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Integral Equations, ε-Fixed Points, Fixed Point Regions, Linear Approximations, and a Family of Kernels

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Abstract

We study a nonlinear Volterra integral equation by first transforming it into an equivalent equation with kernel depending on an (arbitrary) positive number J. The notion of "ε-nearly fixed point" is introduced and a family of resolvent kernels along with a family of corresponding mappings are considered. A part of the paper is devoted to finding a region in which all possible solutions of the equation must reside, yet properties of the family of the resolvents are given. An extensive discussion containing an illustrative example is provided.

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1 Introduction

In this paper we study the behavior of solutions of the Volterra equation

$$x(t) = a(t) - \int_0^t A(t-s)f(s, x(s))ds \tag{1.1}$$

which has been a topic of interest and application for almost a century. Fixed point methods of a variety of complexity have often been employed in which the investigator

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seeks a set G of continuous functions $\phi : [0,\infty) \to \mathbb{R}$ and use the natural mapping defined by (1.1) to map G into G, a fixed point of which would then be a solution of the equation. In our work here, a and f are continuous, while A satisfies the Friedman [7] conditions (A1)–(A3) given in Miller [9, pp. 209–213] as

(A1) $A(t) \in C(0,\infty) \cap L^1(0,1)$.

(A2) $A(t)$ is positive and non-increasing for $t > 0$.

(A3) For each $T > 0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0 < t < \infty$.

Under these conditions the resolvent, R, of A is positive and when $\int_0^\infty A(s)ds = \infty$ (which we always assume here) then

$$\int_0^\infty R(s)ds = 1.$$

Common examples are $A(t-s) = (t-s)^{q-1}, 0 < q < 1$, found in heat problems as well as in many types of fractional differential equations.

We can begin the study of (1.1) as follows.

If, for example, $(C, \|\cdot\|)$ is the normed space of bounded continuous functions $\phi : [0,\infty) \to \mathbb{R}$ with the supremum norm, then (1.1) defines a natural mapping Q on C implies

$$(Q\phi)(t) = a(t) - \int_0^t A(t-s)f(s,\phi(s))ds. \tag{1.2}$$

A fixed point is a function $\psi \in C$ with $Q\psi = \psi$ so that ψ is a solution of (1.1). Notice that all we gain from this result is an existence theorem in the sense that there is a solution of that equation on $[0,\infty)$. Lest we become too impressed, note that for (1.1) we could establish existence using any number of very explicit theorems.

To see the short-comings, we offer Schaefer's fixed point theorem as follows.

Theorem 1.1 (Schaefer [11, p. 29]). *Let $\mathcal{B}$ be a normed space, P a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$. Then either*

(i) the equation $x = \lambda Px$ has a solution for $\lambda = 1$, or

(ii) the set of all such solutions x, for $0 < \lambda < 1$, if any, is unbounded.

Somewhere there is a fixed point. We do not know where or how many. Known existence results would show this without any reference to compactness. And compactness questions directed the investigator as early as 1983 [10] to ε-approximate solutions. We follow [12, p. 36] for a brief presentation of that subject.

Let $(C, \|\cdot\|)$ be a normed space and $g : C \to C$ be a function.

Definition 1.2. A point $x^* \in C$ is an ε-fixed point of g if $\|g(x^*) - x^*\| < \varepsilon$.

The idea in a given problem then is to show that for each $\varepsilon > 0$ there is an ε-fixed point. Such a result is found in [12, p. 36]. Again, we have no indication of the location of that point.

And this is where the present work diverges.

We transform (1.1) into an equivalent equation (sharing all solutions with all solutions of (1.1)). In the process we introduce a parameter J which plays a critical role.

Under specified conditions we show that for any $\varepsilon > 0$ there is a choice of $J > 0$ so that there is an ε-approximate solution of the transformed equation on $[0, \infty)$. **Moreover, we display that fixed point**.

This is simple and it is just the beginning. We advance the work to include fixed point regions which contain all possible ε-fixed points. After this comes a main part in which implications of the ε-approximate solutions are studied and discussed.

2 The Improved Linear Approximation

We will give more detail in an appendix in which it is shown that (1.1) can be transformed into an equivalent equation (sharing solutions)

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds \tag{2.1}$$

with J an arbitrary positive number which will be taken here to be very large and with

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds. \tag{2.2}$$

We will always assume that

$$\int_0^\infty A(s)ds = \infty$$

and that will imply that

$$\int_0^\infty R(s)ds = 1,$$

which holds even as we change J, as will be seen in the appendix.

From (2.1), we extract the linear equation

$$x(t) = z(t) + \int_0^t R(t-s)x(s)ds. \tag{2.3}$$

With the conditions we have here, a simple observation shows that

$$x(t) = a(t)$$

is a solution of (2.3) which we will always designate here by

$$x(t) = \Phi.$$

Had it not been so obvious, when a is continuous on $[0, \infty)$ we could have obtained it by progressive contractions [3] and concluded uniqueness as well. That solution is the unique fixed point of the natural mapping, P, defined by (2.3). In the same way (2.1) defines a natural mapping Q whose fixed point, if any, would solve our original problem (1.1).

The notation P, Q, and Φ will be used through out this section.

Part I: $f(t,x)$ is bounded by M for all $t \geq 0$ and $x \in \mathbb{R}$.

