department of Mathematics,
Mansoura 35516, Egypt
E-mail: shsaker@mans.edu.eg*

**Abstract**

In this paper, by using the Riccati techniques and algebraic inequalities, we will establish some oscillation criteria for a second order half-linear dynamic equation on a discrete time scale.

**AMS subject classification:** 39A10, 39A99, 34C10, 39A11.
**Keywords:** Oscillation, half-linear dynamic equation, discrete time scale, Riccati technique.

## 1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis [17] in order to unify continuous and discrete analysis. Not only can this theory of so-called "dynamic equations" **unify** the theories of differential equations and difference equations, but also it is able to **extend** these classical cases to cases "in between", e.g., to so-called $q$-difference equations. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal

Received March 1, 2006; Accepted August 8, 2006

to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models. A book on the subject of time scales by Bohner and Peterson [6] summarizes and organizes much of the time scale calculus (see also [1, 7]).

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales. We refer the reader to the papers [2–5, 8–15, 20].

For the oscillation of dynamic equations on discrete time scales, Akın–Bohner and Hoffacker [4] considered the second order dynamic equation of Emden–Fowler type

$$x^{\Delta\Delta}(t) + q(t)x^{\gamma}(\sigma(t)) = 0$$

and established necessary and sufficient conditions for oscillation of all solutions when $\gamma > 1$ and $0 < \gamma < 1$ which cannot be applied in the case when $\gamma = 1$. Following this trend, in this paper we consider the second order half-linear dynamic equation

$$(r(x^{\Delta})^{\gamma})^{\Delta}(t) + p(t)x^{\gamma}(t) = 0 \tag{1.1}$$

on the time scale $\mathbb{T}$ which contains only isolated points and is unbounded above and $r$ and $p$ are defined on $\mathbb{T}$. A point is isolated if it is left-scattered and right-scattered. Throughout this paper we assume that

$$r(t) > 0, \quad p(t) \geq 0, \quad \text{and} \quad \gamma \geq 1 \text{ is a quotient of odd positive integers.} \tag{H}$$

We shall also consider the two cases

$$\int_{t_0}^{\infty} \frac{\Delta t}{(r(t))^{1/\gamma}} = \infty \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{(r(t))^{1/\gamma}} < \infty. \tag{1.3}$$

By a solution of (1.1) we mean a nontrivial real-valued function $x$ satisfying equation (1.1) for $t \geq t_0 \geq a$ for some $t_0 \geq a > 0$. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy

$$\sup\{|x(t)| : \ t > t_0\} > 0 \quad \text{for any} \quad t_0 \geq t_x.$$

In this paper we use the Riccati transformation technique and the Hardy, Littlewood and Pólya [16] and Jianchu [18] inequalities

$$x^{\gamma} - y^{\gamma} > \gamma y^{\gamma-1}(x-y) \quad \text{for all} \quad x > y > 0 \quad \text{and} \quad \gamma \geq 1 \tag{1.4}$$

and

$$x^{\gamma} - y^{\gamma} \geq 2^{1-\gamma}(x-y)^{\gamma} \quad \text{for all} \quad x \geq y > 0 \quad \text{and} \quad \gamma \geq 1 \tag{1.5}$$

to obtain some new oscillation criteria for (1.1) when (1.2) or (1.3) holds on time scales with isolated points.

The paper is organized as follows: In the next section we present some basic definitions concerning the calculus on time scales. In Section 3 we develop a Riccati transformation technique to give some sufficient conditions for oscillation of all solutions of (1.1), subject to the condition (1.2). When (1.3) holds, we present some conditions that ensure that all solutions are either oscillatory or convergent to zero. Our results can be applied to different discrete time scales, for example, when $\mathbb{T} = \mathbb{N}_0$, $\mathbb{T} = h\mathbb{N}_0 := \{hk : \ k \in \mathbb{N}_0\}$ for $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0} := \left\{q^k : \ k \in \mathbb{N}_0\right\}$ for $q > 1$, and others.

