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# On the Number of Positive Solutions for a Nonlinear Third Order Boundary Value Problem*

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**Abstract**

In this paper, we consider the nonlinear third order ordinary differential equation $u'''(t) = \lambda a(t)f(u(t)), 0 < t < 1$ with the boundary conditions $\alpha u'(0) - \beta u''(0) = 0, u(1) = u'(1) = 0$. Some sufficient conditions for the nonexistence and existence of at least one, two and $n$ positive solutions for the boundary value problem are established. In doing so the usual restriction that $f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}$ and $f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}$ exist is removed. An example is also given to illustrate the main results.

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**Keywords:** Boundary value problem, third order, positive solution, fixed point theorem.

## 1. Introduction

Recently, the study of the existence of positive solution to third-order boundary value problems has gained much attention and is a rapidly growing field. This class of boundary value problems arises from the investigation for flow of sticky liquid and possesses wide application in liquid mechanics [3, 7].

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This paper deals with the existence and nonexistence of the third order nonlinear ordinary differential equation

$$u'''(t) = \lambda a(t) f(u(t)), 0 < t < 1 \tag{1.1}$$

together with the boundary conditions

$$\alpha u'(0) - \beta u''(0) = 0,\ u(1) = u'(1) = 0, \tag{1.2}$$

where it will be assumed throughout that

(H1) $\lambda > 0$ is positive constant,

(H2) $f : [0, \infty] \to [0, \infty)$ is continuous,

(H3) $a : [0, 1] \to [0, \infty)$ is continuous, and $\int_0^1 a(t)dt > 0$,

(H4) $\alpha, \beta \geq 0, \alpha + \beta > 0$.

We are interested in determining the values of $\lambda$ for which at least one positive solution of problem (1.1), (1.2) exists. The nonexistence of positive solution is also discussed. We also establish some existence criteria of multiple positive solutions when $\lambda = 1$. In doing so the usual restriction that $f_0 = \lim\limits_{u \to 0^+} \dfrac{f(u)}{u}$ and $f_\infty = \lim\limits_{u \to \infty} \dfrac{f(u)}{u}$ exist is removed. An example is also given to illustrate the main results.

We would like to mention some results of Anderson and Davis [2], Anderson [1], Jiang [6], Yao [9], which motivate us to consider the problem (1.1) and (1.2).

In [2], Anderson and Davis considered the third order differential equation (1.1) subjected to the three-point right focal boundary conditions, and established the existence of at least one positive solution for $\lambda$ in a suitable interval. The corresponding discrete case is discussed in [1].

In [6], under the assumption that the nonlinear term is superlinear or sublinear, Jiang obtained the existence of at least one positive solution for the equation (1.1) with one of following six sets of boundary conditions:

$$\begin{gathered} u(0) = u'(0) = u(1) = 0;\ \ u(0) = u'(0) = u'(1) = 0; \\ u(0) = u'(0) = u''(1) = 0;\ \ u(0) = u''(0) = u(1) = 0; \\ u(0) = u''(0) = u'(1) = 0;\ \ u'(0) = u''(0) = u(1) = 0. \end{gathered}$$

In [9], Yao furthermore discussed the existence of single and multiple solutions by considering the properties of $f$ on bounded sets. It is noted that in the papers [2, 6], the authors both assumed that $f_0 = \lim\limits_{u \to 0^+} \dfrac{f(u)}{u}$ and $f_\infty = \lim\limits_{x \to \infty} \dfrac{f(u)}{u}$ exist. Other related results are referred to [4, 8, 10].

The rest of this paper is organized as follows. In Section 2, we shall provide some properties of certain Green's functions which are needed later. For the sake of convenience, we also state Krasnosel'skii's fixed point theorem in a cone. In Section 3 we establish some results for the existence and nonexistence of positive solutions to problem (1.1) and (1.2). In the end of this section, an example is also given to illustrate the main results. In Section 4, we discuss the existence of multiple positive solutions.

## 2. Preliminaries

In this section we will give some lemmas which are useful in proving our main results.

To obtain a solution of boundary value problem (1.1) and (1.2), we let $G(t,s)$ be Green's function of the boundary value problem

$$u'''(t) = 0,\ t \in (0,1), \tag{2.1}$$

$$\alpha u'(0) - \beta u''(0) = 0,\ u(1) = u'(1) = 0. \tag{2.2}$$

Further, it is known [11] that $G(t,s)$ is equal to

$$\begin{cases} \frac{1}{2}\left[(1-t)(1-s)\left(\frac{\alpha t+\beta}{\alpha+\beta}+\beta\right)-(1-s)^2\right], & 0 \leq\ t \leq s \leq 1, \\ \frac{1}{2}\left[(1-t)(1-s)\left(\frac{\alpha t+\beta}{\alpha+\beta}+1\right)-(1-s)^2+(t-s)^2\right], & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.3}$$

For Green's function $G(t,s)$, we have the following results.