In order to avoid constructing a mapping set G we will assume that there is a positive constant M so that

$$0 \leq t < \infty, x \in \mathbb{R} \implies |f(t,x)| \leq M. \tag{2.4}$$

Then for any given $\varepsilon > 0$ we can find a $J > 0$ so that for all such (t,x) we have

$$\frac{|f(t,x)|}{J} \leq \frac{M}{J} < \varepsilon. \tag{2.5}$$

It now follows that for any continuous function $\phi : [0,\infty) \to \mathbb{R}$ then we have

$$(P\phi)(t) = z(t) + \int_0^t R(t-s)\phi(s)ds$$

and

$$(Q\phi)(t) = z(t) + \int_0^t R(t-s)\left[\phi(s)ds - \frac{f(s,\phi(s))}{J}\right]ds$$

so that for any $t > 0$ we have

$$\begin{aligned}|(P\phi)(t) - (Q\phi)(t)| &\leq \int_0^t R(t-s)\frac{|f(s,\phi(s))|ds}{J} \\ &\leq \int_0^t R(s)\varepsilon ds < \varepsilon.\end{aligned}$$

This relation gives the next four results.

Note that P has a unique fixed point Φ and so this relation holds for it yielding

$$|(P\Phi)(t) - (Q\Phi)(t)| < \varepsilon.$$

Theorem 2.1. *Let* (2.4) *hold and* $\varepsilon > 0$ *be given. Let* $J > 0$ *be such that* (2.5) *holds, and let* Φ *be the unique fixed point of* P. *Then for all* $t \geq 0$

$$|(P\Phi)(t) - (Q\Phi)(t)| < \varepsilon$$

so that

$$|\Phi(t) - (Q\Phi)(t)| < \varepsilon.$$

In other words, for any $t \in [0, \infty)$ it is true that the fully established $\Phi(t)$ and the relatively known $Q(\Phi(t))$ are ε close to each other. From this we see that, while Q operating on the known Φ may not be a fixed point, it is very nearly a fixed point. This contributes a solution to an old problem. P resembles a linear approximation to Q with the following properties.

1. Notice that the approximation is uniform and independent of a particular $t \in [0, \infty)$. Classical results often show the approximation weakening with t increasing.

2. Notice that the approximation is global. Classical results often hold only on a short interval.

3. The fixed point of P behaves as a uniformly good approximation to a fixed point of Q in the sense that as $\varepsilon \to 0$ the fixed point of P would become a fixed point of Q if the limit $\varepsilon = 0$ were actually achieved.

We strengthen the result as follows.

Theorem 2.2. *Let* (2.4) *hold and* $\varepsilon > 0$ *be given. Let* $J > 0$ *be such that* (2.5) *holds. If* Q *has a fixed point* ϕ *then*

$$|(P\phi)(t) - (Q\phi)(t)| = |(P\phi)(t) - \phi(t)| < \varepsilon, \quad t \geq 0.$$

We interpret this as saying that if ϕ is a fixed point of Q then it is ε-nearly a fixed point of P.

Example 2.3. We now offer an example showing that these two very simple results faithfully reflect what is happening. In order to see this we will take a trivial example in which the transformation has already been completed and we are looking at the equations for P and Q with a kernel having integral equal to one. We will represent the bounds on $f(t, x)/J$ as just their bounds, $\varepsilon > 0$, and observe what is happening as ε tends to zero. The integral of the kernel times ε comes out front to combine with the initial $a(t)$ in a very interesting way.

It would, of course, be interesting to construct more general examples but the process of transforming the equation means integrating the given integral equation which can only be done in simple cases. We take a to be a positive constant and $J = 1$.

Let the Q equation be

$$x(t) = a - \int_0^t e^{-(t-s)}[x(s) + \varepsilon]ds, \quad \varepsilon > 0$$

so that the P equation is

$$x(t) = a - \int_0^t e^{-(t-s)}x(s)ds.$$

The Q equation integrates as the fixed point

$$x(t) = \frac{a + \varepsilon}{2}\left(e^{-2t} + 1\right) - \varepsilon$$

or,

$$x(t) = \frac{a}{2}\Big(1 + e^{-2t}\Big) + \frac{\varepsilon}{2}\big(e^{-2t} - 1\big)$$

while the P equation integrates as the fixed point

$$x = \frac{a}{2}\Big(1 + e^{-2t}\Big).$$

Notice that the ε in the integrand goes back and joins a in an unexpected way but does join a as $\varepsilon \to 0$. The Q fixed point merges in a unified way throughout t the interval $[0, \infty)$, which would not occur in a complicated case.

The P fixed point approximates the Q fixed point in a manner to delight any investigator.

Before we go to Part II we repeat the idea in the title of this paper concerning *fixed point regions* introduced in several earlier papers [4, p. 297], [5], and [6] which was motivated by two classical results, Schaefer's and Schauder's second theorem.

Schauder's theorem asks for a continuous mapping P of a closed convex nonempty set G into a compact subset of G. A fixed point results but we note that there may be fixed points as well outside G, and these could be disastrous for a given problem. We want to be sure that we know where all fixed points are.

That problem escalates in Schaefer's theorem because we have no idea where or how many fixed points there are. Moreover the already severe problem of compactness in Schauder's result becomes far more demanding in Schaefer's theorem.

Our aim is to eliminate all of these problems including compactness and we did so by asking that P maps Schauder's set G into G^o, the interior of G. We conclude that all fixed points, if any, reside in G. We then call G a fixed point region. Part II uses this idea, as does the next section.

Part II: P has a self-mapping set G, $P : G \to G$, in which $|f(t, x)| < M$ for $(t, x) \in G$.

Now $a(t)$ is still a fixed point of P. Let us note that the method can proceed without that fact as follows. For this section we suppose that there is a closed bounded (by M) convex nonempty set G and that the function $P : G \to G$, while $|f(t, x)| \le M$ in G. The continuity of f and conditions (A1)–(A3) still hold. Under these conditions the mapping is compact and by Schauder's theorem there is a fixed point. Because of the linearity conditions the fixed point is unique (see [3]).

We again note that $\|\Phi - Q\Phi\| < \varepsilon$ from which we see that the fixed point of the linear part behaves as "ε-nearly a fixed point" of the nonlinear equation. That result is then turned around to strengthen the case being made here.