When $\mathbb{T} = \mathbb{N}_0$, our results improve the result established by Thandapani, Ravi and Graef [21] for the second order difference equation

$$\Delta(r(t)\,(\Delta x(t))^\gamma) + p(t)x^\gamma(t) = 0 \quad \text{for} \quad t \in \mathbb{N}_0, \tag{1.6}$$

where $\Delta x(t) = x(t+1) - x(t)$. When $\mathbb{T} = h\mathbb{N}_0$ with $h > 0$ and $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, our results are essentially new for the general second order half linear difference equation

$$\Delta_h(r(t)\,(\Delta_h x(t))^\gamma) + p(t)x^\gamma(t) = 0 \quad \text{for} \quad t \in h\mathbb{N}_0, \tag{1.7}$$

where $\Delta_h x(t) = (x(t+h) - x(t))/h$ and for the second order half-linear $q$-difference equation

$$\Delta_q(r(t)\,(\Delta_q x(t))^\gamma) + p(t)x^\gamma(t) = 0 \quad \text{for} \quad t \in q^{\mathbb{N}_0}, \tag{1.8}$$

where $\Delta_q x(t) = (x(qt) - x(t))/((q-1)t)$.

## 2. Some Preliminaries on Time Scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above. On any time scale $\mathbb{T}$ we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}\,.$$

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess function $\mu$ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of $f$ may be actually replaced by any Banach space) the (delta) derivative is defined in such a way that

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

if $f$ is continuous at $t$ and $t$ is right-scattered. If $t$ is not right-scattered, then the derivative is

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

provided this limit exists. A function $f : [a,b] \to \mathbb{R}$ is said to be rd-continuous if it is right continuous at each right-dense point and there exists a finite left limit at all left-dense points. A simple useful formula is

$$f^\sigma = f + \mu f^\Delta, \quad \text{where} \quad f^\sigma = f \circ \sigma. \tag{2.1}$$

We will make use of the product and quotient rules for the derivative of the product $fg$ and the quotient $f/g$ (where $gg^\sigma \neq 0$) of two differentiable functions $f$ and $g$

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma \quad \text{and} \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{2.2}$$

For $a, b \in \mathbb{T}$ and a differentiable function $f$, the Cauchy integral of $f^\Delta$ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a).$$

An integration by parts formula reads

$$\int_a^b f(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t,$$

and improper integrals are defined as

$$\int_a^\infty f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t.$$

Note that in the case $\mathbb{T} = \mathbb{R}$ we have

$$\sigma(t) = \rho(t) = t, \quad f^\Delta = f', \quad \int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

In this paper we concentrate our work on discrete time scales, for example $\mathbb{N}_0$, $h\mathbb{N}_0$, $q^{\mathbb{N}_0}$. Now let $a, b \in \mathbb{T}$ with $a < b$. In the case $\mathbb{T} = \mathbb{N}_0$ we have

$$\sigma(t) = t + 1, \ \mu(t) \equiv 1, \quad f^\Delta = \Delta f, \quad \int_a^b f(t)\Delta t = \sum_{\nu=a}^{b-1} f(\nu),$$

in the case $\mathbb{T} = h\mathbb{N}_0$ we have

$$\sigma(t) = t + h, \ \mu(t) \equiv h, \quad f^\Delta = \Delta_h f, \quad \int_a^b f(t)\Delta t = \sum_{\nu=a/h}^{b/h-1} hf(\nu h),$$

and in the case $\mathbb{T} = q^{\mathbb{N}_0}$ we have

$$\sigma(t) = qt, \ \mu(t) = (q-1)t, \quad f^\Delta = \Delta_q f, \quad \int_a^b f(t)\Delta t = (q-1) \sum_{\nu=\log_q a}^{\log_q b-1} q^\nu f(q^\nu).$$