**Lemma 2.1.** $0 \leq G(t,s) \leq G(0,s), 0 \leq t \leq s \leq 1.$

*Proof.* By (2.3), it is easy to see that if $t \leq s$, then

$$G_t(t,s) = -(1-s)\frac{\alpha t+\beta}{\alpha+\beta} \leq 0,$$

and if $t \geq s$, then

$$G_t(t,s) = -(1-t)\frac{\alpha s+\beta}{\alpha+\beta} \leq 0.$$

The proof is complete. ■

**Lemma 2.2.** $G(t,s) \geq q(t)G(0,s)$ for $0 \leq t, s \leq 1$, where $q(t) = \dfrac{(1-t)^2\beta}{\alpha+2\beta}$.

*Proof.* If $t \leq s$, then

$$\frac{G(t,s)}{G(0,s)} = \frac{(1-t)(\frac{\alpha t+\beta}{\alpha+\beta}+1)-(1-s)}{(\frac{\beta}{\alpha+\beta}+1)-(1-s)} \geq \frac{(1-t)(\alpha t+\beta)}{\beta+s(\alpha+\beta)} \geq \frac{(1-t)^2\beta}{\alpha+2\beta},$$

and if $t \geq s$, then

$$\frac{G(t,s)}{G(0,s)} = \frac{(1-t)^2(\alpha s+\beta)}{(1-s)(\beta+(\alpha+\beta)t)} \geq \frac{\beta(1-t)^2}{\beta+t(\alpha+\beta)} \geq \frac{(1-t)^2\beta}{\alpha+2\beta}.$$

The proof is complete. ■

Let $X = C[0,1]$ with the norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|,\ u \in X.$$

Clearly, $X$ is a Banach space. If we let

$$P = \{u \in X \mid u(0) > 0, u(t) \text{ is nonincreasing}, u(t) \geq q(t)\|u\|, t \in [0,1]\},$$

then it is easy to see that $P$ is a positive cone in $X$, and that if $u \in P$, then $\|u\| = u(0)$.

Define the operator $T : P \to X$ by

$$Tu(t) = \lambda \int_0^1 G(t,s)a(s)f(u(s))ds,\ 0 \leq t \leq 1, u \in P.$$

**Lemma 2.3.** $T(P) \subset P$.

*Proof.* Since $G(t,s) \geq 0$, and

$$\begin{aligned}(Tu)(t) &= \lambda \int_0^1 G(t,s)a(s)f(u(s))ds \geq \lambda q(t) \int_0^1 G(0,s)a(s)f(u(s))ds \\ &\geq q(t)\lambda \max_{0\leq t\leq 1} \int_0^1 G(t,s)a(s)f(u(s))ds = q(t)\|Tu\|.\end{aligned}$$

Thus $T(P) \subset P$. The proof is complete. ■

By standard arguments, it is easy to see that $T : P \to P$ is a completely continuous operator.

It is clear that solving the boundary value problem (1.1) and (1.2) is equivalent to finding a solution of

$$Tu = u, \quad u \in P,$$

that is, to finding a fixed point of $T$ in $P$.

In order to prove our main results, the following well-known fixed point theorem is needed.

**Lemma 2.4. (Krasnosel'skii's Fixed Point Theorem [5])** Let $P$ be a cone in a Banach space $X$. Assume $\Omega_1, \Omega_2$ are open subsets of $X$ with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. Assume

$$L : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$$

is a completely continuous operator such that either ($K_1$)

$$\|Lu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_1 \text{ and } \|Lu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_2,$$

or ($K_2$)

$$\|Lu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_1 \text{ and } \|Lu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_2.$$

Then $L$ has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

## 3. Existence and Nonexistence Results

First, we define some important constants:

$$A = \int_0^1 G(0,s)a(s)q(s)ds, \qquad B = \int_0^1 G(0,s)a(s)ds,$$

$$F_0 = \lim_{u \to 0^+} \sup \frac{f(u)}{u}, \qquad f_0 = \lim_{u \to 0+} \inf \frac{f(u)}{u}.$$

$$F_\infty = \lim_{u \to +\infty} \sup \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to +\infty} \inf \frac{f(u)}{u}.$$

**Theorem 3.1.** Suppose that $Af_\infty > BF_0$. Then for each

$$\lambda \in \left((Af_\infty)^{-1}, (BF_0)^{-1}\right), \tag{3.1}$$

the problem (1.1) and (1.2) has at least one positive solution. Here we assume that $(Af_\infty)^{-1} = 0$ if $f_\infty = \infty$ and $(BF_0)^{-1} = \infty$ if $F_0 = 0$.