Theorem 2.4. *Let the assumptions on G of Part II hold. Also, let $\varepsilon > 0$ be given and determine J so large that $M/J < \varepsilon$. Then for any continuous ϕ on $[0, \infty)$ residing in G and any $t \in [0, \infty)$ we have*

$$|(P\phi)(t) - (Q\phi)(t)| < \varepsilon.$$

If Φ is the unique fixed point of P residing in G then

$$|\Phi(t) - Q(\Phi(t))| < \varepsilon, \quad t \geq 0.$$

We interpret the theorem by saying that if Φ is the unique fixed point of P residing in G then it is almost a fixed point of Q.

We may slightly strengthen the result as follows.We still do not know for sure where that approximate fixed point is.

Theorem 2.5. *Let the assumptions on G of Part II hold and suppose that Q has a fixed point ϕ residing in G. Then for each $t \in [0, \infty)$ we have*

$$|(P\phi)(t) - (Q\phi)(t)| = |(P\phi)(t) - \phi(t)| < \varepsilon.$$

We interpret this theorem by saying that any fixed point of Q is ε-nearly a fixed point of P.

We wish to show a process of studying (2.1) to obtain G and bring home the fact that we never have to contend with f in that process. Frequently the function $a(t)$ is, in fact, a constant which is the initial value of the solution. In particular this happens in the case of fractional equations of Caputo type for which the fractional kernels satisfy (A1)–(A3). In the theorem, below, we take $a(t) = c \in \mathbb{R}$ and assume that f satisfies a sign condition. It is worth mentioning that we do not require any Lipschitz type conditions.

Theorem 2.6. *Consider the equation*

$$x(t) = c - \int_0^t A(t-s) f(s, x(s))\, ds, \quad t \geq 0, \tag{2.6}$$

with A satisfying (A_1)-(A_3) and $xf(t,x) \geq 0, t \geq 0, x \in \mathbb{R}$. Let $T > 0$ be arbitrary and set

$$G_T = \{\phi \in C([0,T], \mathbb{R}) : \|\phi\| \leq c_0\}$$

with $|c| < c_0$. Then the set G_T is a fixed point region of the equation (2.6) *on* $[0,T]$. *In particular, the set :*

$$G_0 = \{\phi \in C([0,\infty), \mathbb{R}), \|\phi\| \leq c_0\}$$

is a fixed point region for Q on $[0, \infty)$.

Proof. By continuity, the function $|f|$ attains a nonnegative maximum on the compact set $[0,T] \times [-c_0, c_0]$, say M_T. Consider a $J > 0$ such that

$$\frac{M_T}{J} \leq |c| < c_0,$$

and transform (2.6) into

$$x(t) = c - c\int_0^t R(t-s)ds + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right] ds, t \in [0,T],$$

where R is the corresponding resolvent kernel (see the Appendix). With this in hand, and taking into consideration that the quantities x and $\dfrac{f(s,x)}{J}$ have the same sign, we see that for $\phi \in G_T$, $s \in [0,t]$ it holds

$$\begin{aligned}\left|x(s) - \frac{f(s,x(s))}{J}\right| &\leq \max\left\{|x(s)|, \left|\frac{f(s,x(s))}{J}\right|\right\} \\ &\leq \max\left\{c_0, \frac{M_T}{J}\right\} \leq c_0,\end{aligned}$$

and so, in view of (5.1) (Appendix), for the natural mapping Q we have for $t \in [0,T]$

$$\begin{aligned}|(Q\phi)(t)| &\leq |c|\left[1 - \int_0^t R(t-s)ds\right] + \int_0^t R(t-s)\left|x(s) - \frac{f(s,x(s))}{J}\right| ds \\ &\leq |c|\left[1 - \int_0^t R(t-s)ds\right] + c_0\int_0^t R(t-s)ds \\ &\leq [c_0 - |c|]\int_0^T R(s)ds + |c| := d_T \\ &< c_0 - |c| + |c| = c_0,\end{aligned}$$

from which it follows that $Q(G_T) \subseteq \overline{B(0,d_T)} \subset G_T^0$. Consequently, as any fixed point of Q cannot leave G_T (see [4]), the set G_T is a fixed point region of the mapping Q. Since c_0 does not depend on $T > 0$ we may infer that

$$G = \{\phi : [0,\infty) \to \mathbb{R}, \|\phi\| \leq c_0\}$$

is a fixed point region for Q on $[0,\infty)$. □

To illustrate the result of Theorem 2.6 we cite a simple example, yet a more general one (formulated as a Proposition). Instead of simply stating the results obtained, we prefer to present the steps of the proof in some detail so that the procedure will be apparent.

Example 2.7. Consider (2.6) with $c = 6$, $f(t,x) := \operatorname{sgn} x\sqrt{|x| + t}$, i.e., consider the equation

$$x(t) = 6 - \int_0^t A(t-s)\operatorname{sgn} x\sqrt{|x| + s}ds, \ t \geq 0.$$

Let $T > 0$ be arbitrary and set

$$G_T = \{x \in C([0,T], \mathbb{R}) : \|x\| \leq 7\},$$

so

$$M_T = \max\left\{|f(t,x)| : t \in [0,T], |x| \leq \sqrt{7}\right\} = \sqrt{7+T}.$$

Now for $J = \sqrt{8+T}$ we have

$$\left| x(s) - \frac{f(s, x(s))}{J} \right| \le \max\left\{7, \frac{\sqrt{7+T}}{\sqrt{8+T}}\right\} = 7,$$

so for any $x \in G_T$ we have

$$\begin{aligned} |Q(x)(t)| &\le 6\left[1 - \int_0^t R(t-s)ds\right] + 7\int_0^t R(t-s)ds \\ &= 6 + \int_0^t R(s)ds \\ &\le 6 + \int_0^T R(s)ds := d_T < 6 + \int_0^\infty R(s)ds = 7, \end{aligned}$$

thus $Q(G_T) \subseteq \overline{B(0, d_T)} \subset G_T^o$. As $d_T < 7$ for all $T > 0$, we may conclude that the set

$$G_0 = \{x \in C([0,\infty), \mathbb{R}) : \|x\| \le 7\},$$

is a fixed point region containing all possible global solutions of the equation. Clearly, f is not Lipschitz, furthermore it does not satisfy any linearity condition.