# 3. Main Results

Throughout this paper, we use the notation

$$K_+ = \max\{K, 0\} \quad \text{for all} \quad K \in \mathbb{R}.$$

In this section, by using the Riccati substitution and the inequalities (1.4) and (1.5) we establish some new oscillation criteria for (1.1) when $\gamma \geq 1$ and then we deduce some oscillation criteria for equations (1.6)–(1.8). A solution $x$ of (1.1) is called oscillatory if for any $t_1 \in [a, \infty)$ there exists $t_2 \in [t_1, \infty)$ such that $x(t_2)x(\sigma(t_2)) \leq 0$. The dynamic equation (1.1) is called oscillatory if all its solutions are oscillatory. If the solution $x$ is not oscillatory, then it is said to be nonoscillatory. The solution $x$ is nonoscillatory if it is eventually positive or negative, i.e., there exists $t_1 \in [a, \infty)$ such that $x(t)x^\sigma(t) > 0$ for all $t \in [t_1, \infty)$.

## 3.1. First we Consider the Case When (1.2) Holds

**Theorem 3.1.** Assume that (H) and (1.2) hold. Furthermore, assume that

$$r^\Delta(t) \geq 0 \quad \text{for all} \quad t \in \mathbb{T} \tag{3.1}$$

and there exists a positive differentiable function $\delta$ such that

$$\limsup_{t\to\infty} \int_a^t \left[ \delta(s)p(s) - \frac{r(s)\left(\left(\delta^\Delta(s)\right)_+\right)^2}{2^{3-\gamma}\gamma s^{\gamma-1}\delta(s)} \right] \Delta s = \infty. \tag{3.2}$$

Then every solution of equation (1.1) is oscillatory on $[a, \infty)$.

*Proof.* Suppose to the contrary that $x$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x$ is an eventually positive solution of (1.1) such that $x(t) > 0$ for all $t \geq t_0 > a$. We shall consider only this case, since the substitution $\tilde{x} = -x$ transforms equation (1.1) into an equation of the same form. In view of (1.1) we have

$$\left(r\left(x^\Delta\right)^\gamma\right)^\Delta(t) = -p(t)x^\gamma(t) \leq 0 \tag{3.3}$$

for all $t \geq t_0$, and so $r(x^\Delta)^\gamma$ is an eventually nonincreasing function, hence is either eventually nonnegative or eventually negative. Suppose there exists $t_1 \geq t_0$ such that $(r(x^\Delta)^\gamma)(t_1) =: c < 0$. Then from (3.3) we have $(r(x^\Delta)^\gamma)(t) \leq (r(x^\Delta)^\gamma(t_1) = c$ for all $t \geq t_1$ and hence

$$x^\Delta(t) \leq c^{\frac{1}{\gamma}} \left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}} \quad \text{for all} \quad t \geq t_1,$$

which implies by (1.2) that

$$x(t) \leq x(t_1) + c^{\frac{1}{\gamma}} \int_{t_1}^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the fact that $x(t) > 0$ for all $t \geq t_0$. Hence $r(x^\Delta)^\gamma$ is eventually nonnegative. Therefore, we see that there is some $t_0$ such that

$$x > 0, \quad x^\Delta \geq 0, \quad (r(x^\Delta)^\gamma)^\Delta \leq 0 \quad \text{on} \quad [t_0, \infty). \tag{3.4}$$

Define the function $w$ by

$$w = \delta \frac{r(x^\Delta)^\gamma}{x^\gamma} \quad \text{on} \quad [t_0, \infty). \tag{3.5}$$

Then $w(t) \geq 0$ for all $t \geq t_0$, and using (2.2) yields that

$$w^\Delta = (r(x^\Delta)^\gamma)^\sigma \left(\frac{\delta}{x^\gamma}\right)^\Delta + \frac{\delta}{x^\gamma}(r(x^\Delta)^\gamma)^\Delta.$$

Using the calculus from Section 2, we obtain

$$w^\Delta = \frac{\delta}{x^\gamma}(r(x^\Delta)^\gamma)^\Delta + (r(x^\Delta)^\gamma)^\sigma \left(\frac{x^\gamma \delta^\Delta - \delta(x^\gamma)^\Delta}{x^\gamma (x^\sigma)^\gamma}\right). \tag{3.6}$$