*Proof.* Choose $\varepsilon > 0$ sufficiently small such that

$$(F_0 + \varepsilon)\lambda B \leq 1. \tag{3.2}$$

By the definition of $F_0$, we see that there exists an $l_1 > 0$, such that

$$f(u) \leq (F_0 + \varepsilon)u \text{ for } 0 < u \leq l_1.$$

So if $u \in P$ with $\|u\| = l_1$, we have

$$\begin{aligned}
\|Tu\| = (Tu)(0) &= \lambda \int_0^1 G(0,s)a(s)f(u(s))ds \\
&\leq \lambda \int_0^1 G(0,s)a(s)(F_0 + \varepsilon)u(s)ds \\
&\leq \lambda(F_0 + \varepsilon)\|u\| \int_0^1 G(0,s)a(s)ds \\
&= \lambda B(F_0 + \varepsilon)\|u\| \leq \|u\|.
\end{aligned}$$

Thus, if we let
$$\Omega_1 = \{u \in X \mid \|u\| < l_1\},$$
then
$$\|Tu\| \le \|u\| \text{ for } u \in P \cap \partial\Omega_1.$$
Now choose $c \in (0, \gamma)$ and $\varepsilon > 0,$ such that

$$\lambda \ge \left\{ (f_\infty - \varepsilon) \int_0^c G(0,s)a(s)q(s)ds \right\}^{-1},$$

where $\gamma \in \left(\frac{3}{4}, 1\right)$. There exists $l_3 > 0$, such that

$$f(u) \ge (f_\infty - \varepsilon)u \text{ for } u \ge l_3.$$

Let $c' = \dfrac{\beta(1-c)^2}{\alpha + 2\beta}$ and $l_2 = \max\left\{\dfrac{l_3}{c'}, 2l_1\right\}$. If $u \in P$ with $\|u\| = l_2$, in view of the definition of $q(t)$, we obtain that

$$u(t) \ge q(t)\|u\| \ge q(c)\|u\| = \frac{\beta(1-c)^2}{\alpha+2\beta}\|u\| = c'l_2 \ge l_3 \text{ for } t \in [0, c],$$

and

$$\begin{aligned}
\|Tu\| = Tu(0) &= \lambda \int_0^1 G(0,s)a(s)f(u(s))ds \\
&\ge \lambda \int_0^c G(0,s)a(s)(f_\infty - \varepsilon)u(s)ds \\
&\ge \lambda(f_\infty - \varepsilon)\|u\| \int_0^c G(0,s)a(s)q(s)ds \ge \|u\|.
\end{aligned}$$

Let
$$\Omega_2 = \{u \in X \mid \|u\| < l_2\}.$$
Then $\Omega_1 \subset \overline{\Omega_2}$ and
$$\|Tu\| \ge \|u\| \text{ for } u \in P \cap \partial\Omega_2.$$
Condition ($K_1$) of Lemma 2.4 is satisfied. So there exists a fixed point of $T$ in $P$. This completes the proof. ■

**Theorem 3.2.** Assume that $Af_0 > BF_\infty$. Then for each

$$\lambda \in \left((Af_0)^{-1}, (BF_\infty)^{-1}\right), \tag{3.3}$$

the boundary value problem (1.1), (1.2) has at least one positive solution. Here we suppose $(Af_0)^{-1} = 0$ if $f_0 = \infty$ and $(BF_\infty)^{-1} = \infty$ if $F_\infty = 0$.

*Proof.* Choose $\varepsilon > 0$ sufficiently small such that

$$(f_0 - \varepsilon)\lambda A \geq 1.$$

From the definition of $f_0$, we see that there exists $l_1 > 0$, such that

$$f(u) \geq (f_0 - \varepsilon)u \text{ for } 0 < u \leq l_1.$$

For $u \in P$ with $\|u\| = l_1$, we have

$$\begin{aligned}
\|Tu\| = (Tu)(0) = \lambda \int_0^1 G(0,s)a(s)f(u(s))ds \\
\geq \lambda \int_0^1 G(0,s)a(s)(f_0 - \varepsilon)u(s)ds \\
\geq \lambda(f_0 - \varepsilon)\|u\| \int_0^1 G(0,s)a(s)q(s)ds \\
= \lambda(f_0 - \varepsilon)\|u\|A \geq \|u\|.
\end{aligned}$$