To emphasize that the procedure above can be effective to more general cases than a being constant, we now show that it can also be applied in the case of fractional equations of Riemann-Liouville type where $a(t) = x^0 t^{q-1}$, i.e., in the case of equation

$$x(t) = x^0 t^{q-1} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \quad t > 0, \tag{2.7}$$

with $\left|x^0\right| \ne 0$ and $q \in (0,1)$. The condition posed on f in the next proposition arises as a natural consequence of the fact that if x is a solution of (2.7) and $\varepsilon > 0$ is given then

$$\left(x^0 - \varepsilon\right) t^{q-1} < x(t) < \left(x^0 + \varepsilon\right) t^{q-1}$$

near zero (see [1, p. 249]).

Proposition 2.8. *Let f satisfy $xf(t,x) \ge 0, t > 0, x \in \mathbb{R}$, and assume that there exists an $\varepsilon > 0$ such that the function $\widehat{f}(s,z) := s^{1-q} f\left(s, s^{q-1}z\right), s > 0, z \in \mathbb{R}$ is bounded and continuous on the set*

$$D_t := \left\{(s,z) : |z| < \left|x^0\right| + \varepsilon, s \in (0,t)\right\}$$

for any $t > 0$. Then, for an arbitrary $T > 0$, the set

$$G_T := \left\{x(t) = \phi(t) t^{q-1} : \phi \in C([0,T], \mathbb{R}), |\phi(t)| < \left|x^0\right| + \varepsilon, t \in (0,T]\right\}$$

is a fixed point region of (2.7) on $(0,T]$. In turn, the set

$$G_0 := \left\{x(t) = \phi(t) t^{q-1} : \phi \in C([0,\infty)), \mathbb{R}), |\phi(t)| < \left|x^0\right| + \varepsilon, t > 0\right\}$$

is a fixed point region of (2.7) on $(0,\infty)$.

Proof. Let $T>0$ be arbitrary and set $M_T := \sup_{(s,z)\in D_T} \left|\widehat{f}(s,z)\right|$. For $J>0$ with

$$\frac{M_T}{J\Gamma(q)} < \left|x^0\right| + \varepsilon.$$

we may transform (2.7) to

$$x(t) = x^0\left[t^{q-1} - \int_0^t R(t-s)\, s^{q-1} ds\right] + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J\Gamma(q)}\right] ds, \tag{2.8}$$

and note that here we have $A(t) = t^{q-1}$ so the kernel R satisfies the resolvent equation

$$R(t) = Jt^{q-1} - \int_0^t R(t-s) J s^{q-1} ds = J\left[t^{q-1} - \int_0^t R(t-s) s^{q-1} ds\right],$$

from which, in view of $R(t)>0$, we have

$$t^{q-1} - \int_0^t R(t-s)\, s^{q-1} ds \geq 0,$$

and

$$1 - t^{1-q}\int_0^t R(t-s)\, s^{q-1} ds \geq 0, \quad t>0. \tag{2.9}$$

Due to the singular function $x^0 t^{q-1}$ which blows up at $t=0$, we are interested in solutions of the type $x(t) = t^{q-1}\phi(t), t>0$ with ϕ being continuous on $[0,T]$ and such that $\lim_{t\to 0}\phi(t) = \phi(0) = x^0 \neq 0$, so multiplying equation (2.8) by t^{1-q} we may look for solutions ϕ of the equation

$$\phi(t) = x^0\left[1 - t^{1-q}\int_0^t R(t-s)\, s^{q-1} ds\right] + t^{1-q}\int_0^t R(t-s)\left[s^{q-1}\phi(s) - \frac{f(s, s^{q-1}\phi(s))}{J\Gamma(q)}\right] ds, \tag{2.10}$$

which are continuous on the closed interval $[0,T]$. We, then, consider the set

$$\widehat{G_T} := \left\{\phi \in C([0,T],\mathbb{R}) : \|\phi\| \leq \left|x^0\right| + \varepsilon\right\}$$

and let Q be the natural mapping defined by the right-hand-side of (2.10), i.e., we define Q by $\phi \in \widehat{G_T}$ implies

$$Q\phi(t) \;:\; = x^0\left[1 - t^{1-q}\int_0^t R(t-s)\, s^{q-1} ds\right]$$

$$+t^{1-q}\int_0^t R(t-s)\left[s^{q-1}\phi(s)-\frac{f(s,s^{q-1}\phi(s))}{J\Gamma(q)}\right]ds.$$

Notice that by the definition of the resolvent kernel R and the assumption on $\widehat{f}$ it follows that Q is well defined and $Q\phi$ is a continuous function on $[0,T]$. Furthermore, in view of the sign condition of f, for $\phi\in G$ and $s\in(0.T)$ it holds

$$\begin{aligned}\left|\phi(s)-\frac{s^{1-q}f(s,s^{q-1}\phi(s))}{J\Gamma(q)}\right| &\leq \max\left\{|\phi(s)|,\frac{s^{1-q}|f(s,s^{q-1}\phi(s))|}{J\Gamma(q)}\right\}\\ &\leq \max\left\{\|\phi\|_T,\frac{M_T}{J\Gamma(q)}\right\}\leq\left|x^0\right|+\varepsilon,\end{aligned}$$

so, taking (2.9) into consideration, we have for $t\in[0,T]$

$$\begin{aligned}|Q\phi(t)| &\leq \left|x^0\right|\left[1-t^{1-q}\int_0^t R(t-s)s^{q-1}ds\right]\\ &\quad +t^{1-q}\int_0^t R(t-s)\left|s^{q-1}\phi(s)-\frac{f(s,s^{q-1}\phi(s))}{J\Gamma(q)}\right|ds\\ &= \left|x^0\right|\left[1-t^{1-q}\int_0^t R(t-s)s^{q-1}ds\right]\\ &\quad +t^{1-q}\int_0^t R(t-s)s^{q-1}\left|\phi(s)-\frac{s^{1-q}f(s,s^{q-1}\phi(s))}{J\Gamma(q)}\right|ds\\ &< \left(\left|x^0\right|+\varepsilon\right)\left[1-t^{1-q}\int_0^t R(t-s)s^{q-1}ds\right]\\ &\quad +\left(\left|x^0\right|+\varepsilon\right)t^{1-q}\int_0^t R(t-s)s^{q-1}ds\\ &= \left|x^0\right|+\varepsilon.\end{aligned}$$