In view of (1.1) and (3.6), we find

$$w^\Delta = -\delta p + \frac{\delta^\Delta}{\delta^\sigma} w^\sigma - \frac{\delta (r(x^\Delta)^\gamma)^\sigma (x^\gamma)^\Delta}{x^\gamma (x^\sigma)^\gamma}. \tag{3.7}$$

From (2.1) we have

$$\frac{(x^\sigma)^\gamma - x^\gamma}{\mu} = (x^\gamma)^\Delta.$$

By using the inequality (1.4), we obtain

$$(x^\gamma)^\Delta = \frac{(x^\sigma)^\gamma - x^\gamma}{\mu} \geq \frac{\gamma x^{\gamma-1}}{\mu}(x^\sigma - x) = \gamma x^{\gamma-1} x^\Delta. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$w^\Delta \leq -\delta p + \frac{(\delta^\Delta)_+}{\delta^\sigma} w^\sigma - \frac{\delta (r(x^\Delta)^\gamma)^\sigma \gamma x^{\gamma-1} x^\Delta}{x^\gamma (x^\sigma)^\gamma}. \tag{3.9}$$

Now we show that (3.1) and (3.4) imply that

$$x^{\Delta\Delta} \leq 0 \quad \text{on} \quad [t_0, \infty). \tag{3.10}$$

We assume there exists $t \geq t_0$ with $x^{\Delta\Delta}(t) > 0$. Then we find, by using (2.2), (3.1), (3.4), and (1.4), that

$$\begin{aligned} 0 \geq \left(r(x^\Delta)^\gamma\right)^\Delta(t) &= r^\Delta(t)\left(x^\Delta(t)\right)^\gamma + r^\sigma(t)\left[(x^\Delta)^\gamma\right]^\Delta(t) \\ \geq r^\sigma(t)\left[(x^\Delta)^\gamma\right]^\Delta(t) &= r^\sigma(t)\frac{\left((x^\Delta)^\sigma(t)\right)^\gamma - (x^\Delta(t))^\gamma}{\mu(t)} \\ > r^\sigma(t)\gamma(x^\Delta(t))^{\gamma-1}\frac{(x^\Delta)^\sigma(t) - x^\Delta(t)}{\mu(t)} &= r^\sigma(t)\gamma(x^\Delta(t))^{\gamma-1}x^{\Delta\Delta}(t) \\ \geq 0, \end{aligned}$$

a contradiction. Hence (3.10) holds. Thus $x^{\Delta}$ is positive and nonincreasing. Using this, we have for $t \geq 2t_0$

$$x(t) = x(t_0) + \int_{t_0}^{t} x^{\Delta}(s)\Delta s \geq \int_{t_0}^{t} x^{\Delta}(t)\Delta s = (t - t_0)x^{\Delta}(t) \geq \frac{t}{2}x^{\Delta}(t),$$

and this implies that on $[t_0, \infty)$ we have

$$\gamma x^{\gamma-1} \geq h(x^{\Delta})^{\gamma-1}, \quad \text{where} \quad h(t) = \gamma \left(\frac{t}{2}\right)^{\gamma-1} \text{ for } t \in \mathbb{T}. \tag{3.11}$$

By the last part of (3.4), we have

$$x^{\Delta} \geq \frac{(r^{\sigma})^{1/\gamma}}{r^{1/\gamma}}(x^{\Delta})^{\sigma}. \tag{3.12}$$

It follows from (3.11) and (3.12) that on $[t_0, \infty)$

$$\gamma x^{\gamma-1}x^{\Delta} \geq h(x^{\Delta})^{\gamma} \geq h\frac{r^{\sigma}}{r}\left((x^{\Delta})^{\sigma}\right)^{\gamma}. \tag{3.13}$$

Substituting (3.13) in (3.9) and using (3.4), we obtain

$$w^{\Delta} \leq -\delta p + \frac{(\delta^{\Delta})_+}{\delta^{\sigma}}w^{\sigma} - \frac{h\delta}{(\delta^{\sigma})^2 r}(w^{\sigma})^2. \tag{3.14}$$