So we let

$$\Omega_1 = \{u \in X \mid \|u\| < l_1\}.$$

Then

$$\|Tu\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_1.$$

Now choose $\varepsilon \in (0,1)$ such that

$$\lambda(F_\infty + \varepsilon)B < 1.$$

There exists $l_3 > 0$, such that

$$f(u) \leq (F_\infty + \varepsilon)u \text{ for } x \geq l_3.$$

Let

$$M = \max_{0 \leq u \leq l_3} f(u).$$

Then

$$f(u) \leq M + (F_\infty + \varepsilon)u \text{ for } u \geq 0.$$

Let

$$l_2 = \max\{2l_1, \lambda MB(1 - \lambda(F_\infty + \varepsilon)B)^{-1}\}. \tag{3.4}$$

Note that (3.4) implies that

$$\lambda MB + \lambda(F_\infty + \varepsilon)Bl_2 \leq l_2.$$

If $u \in P$ with $\|u\| = l_2$, we have

$$\begin{aligned}(Tu)(0) &= \lambda \int_0^1 G(0,s)a(s)f(u(s))ds \\ &\le \lambda \int_0^1 G(0,s)a(s)(M + (F_\infty + \varepsilon)u(s))ds \\ &\le \lambda MB + \lambda(F_\infty + \varepsilon) \int_0^1 G(0,s)a(s)u(s)ds \\ &\le \lambda MB + \lambda(F_\infty + \varepsilon)Bl_2 \le l_2,\end{aligned}$$

which implies $\|Tu\| \le \|u\|$. Thus, if we let $\Omega_2 = \{u \in X \mid \|u\| < l_2\}$, then $\Omega_1 \subset \overline{\Omega_2}$, and

$$\|Tu\| \le \|u\| \text{ for } u \in P \cap \partial\Omega_2.$$

Condition ($K_2$) of Lemma 2.4 is satisfied, so there exists a fixed point of $T$ in $P$ and the proof is complete. ■

The following two results give sufficient conditions for the boundary value problem (1.1), (1.2) to have no positive solution.

**Theorem 3.3.** Suppose that the following condition holds:

$$\lambda Bf(u) < u \text{ for } u \in (0, \infty).$$

Then the problem (1.1), (1.2) has no positive solution.

*Proof.* Assume to the contrary that $u$ is a positive solution of (1.1), (1.2). Then

$$\begin{aligned}u(0) &= \lambda \int_0^1 G(0,s)a(s)f(u(s))ds < B^{-1} \int_0^1 G(0,s)a(s)u(s)ds \\ &\le B^{-1}u(0) \int_0^1 G(0,s)a(s)ds \le u(0),\end{aligned}$$

which is a contradiction and completes the proof. ■

**Theorem 3.4.** Suppose that $\lambda Af(u) > u$ for $u \in (0, \infty)$. Then the problem (1.1) and (1.2) has no positive solution.

The proof of Theorem 3.4 is similar to that of Theorem 3.3 and therefore omitted.

**Example 3.5.** Consider the boundary value problem

$$u'''(t) = \lambda\,(10 + t)\,\frac{1 + 8u(t)}{1 + u(t)}u(t)(2 + \sin u(t)),\ 0 < t < 1, \tag{3.5}$$

$$u'(0) - 20u''(0) = 0,\ u(1) = u'(1) = 0. \tag{3.6}$$

Then, $F_0 = f_0 = 2$, $F_\infty = 24$, $f_\infty = 8$, and

$$2u < f(u) < 24u \quad \text{for } u > 0.$$

It is easy to see that $G(0,s) = \dfrac{1}{42}(1-s)(20+21s)$. By direct calculation, we obtain that

$$A = 1.21429 \quad \text{and} \quad B = 3.33532.$$

From Theorem 3.2 we see that if

$$0.102941 \approx \frac{1}{8A} < \lambda < \frac{1}{2B} \approx 0.149911,$$

then the problem (3.5), (3.6) has at least one positive solution. From Theorem 3.3 we have that if

$$\lambda < \frac{1}{24B} \approx 0.012493,$$

then the problem (3.5), (3.6) has no positive solution. By Theorem 3.4, if

$$\lambda > \frac{1}{2A} \approx 0.411765,$$

then the problem (3.5), (3.6) has no positive solution.

## 4. Existence of Multiple Positive Solutions

In this section we will consider the existence of multiple positive solutions of the equation

$$u'''(t) = a(t)f(u(t)),\ 0 < t < 1, \tag{4.1}$$

under the boundary conditions

$$\alpha u'(0) - \beta u''(0) = 0,\ u(1) = u'(1) = 0. \tag{4.2}$$

As in the previous section, we define the constant $B$ in the same way as in Section 3, and for $r \in (0,1)$, define

$$L(r) = \int_0^r G(0,s)a(s)ds.$$

The following two lemmas will be used to prove our main results.