Following the argumentation in the last lines of Theorem 2.6 we may conclude that $\widehat{G_T}$ is a fixed point region for the equation (2.10). This, in turn, implies that the sets G_0 and G_T are fixed point regions for the equation (2.7) on $(0,\infty)$ and $[0,T]$, respectively. □

Two things are worth noticing here. The first one is that, in fact, we need the sign property to hold only on the set $(0,\infty)\times[-c_0,c_0]$ since all our work is limited on this strip. The second one is that the above procedure may yield fixed point regions for any equation of the type of (1.1) with f satisfying the sign condition and $a(t)\geq 0$, nonincreasing and such that

$$a(t)-\int_0^t R(t-s)a(s)ds\geq 0,\quad t>0, \tag{2.11}$$

with $J>0$ being sufficiently large. Note that, when $x^0,c>0$, then both the last two results can be obtained as special cases of the next one. The proof may be achieved by following the arguments in the proof of Theorem 2.6.

Theorem 2.9. *Assume that the continuous function $a(t)$, $t > 0$ is non-negative and non-increasing and let f be continuous and satisfy $xf(t,x) \geq 0$, $(t,x) \in (0,\infty) \times [-c_0, c_0]$ for some $c_0 > a(0)$. Let $T > 0$ be arbitrary and take $J_0 > 0$ be such that*

$$\frac{M_T}{J_0} < c_0,$$

where $M_T := \sup\{|f(t,z)| : (t,z) \in [0,T] \times [-c_0, c_0]\}$. If for some $J > J_0$ the inequality (2.11) *is satisfied, then the set*

$$G_T = \{\phi \in C([0,T], \mathbb{R}) : \|\phi\| \leq c_0\}$$

is a fixed point region for the equation (1.1) *on* $[0,T]$.

3 Fixed Point Regions and a Family of Kernels

In this section we look closer at the transformation Q and the resolvent R mentioned in the previous sections. In fact, as $J > 0$ may vary, a class of resolvents and a corresponding class of transformations are created, so we adopt a more detailed notation. Recall that we deal with the equation

$$x(t) = a(t) - \int_0^t A(t-s) f(s, x(s))\, ds, \quad t \geq 0, \tag{3.1}$$

and, following the procedure presented in the Appendix, we consider an (arbitrary) $J > 0$ and transform (3.1) into

$$x(t) = z_J(t) + \int_0^t R_J(t-s) \left[x(s) - \frac{f(s, x(s))}{J} \right] ds, \tag{3.2}$$

with

$$z_J(t) = a(t) - \int_0^t R_J(t-s)\, a(s)\, ds, \quad t \geq 0, \tag{3.3}$$

and R_J the corresponding resolvent kernel of JA which is the unique solution to the resolvent equation

$$R_J(t) = JA(t) - \int_0^t JA(t-s)\, R_J(s)\, ds. \tag{3.4}$$

It is crucial to see that, regarding to the given kernel A, for any arbitrarily chosen $J > 0$ there corresponds a unique resolvent kernel (denoted by R_J), which is the (unique) solution of (3.4). We note that this transformation is valid also for the equation

$$x(t) = a(t) - \int_0^t k(t,s) f(s, x(s))\, ds, \quad t \geq 0,$$

under very general conditions concerning the kernel $k(t,s)$ (see [9, Ch. IV]) which do not require $k(t,s)$ to be of convolution type and allow $k(t,s)$ to have singularities, yet when $k(t,s)$ is continuous then so is R_J (see [9, p. 202]). It turns out that, for a given kernel $k(t,s)$ satisfying such general conditions, a family of resolvent kernels $\{R_J(t), t>0\}_{J>0}$ may be constructed. In our work here, we focus on kernels of convolution type that satisfy conditions (A1)–(A3) for which some interesting properties hold (see, also, the Appendix).

It is known (see [9, p. 192]) that the equations (3.1) and (3.2) share solutions, for any value of $J>0$. Hence, if a solution x of the equation (3.2) with $J=J_0$ is found, then this function x is also a solution of (3.1) and vice versa. Furthermore, this x is also a solution of (3.2) for any positive J. In other words, once a solution x of (3.2) with a specific (arbitrary) $J=J_0>0$ is found, then this function x also satisfies (3.1), yet an infinite number of equations, namely (3.2) for any $J>0$.