Using the fact that $\alpha u - \beta u^2 \leq \alpha^2/(4\beta)$ for $\beta > 0$, we conclude from (3.14) that

$$w^{\Delta} \leq -\left[\delta p - \frac{\left((\delta^{\Delta})_+\right)^2 r}{4h\delta}\right]. \tag{3.15}$$

Integrating (3.15) from $t_0$ to $t$, we obtain

$$-w(t_0) \leq w(t) - w(t_0) \leq -\int_{t_0}^{t}\left[\delta(s)p(s) - \frac{r(s)\left((\delta^{\Delta}(s))_+\right)^2}{4h(s)\delta(s)}\right]\Delta s,$$

which yields

$$\int_{t_0}^{t}\left[\delta(s)p(s) - \frac{r(s)\left((\delta^{\Delta}(s))_+\right)^2}{2^{3-\gamma}\gamma s^{\gamma-1}\delta(s)}\right]\Delta s \leq w(t_0)$$

for all $t \geq t_0$. This is contrary to (3.2). The proof is complete. ■

From Theorem 3.1, we can obtain different conditions for oscillation of all solutions of (1.1) by different choices of $\delta$. For instance, let $\delta(t) = t$ for $t \in \mathbb{T}$. By Theorem 3.1, we have the following result.

**Corollary 3.2.** Assume that (H), (1.2), and (3.1) hold. Furthermore, assume that

$$\limsup_{t\to\infty}\int_a^t\left[sp(s)-\frac{r(s)}{2^{3-\gamma}\gamma s^\gamma}\right]\Delta s=\infty.$$

Then every solution of (1.1) is oscillatory on $[a,\infty)$.

The following theorem gives the Kamenev-type oscillation criteria for (1.1).

**Theorem 3.3.** Assume that (H), (1.2), and (3.1) hold. If there exists a positive differentiable function $\delta$ such that

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_a^t(t-s)^m\left[\delta(s)p(s)-\frac{r(s)\left((\delta^\Delta(s))_+\right)^2}{2^{3-\gamma}\gamma s^{\gamma-1}\delta(s)}\right]\Delta s=\infty \tag{3.16}$$

for some odd $m\in\mathbb{N}$, then every solution of (1.1) is oscillatory on $[a,\infty)$.

*Proof.* The proof is similar to the proof of [20, Theorem 3.2] by using the inequality (3.15) and hence is omitted. ■

From Theorem 3.1, we can give sufficient conditions for oscillation of (1.1) on any discrete time scale. For example, in light of the formulas given at the end of Section 2, we can deduce the following oscillation criteria for equations (1.6)–(1.8). The corresponding Kamenev-type oscillation criteria can be deduced from Theorem 3.3.

**Corollary 3.4.** Assume $r(\nu+1)\ge r(\nu)>0$ and $p(\nu)\ge 0$ for all $\nu\in\mathbb{N}_0$ such that

$$\sum_{\nu=0}^{\infty}\frac{1}{(r(\nu))^{1/\gamma}}=\infty. \tag{3.17}$$

Furthermore, assume that there exist positive numbers $\delta(\nu)$ for all $\nu\in\mathbb{N}_0$ with

$$\limsup_{t\to\infty}\sum_{\nu=0}^{t}\left[\delta(\nu)p(\nu)-\frac{r(\nu)((\delta(\nu+1)-\delta(\nu))_+)^2}{2^{3-\gamma}\gamma\nu^{\gamma-1}\delta(\nu)}\right]=\infty.$$

Then every solution of equation (1.6) is oscillatory.

**Corollary 3.5.** Let $h>0$. Assume $r(\nu h+h)\ge r(\nu h)>0$ and $p(\nu h)\ge 0$ for all $\nu\in\mathbb{N}_0$ such that

$$\sum_{\nu=0}^{\infty}\frac{1}{(r(\nu h))^{1/\gamma}}=\infty. \tag{3.18}$$

Furthermore, assume that there exist positive numbers $\delta(\nu h)$ for all $\nu\in\mathbb{N}_0$ with

$$\limsup_{t\to\infty}\sum_{\nu=0}^{t}\left[\delta(\nu h)p(\nu h)-\frac{r(\nu h)((\delta(\nu h+h)-\delta(\nu h))_+)^2}{2^{3-\gamma}\gamma\nu^{\gamma-1}h^{\gamma+1}\delta(\nu h)}\right]=\infty.$$

Then every solution of equation (1.7) is oscillatory.