**Lemma 4.1.** Assume that there exist $c > 0$ such that $f(u) \le \dfrac{c}{B}$ for $u \in [0,c]$. Then

$$\|Tu\| \le c \quad \text{for } u \in P \text{ with } \|u\| = c.$$

*Proof.* If $u \in P$ with $\|u\| = c$, then

$$\|Tu\| = (Tu)(0) = \int_0^1 G(0,s)a(s)f(u(s))ds \le \frac{c}{B}\int_0^1 G(0,s)a(s)ds = c,$$

so the proof is complete. ■

**Lemma 4.2.** Assume that there is $c > 0$ such that $f(u) \geq \dfrac{c}{L(r)}$ for $u \in [cq(r), c]$. Then

$$\|Tu\| \geq c \quad \text{for } u \in P \text{ with } \|u\| = c.$$

*Proof.* If $u \in P$ with $\|u\| = c$, then for $t \in [0, r]$ we have

$$u(t) \geq q(t)\|u\| \geq q(r)\|u\| = cq(r).$$

Therefore, we have

$$\begin{aligned}\|Tu\| = (Tu)(0) = \int_0^1 G(0,s)a(s)f(u(s))ds \geq \int_0^r G(0,s)a(s)f(u(s))ds \\ \geq \frac{c}{L(r)} \int_0^r G(0,s)a(s)ds = c,\end{aligned}$$

so the proof of the lemma is complete. ■

Now we are ready to present our main results in this section.

**Theorem 4.3.** Assume that there are constants $0 < c_1 < c_2 < c_3 < c_4$ and $r_2, r_3 \in (0, 1)$ such that

(i) $f(u) \leq \dfrac{c_i}{B}$ for $z \in [0, c_i]$, $i = 1, 4$, and

(ii) $f(u) \geq \dfrac{c_i}{L(r_i)}$ for $z \in [c_i q(r_i), c_i\,]$, $i = 2, 3$.

Then the problem (4.1), (4.2) has at least two positive solutions.

*Proof.* If

$$\Omega_i = \{x \in X \mid \|x\| < c_i\}, \quad i = 1, 2, 3, 4,$$

then, from Lemma 4.1 and 4.2, we have

$$\|Tu\| \leq \|u\| \text{ for } u \in P \cap \partial\Omega_i, \quad i = 1, 4$$

and

$$\|Tu\| \geq \|u\| \text{ for } u \in P \cap \partial\Omega_i, \quad i = 2, 3.$$

Now from Lemma 2.4 we see that $T$ has two fixed points, one in each of two sets $P \cap (\overline{\Omega}_4 \backslash \Omega_3)$ and $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This completes the proof of the theorem. ■

Similarly, we can prove the following result.

**Theorem 4.4.** Assume that there are constants $0 < c_1 < c_2 < c_3 < c_4$ and $r_1, r_4 \in (0, 1)$ such that

(i) $f(z) \le \dfrac{c_i}{B}$, $z \in [0, c_i]$, $i = 2, 3$, and

(ii) $f(z) \ge \dfrac{c_i}{L(r_i)}$, $z \in [c_i q(r_i), c_i\,],\ i = 1, 4.$

Then the problem (4.1), (4.2) has at least two positive solutions.

Theorem 4.3 and Theorem 4.4 are given for the existence of double positive solutions. We can obtain many other similar results. For arbitrary positive integer $n$, we can impose appropriately conditions on $f$ so that problem (4.1), (4.2) has at least $n$ positive solutions. Here is one such result.

**Theorem 4.5.** Assume that there are constants $0 < c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < \cdots < c_{2n-1} < c_{2n}$ and $r_2, r_3, \ldots, r_{4k-2}, r_{4k-1}, \ldots, r_{2n-1}, r_{2n} \in (0, 1)$ such that

(i) $f(z) \le \dfrac{c_i}{B}$ for $z \in [0, c_i]$, $i = 1, 4, \ldots, 4k-3, 4k-3 + \dfrac{3(1 + (-1)^n)}{2}$, and

(ii) $f(z) \ge \dfrac{c_i}{L(r_i)}$ for $z \in [c_i q(r_i), c_i\,]$ and $i = 2, 3, \ldots, 4k-2, 4k-2+ \dfrac{1 + (-1)^n}{2}$,

where $k = 1, 2, \ldots, \left[\dfrac{n+1}{2}\right]$.

Then the problem (4.1), (4.2) has at least $n$ positive solutions.

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