We now fix an arbitrary $T>0$ and work in the Banach space

$$C_T := \left(C\left([0,T],\mathbb{R}\right), \|\cdot\|_T\right)$$

consisting of continuous real functions defined on $[0,T]$, and equipped with the usual sup-norm denoted by $\|\cdot\|_T$. One of our aims is to investigate fixed point regions of the mapping Q_J in the space C_T, (which is equivalent to spotting continuous solutions to equations (3.1) and (3.2) on $[0,T]$) with hope that, as $T>0$ is arbitrarily chosen, our results may lead to information on global solutions. For convenience, we are going to use the term "fixed point regions" for both, mappings and corresponding equations, so for $J>0$ we let $P_J, Q_J : C_T \to C_T$ be defined, respectively, by

$$P_J(\phi)(t) := z_J(t) + \int_0^t R_J(t-s)\,\phi(s)\,ds, \quad t\in[0,T],$$

and

$$Q_J(\phi)(t) := z_J(t) + \int_0^t R_J(t-s)\left[\phi(s) - \frac{f(s,\phi(s))}{J}\right] ds, t\in[0,T], \qquad (3.5)$$

with z_J and Q_J given in (3.3) and (3.5), respectively. We may easily see that the mapping P_J is a contraction in the Banach space C_T, yet that its unique fixed point is the function a (regardless of the choice of $J>0$). Recalling that in order to spot a solution of (3.1) on $[0,T]$ it suffices to find a fixed point of (3.5) for a single value of J, we consider an arbitrary $J>0$ and let ϕ be a fixed point of (3.5). Note that since we have assumed that f is bounded by $M>0$, for $t\in[0,T]$ it holds

$$\begin{aligned}|Q_J(\phi)(t) - P_J(\phi)(t)| &= |\phi(t) - P_J(\phi)(t)| \\ &\le \left|\int_0^t R_J(t-s)\,\frac{f(s,\phi(s))}{J} ds\right| \le \frac{M}{J}\left|\int_0^T R_J(s)\,ds\right|,\end{aligned}$$

and so, taking into consideration that a is the unique fixed point of P_J, we have for $t \in [0,T]$

$$\begin{aligned}
|\phi(t) - a(t)| &= |Q_J(\phi)(t) - P_J(a)(t)| \\
&= \left| \int_0^t R_J(t-s) \left[\phi(s) - a(s) - \frac{f(s,\phi(s))}{J} \right] ds \right| \\
&\leq \int_0^t R_J(t-s) |\phi(s) - a(s)| \, ds + \int_0^t R_J(t-s) \left| \frac{f(s,\phi(s))}{J} \right| ds \\
&\leq \|\phi - a\|_T \int_0^t R_J(t-s) \, ds + \frac{M}{J} \int_0^t R_J(t-s) \, ds,
\end{aligned}$$

from which we take

$$\|\phi - a\|_T \leq \|\phi - a\|_T \int_0^T R_J(s) \, ds + \frac{M}{J} \int_0^T R_J(s) \, ds.$$

Since for the resolvent kernel R_J we have $0 < \int_0^T R_J(s) \, ds < 1$ for any $T > 0$ (see (5.1) in the Appendix), the last inequality yields

$$\|\phi - a\|_T \leq \frac{M \int_0^T R_J(s) \, ds}{J \left[1 - \int_0^T R_J(s) \, ds \right]} := W_T(J). \tag{3.6}$$

From the above discussion we have the next result.

Proposition 3.1. *Let $T > 0$ be given, $M > 0$ be a bound of f on $[0,T] \times \mathbb{R}$, and $J > 0$ be an (arbitrary) positive number. Then a fixed point region of the equations* (3.1) *and* (3.2) *is the ball $B(a; W_T(J))$.*

Clearly, as $0 < \int_0^t R_J(s)ds < 1$ for any $t, J > 0$, the quantity $W_T(J)$ introduced in (3.6) is meaningful for any positive number J, so a corresponding function $W_T(J)$ may be defined on $(0,\infty)$ by (3.6). This function is positive (since $R_J(t)$ is). Moreover, as established in the next lemma, it attains a positive minimum on $[0,\infty)$ (with its value at 0 considered to be the limit for $J \to 0+$) provided that

$$\int_0^t A(t-s) f(s, a(s)) \, ds \not\equiv 0. \tag{3.7}$$

This condition will be assumed to hold throughout the rest of the paper without any further mention. Note that in the opposite case, the function a is a solution of (3.1) and vice versa.

Lemma 3.2. *Let $T > 0$ be given. Then the function $W_T(J) : (0,\infty) \to \mathbb{R}$ given by*

$$W_T(J) := \frac{M \int_0^T R_J(s)\, ds}{J \left[1 - \int_0^T R_J(s)\, ds\right]}, \quad J > 0,$$

is well defined and positive. Moreover, the number

$$W_T := \inf_{J>0} \frac{M \int_0^T R_J(s)\, ds}{J \left[1 - \int_0^T R_J(s)\, ds\right]} \in (0,+\infty). \tag{3.8}$$

Proof. Assume that the infimum is zero. Then there exists a sequence of positive numbers (J_n) tending either to some real number $J_0 \geq 0$ or to $+\infty$ with $\lim_{n\to\infty} W_T(J_n) = 0$. In view of (3.6), this implies that $||\phi - a||_T = 0$, meaning that a is a fixed point of (3.1), which contradicts (3.7). □

Since (3.1) shares solutions with (3.2) for any $J > 0$, we may state the following result concerning the fixed point region of equation (3.1) (on $[0,T]$).

Theorem 3.3. *Let f be bounded by M. Then for a given $T > 0$, a fixed point region* (3.1) *(on $[0,T]$) is the ball $B(a; W_T)$ where W_T is defined by* (3.8).