**Corollary 3.6.** Let $q > 1$. Assume $r(q^{\nu+1}) \geq r(q^{\nu}) > 0$ and $p(q^{\nu}) \geq 0$ for all $\nu \in \mathbb{N}_0$ such that

$$\sum_{\nu=0}^{\infty} \frac{q^{\nu}}{(r(q^{\nu}))^{1/\gamma}} = \infty. \tag{3.19}$$

Furthermore, assume that there exist positive numbers $\delta(q^{\nu})$ for all $\nu \in \mathbb{N}_0$ with

$$\limsup_{t\to\infty} \sum_{\nu=0}^{t} \left[ q^{\nu}\delta(q^{\nu})p(q^{\nu}) - \frac{r(q^{\nu})((\delta(q^{\nu+1}) - \delta(q^{\nu}))_+)^2}{2^{3-\gamma}\gamma q^{\nu\gamma}(q-1)^2\delta(q^{\nu})} \right] = \infty.$$

Then every solution of equation (1.8) is oscillatory.

**Remark 3.7.** Thandapani, Ravi and Graef [21] considered the second order half-linear difference equation

$$\Delta((\Delta x(n))^{\gamma}) + p(n)x^{\gamma}(n) = 0 \quad \text{for} \quad n \in \mathbb{N}_0$$

and proved that every solution is oscillatory if

$$\sum_{n=0}^{\infty} p(n) = \infty.$$

But one can easily see that this result cannot be applied to the discrete half-linear Euler difference equation when $p(n) = \alpha/n^2$ and also cannot be applied to the equations (1.7) and (1.8). So our results extend and improve the results in [21].

It is clear that the condition (3.1) plays an important rôle in the proof of Theorem 3.1. In the following we establish some new oscillation criteria without the condition (3.1) which then can be applied in the general case when $r$ is any positive real-valued rd-continuous function. This improves Theorem 3.1.

**Theorem 3.8.** Assume that (H) and (1.2) hold. If there exists a positive differentiable function $\delta$ such that

$$\limsup_{t\to\infty} \int_a^t \left[ \delta(s)p(s) - \frac{r(s)\left((\delta^{\Delta}(s))_+\right)^2}{2^{3-\gamma}(\mu(s))^{\gamma-1}\delta(s)} \right] \Delta s = \infty, \tag{3.20}$$

then every solution of (1.1) is oscillatory on $[a, \infty)$.

*Proof.* Proceeding as in the proof of Theorem 3.1, we assume to the contrary that $x$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x$ is an eventually positive solution of (1.1) such that $x(t) > 0$ for all $t \geq t_0 > a$. Then, as in the proof of Theorem 3.1, we see that (3.4) holds. Define again the function $w$ by (3.5). Then $w(t) > 0$ and (3.7) holds. By using the inequality (1.5) we have

$$\begin{aligned} (x^{\gamma})^{\Delta} &= \frac{(x^{\gamma})^{\sigma} - x^{\gamma}}{\mu} \geq \frac{2^{1-\gamma}}{\mu}(x^{\sigma} - x)^{\gamma} \\ &= \frac{2^{1-\gamma}}{\mu}(\mu x^{\Delta})^{\gamma} = \frac{\mu^{\gamma-1}}{2^{\gamma-1}}(x^{\Delta})^{\gamma}. \end{aligned}$$

From this and (3.7) it follows by using (3.4) that

$$w^{\Delta} \leq -\delta p + \frac{(\delta^{\Delta})_+}{\delta^{\sigma}} w^{\sigma} - \frac{2^{1-\gamma}\mu^{\gamma-1}\delta(r(x^{\Delta})^{\gamma})^{\sigma}(x^{\Delta})^{\gamma}}{(x^{2\gamma})^{\sigma}}.$$

Now, by using (3.12), we have

$$w^{\Delta} \leq -\delta p + \frac{(\delta^{\Delta})_+}{\delta^{\sigma}} w^{\sigma} - \frac{2^{1-\gamma}\mu^{\gamma-1}\delta}{(\delta^{\sigma})^2 r}(w^{\sigma})^2.$$

The remainder of the proof is similar to that of the proof of Theorem 3.1 and hence is omitted. ■

By choosing $\delta(t) = t$ and $\delta(t) \equiv 1$ for $t \in \mathbb{T}$, Theorem 3.8 yields the following two corollaries.

**Corollary 3.9.** Assume that (H) and (1.2) hold. If

$$\limsup_{t\to\infty} \int_a^t \left[ sp(s) - \frac{r(s)}{2^{3-\gamma}(\mu(s))^{\gamma-1}s} \right] \Delta s = \infty,$$

then every solution of (1.1) is oscillatory on $[a, \infty)$.

**Corollary 3.10. (Leighton–Wintner)** Assume that (H) holds. If

$$\int_{t_0}^{\infty} \frac{\Delta t}{(r(t))^{1/\gamma}} = \infty \quad \text{and} \quad \int_a^{\infty} p(s)\Delta s = \infty,$$

then every solution of (1.1) is oscillatory on $[a, \infty)$.

The following theorem gives a Kamenev-type oscillation criteria for (1.1) which does not require the condition $r^{\Delta} \geq 0$ and hence improves Theorem 3.3.

**Theorem 3.11.** Assume that (H) and (1.2) hold. If there exists a positive differentiable function $\delta$ such that

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_a^t (t-s)^m \left[ \delta(s)p(s) - \frac{r(s)(\delta^{\Delta}(s))_+^2}{2^{3-\gamma}(\mu(s))^{\gamma-1}\delta(s)} \right] \Delta s = \infty \tag{3.21}$$

for some odd $m \in \mathbb{N}$, then every solution of (1.1) is oscillatory on $[a, \infty)$.

From Theorem 3.8 we can establish new sufficient conditions for oscillation of (1.6)–(1.8) without the condition on $r$. These improve Corollaries 3.4–3.6. Also, Kamenev-type oscillation criteria can be established from Theorem 3.11.

**Corollary 3.12.** Assume $r(\nu) > 0$ and $p(\nu) \geq 0$ for all $\nu \in \mathbb{N}_0$ such that (3.17) holds. Furthermore, assume that there exist positive numbers $\delta(\nu)$ for all $\nu \in \mathbb{N}_0$ with

$$\limsup_{t\to\infty} \sum_{\nu=0}^{t} \left[ \delta(\nu)p(\nu) - \frac{r(\nu)((\delta(\nu+1) - \delta(\nu))_+)^2}{2^{3-\gamma}\delta(\nu)} \right] = \infty.$$

Then every solution of equation (1.6) is oscillatory.

**Corollary 3.13.** Let $h > 0$. Assume $r(\nu h) > 0$ and $p(\nu h) \geq 0$ for all $\nu \in \mathbb{N}_0$ such that (3.18) holds. Furthermore, assume that there exist positive numbers $\delta(\nu h)$ for all $\nu \in \mathbb{N}_0$ with

$$\limsup_{t\to\infty} \sum_{\nu=0}^{t} \left[ \delta(\nu h)p(\nu h) - \frac{r(\nu h)((\delta(\nu h + h) - \delta(\nu h))_+)^2}{2^{3-\gamma}h^{\gamma+1}\delta(\nu h)} \right] = \infty.$$

Then every solution of equation (1.7) is oscillatory.

**Corollary 3.14.** Let $q > 1$. Assume $r(q^\nu) > 0$ and $p(q^\nu) \geq 0$ for all $\nu \in \mathbb{N}_0$ such that (3.19) holds. Furthermore, assume that there exist positive numbers $\delta(q^\nu)$ for all $\nu \in \mathbb{N}_0$ with

$$\limsup_{t\to\infty} \sum_{\nu=0}^{t} \left[ q^\nu \delta(q^\nu)p(q^\nu) - \frac{r(q^\nu)((\delta(q^{\nu+1}) - \delta(q^\nu))_+)^2}{2^{3-\gamma}q^{\nu\gamma}(q-1)^{\gamma+1}\delta(q^\nu)} \right] = \infty.$$

Then every solution of equation (1.8) is oscillatory.