Proof. It follows immediately from Proposition 3.1 and Lemma 3.2 and the observation that a function x is a fixed point of the equation (3.1) if and only if it is a fixed point of (3.2) for some $J > 0$. □

Though it is rather hard to expect precise calculation of the number W_T, however we could estimate W_T by approximating values of the function $W_T(J)$ for a finite number of convenient J's and take their minimum. For example, we could consider approximating the resolvent kernels R_{k_n} satisfying

$$R_{k_n}(t) = k_n A(t) - \int_0^t k_n A(t-s) R_{k_n}(s)\, ds,$$

taken for finitely many $J = k_n > 0$, and estimate W_T by

$$W_T \simeq \min\{W_T(k_n)\}.$$

By a first look at the result of Theorem 3.3 one may note that it cannot be used to spot fixed point regions of global solutions (i.e., solutions defined on the whole half-axis $[0,\infty)$) to equation (3.1). This is due to the fact that the integral of the resolvent kernel R_J tends to one as $T \to \infty$, so the denominator of $W_T(J)$ tends to zero (for any value of $J > 0$) while the numerator tends to M, thus the radius of the fixed point region tends to infinity. However, one may interpret the number W_T as some kind of information of the (maximum) growth of a (possibly global) solution: any solution $x(t)$ of (3.1) is bounded by the function $|a| + W_t$, and, of course, by any of the functions

$$w_J(t) := |a(t)| + \frac{M \int_0^t R_J(s)\, ds}{J\left[1 - \int_0^t R_J(s)\, ds\right]}, \quad t \geq 0 \quad (J > 0).$$

In fact, the growth of solutions is dictated by $|a|$ and the growth of R_J.

While in our work here we are mostly interested in positive kernels A with infinite integral, however it should be noticed that our results can also be applied to L^1 kernels which satisfy (A1)–(A3), since our transformation is still valid and the assumptions of Proposition 3.1 and Lemma 3.2 are satisfied. Of course, in such a case a fixed point region may easily be found directly from equation (3.1) since, for any $t \geq 0$ it holds

$$|x(t) - a(t)| \leq \int_0^t A(t-s)\, |f(s, x(s))|\, ds \leq M \int_0^\infty A(s)\, ds = Mk,$$

implying

$$\|x - a\|_T \leq M \int_0^T A(s)\, ds \leq Mk,$$

so the ball $B(a; Mk)$ (in $C([0,\infty), \|\cdot\|)$) is a fixed point region for global solutions of (3.1). It is of interest to see that, regarding global solutions of (3.1), Theorem 3.3 gives exactly the same fixed point region.

Corollary 3.4. *Let $A \in L^1$ satisfy (A1)–(A3) with $\int_0^\infty A(s)\, ds = k \in \mathbb{R}$ and $|f|$ bounded by M. Then a fixed point region for global solutions of the equation* (3.1)*, is the ball $B(a; Mk)$.*

Proof. Since $\int_0^\infty A(s)\, ds = k$, we have $\int_0^\infty JA(s)\, ds = Jk$ and (see [9, Th. 6.2, p. 212])

$$\int_0^\infty R_J(s)\, ds = \frac{Jk}{Jk+1} < 1, \quad J > 0.$$

It follows that the denominator in (3.6) is never zero and we may let $T \to \infty$ in (3.6), to obtain fixed point regions for global solutions. We find

$$\begin{aligned} W_\infty(J) \quad &: = \lim_{T\to\infty} W_T(J) = \lim_{T\to\infty} \frac{M \int_0^T R_J(s)\, ds}{J\left[1 - \int_0^T R_J(s)\, ds\right]} \\ &= \frac{M \frac{Jk}{Jk+1}}{J\left[1 - \frac{Jk}{Jk+1}\right]} = \frac{\frac{M}{J}\frac{Jk}{Jk+1}}{1 - \frac{Jk}{Jk+1}} = \frac{\frac{Mk}{Jk+1}}{\frac{1}{Jk+1}} = Mk. \end{aligned}$$

As $W_\infty(J)$ does not depend on J, we have

$$\|\phi - a\|_\infty \leq W_\infty := Mk,$$

which completes the proof. □

We now return to $A \notin L^1$ and give some interesting properties of the class of resolvent kernels $\{R_J(t)\}_{J>0}$ when the function $A(t)$ is completely monotone. In such a case the resolvent R_J is also completely monotone, and, in particular, R_J is nonincreasing (see [8, 9]). Though more general results are known to hold for (iv)-(v), the proofs below are based on the resolvent equation and they are quite simple. It is worth mentioning that fractional kernels $A(t) = t^{q-1}$, $q \in (0,1)$ are completely monotone and satisfy (A1)–(A3) (see [9, p. 221]), hence the results of the next proposition do apply on the resolvents of the fractional kernels.

Proposition 3.5. *Assume that A is completely monotone and satisfies (A1)–(A3) with $A \notin L^1(0,\infty)$. Let R_J be the resolvent kernel of JA for $J > 0$.*

(i) For any $T > 0$ we have

$$\lim_{J\to\infty} \int_0^T R_J(s)\,ds = 1. \tag{3.9}$$

(ii) It holds

$$\lim_{J\to\infty} R_J(t) = 0, t > 0, \quad \lim_{J\to\infty} R_J(0) = \infty. \tag{3.10}$$

(iii) For any $t > 0$ we have

$$\lim_{J\to\infty} \int_0^t R_J(t-s)\,A(s)\,ds = A(t).$$

(iv) For any $J > 0$ we have

$$\int_0^t R_J(t-s)\,A(s)\,ds \approx A(t) \quad \text{as } t \to \infty.$$

(iv) If $\lim_{t\to\infty} A(t) = 0$ then,

$$\lim_{t\to\infty} R_J(t) \int_0^t A(s)\,ds = 0.$$

Proof. (i) Assume that there exists some $T > 0$ for which $\liminf_{J\to\infty} \int_0^T R_J(s)\,ds = \ell_T \in [0,1)$. Then there exists a sequence $J_n \to +\infty$ with $\lim_{n\to\infty} \int_0^T R_{J_n}(s)\,ds = \ell_T$, and so $\lim_{n\to\infty} \frac{\int_0^T R_{J_n}(s)\,ds}{1 - \int_0^T R_{J_n}(s)\,ds} = \frac{\ell_T}{1-\ell_T} \in (0,\infty)$. Then from (3.6) we find

$$\|\phi - a\|_T \le \lim_{n\to\infty} \frac{M \int_0^T R_{J_n}(s)\,ds}{J_n \left[1 - \int_0^T R_{J_n}(s)\,ds\right]} = \lim_{n\to\infty} \frac{M}{J_n} \lim_{n\to\infty} \frac{\ell_T}{1-\ell_T} = 0,$$

which implies that $\phi \equiv a$, i.e., the function a is a fixed point of both equations (3.1) and (3.2), which contradicts (3.7).