### 3.2. Next we Consider the Case When (1.3) Holds

Now we give some sufficient conditions when (1.3) holds, which guarantee that every solution of (1.1) oscillates or converges to zero.

**Theorem 3.15.** Assume (H), (1.3), and (3.1). Let $\delta$ be such that (3.2) holds. If

$$\int_a^\infty \left[ \frac{1}{r(t)} \int_a^t p(s)\Delta s \right]^{\frac{1}{\gamma}} \Delta t = \infty, \tag{3.22}$$

then every solution $x$ of (1.1) is oscillatory or satisfies $x(t) \to 0$ as $t \to \infty$.

*Proof.* We proceed as in Theorem 3.1 and assume that (1.1) has a nonoscillatory solution such that $x(t) > 0$ for $t \geq t_0 > a$. From the proof of Theorem 3.1 we see that there exist two possible cases of the sign of $x^\Delta$. The proof when $x^\Delta$ is eventually positive is similar to that of the proof of Theorem 3.1 and hence is omitted. Next, suppose that

$x^\Delta(t) < 0$ for $t \geq t_1$. Then $x$ is decreasing and $\lim_{t\to\infty} x(t) =: b \geq 0$. We assert that $b = 0$. If not, then $x(t) > b > 0$ for $t \geq t_2 > t_1$. Define the function

$$u = r(x^\Delta)^\gamma.$$

Then, from (1.1) for $t \geq t_2$, we obtain

$$u^\Delta = -px^\gamma \leq -b^\gamma p.$$

Hence, for $t \geq t_2$, we have

$$u(t) \leq u(t_2) - b^\gamma \int_{t_2}^t p(s)\Delta s \leq -b^\gamma \int_{t_2}^t p(s)\Delta s$$

since $u(t_2) = r(t_2)(x^\Delta(t_2))^\gamma < 0$. We may integrate the last inequality from $t_2$ to $t$ to obtain

$$\int_{t_2}^t x^\Delta(s)\Delta s \leq -b \int_{t_2}^t \left[\frac{1}{r(s)} \int_{t_2}^s p(\tau)\Delta\tau\right]^{\frac{1}{\gamma}} \Delta s.$$

By condition (3.22) we get $x(t) \to -\infty$ as $t \to \infty$, and this is a contradiction to the fact that $x(t) > 0$ for $t \geq t_0$. Thus $b = 0$ and then $x(t) \to 0$ as $t \to \infty$. The proof is complete. ■

The following theorems are immediate and the proofs are omitted.

**Theorem 3.16.** Assume (H), (1.3), and (3.1). Let $\delta$ be such that (3.16) and (3.22) hold. Then every solution $x$ of (1.1) is oscillatory or satisfies $x(t) \to 0$ as $t \to \infty$.

**Theorem 3.17.** Assume (H) and (1.3). Let $\delta$ be such that (3.20) and (3.22) hold. Then every solution $x$ of (1.1) is oscillatory or satisfies $x(t) \to 0$ as $t \to \infty$.

**Theorem 3.18.** Assume (H) and (1.3). Let $\delta$ be such that (3.21) and (3.22) hold. Then every solution $x$ of (1.1) is oscillatory or satisfies $x(t) \to 0$ as $t \to \infty$.

## 4. Conclusion

In this paper, by using the Riccati technique and Hardy–Littlewood–Pólya and Jianchu inequalities, we have established some new oscillation criteria for second order half-linear dynamic equations on discrete time scales. Our results are not only for difference equations but also can be applied to any discrete time scale. The results are proved in the case $\gamma \geq 1$ and cannot be applied in the case when $\gamma < 1$. So it would be interesting to extend the above results to include this case and this will be of our interest in future work.

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