(ii) For the sake of contradiction we assume that there exists some $\widehat{t} > 0$ with $\limsup_{J\to\infty} R_J\left(\widehat{t}\right) = \delta > 0$. This implies that for the positive number $\varepsilon = \delta$ there exists a sufficiently large J_0 and an increasing sequence (J_n) with $J_n \to \infty$ such that

$$\frac{\delta}{2} < R_{J_n}\left(\widehat{t}\right) < \frac{3\delta}{2}, \quad \text{for all } J_n > J_0.$$

Then by the fact that each R_J is non-increasing for any J_n we would have that

$$\frac{\delta}{2}\frac{\widehat{t}}{2} < R_{J_n}\left(\widehat{t}\right)\int_{\widehat{t}/2}^{\widehat{t}} ds \le \int_{\widehat{t}/2}^{\widehat{t}} R_{J_n}(s)\, ds$$

thus

$$\int_0^{\widehat{t}} R_{J_n}(s)\, ds - \int_0^{\frac{\widehat{t}}{2}} R_{J_n}(s)\, ds = \int_{\widehat{t}/2}^{\widehat{t}} R_{J_n}(s)\, ds > \frac{\delta}{2}\frac{\widehat{t}}{2} > 0,$$

from which, by $J_n \to \infty$ and in view of (3.9), we reach the contradiction $0 < \frac{\delta}{2}\frac{\widehat{t}}{2} \le 0$, and this proves (3.10). The limit at $t = 0$ is immediate since, either $R_J(0) = JA(0)$ when A is defined at 0, or $A(0) = \infty$ when A is singular at 0 (recall that A is positive and non-increasing so $A(0) \in (0, \infty]$).

(iii) From (3.4), we have

$$\lim_{J\to\infty}\int_0^t R_J(t-s)\, A(s)\, ds = \lim_{J\to\infty}\left[A(t) - \frac{R_J(t)}{J}\right] = A(t) - 0 = A(t).$$

(iv) As, for any $J > 0$ the function $R_J(t)$ is positive, nondecreasing and in L^1, it follows that $\lim_{t\to\infty} R_J(t) = 0$, so from (3.4), we take

$$\lim_{t\to\infty}\left[A(t) - \int_0^t R_J(t-s)\, A(s)\, ds\right] = \lim_{t\to\infty}\frac{R_J(t)}{J} = 0.$$

(iv) Since $R_J(t)$ is positive and nondecreasing, from (3.4), we have

$$\begin{aligned} A(t) &= \frac{R_J(t)}{J} + \int_0^t R_J(t-s)\, A(s)\, ds > \int_0^t R_J(t-s)\, A(s)\, ds \\ &= \int_0^t R_J(s)\, A(t-s)\, ds \ge R_J(t)\int_0^t A(t-s)\, ds \\ &= R_J(t)\int_0^t A(s)\, ds \ge 0, \end{aligned}$$

and the result follows. □

Regarding the class of resolvent kernels $\{R_J\}_{J>0}$, in view of (i) and (ii) above the limiting process as $J \to \infty$ leads to

$$R_\infty(t) := \lim_{J\to\infty} R_J(t) = \begin{cases} \infty, & t = 0 \\ 0, & t > 1, \end{cases}$$

while

$$1 = \lim_{J\to\infty} \int_0^T R_J(s)\,ds := \int_0^T R_\infty(s)\,ds,$$

a Dirac-type kernel R_∞ with integral 1 on any interval $[0,T]$ $T > 0$ as well as on $[0,\infty)$.

Before closing this section we mention a case where a fixed point region can easily be detected by the use of the transformed equation (3.2) (with a properly chosen $J > 0$) while the requirement of boundedness of the function f is replaced by a certain type of linearity condition. Clearly, when f is not bounded neither Proposition 3.1 nor Theorem 3.3 can be applied. However, equation (3.2) with a proper choice of J offers a remarkably simple estimation of a fixed point region by means of the corresponding resolvent kernel. To be specific, when there exist positive numbers $K, m > 0$ such that f satisfies

$$m \le \frac{f(t,x)}{x} \le K, \;\; x \neq 0, t \ge 0,$$

then taking $J = m + K$ we have

$$\begin{aligned} \frac{m}{K+m} &\le \frac{f(t,x)}{Jx} = \frac{1}{J}\frac{f(t,x)}{x} \le \frac{K}{K+m} \\ 0 &\le 1 - \frac{f(t,x)}{Jx} \le 1 - \frac{m}{K+m}, \end{aligned}$$

so, for a solution ϕ of (3.2), we have for $t \in [0,T]$

$$\begin{aligned} |\phi(t)| &\le |z_J(t)| + \int_0^t R_J(t-s)\,|\phi(s)| \left|1 - \frac{f(s,\phi(s))}{J\phi(s)}\right| ds \\ &\le |z_J(t)| + \int_0^t R_J(t-s)\,|\phi(s)| \frac{K}{K+m} ds \\ &\le \|z_J\|_T + \|\phi\|_T \frac{K}{K+m}, \end{aligned}$$

from which

$$\|\phi\|_T \le \frac{K+m}{m} \|z_J\|_T .$$

If, in advance, the function a is bounded on $[0,\infty)$, then, by definition, z_J is also bounded by $2\|a\|$. Consequently, any fixed point of (3.2), [hence of (3.1)] belongs to $B\left(0; 2\frac{K+m}{m}\right)$. It is clear that as the function f is not bounded the main result of this paper cannot be applied but the above consideration gives a remarkably simple estimation of a fixed point region by means of the resolvent kernel corresponding to the specific $J = m + K$. Certainly, the best choice of J is when $K = \sup\left\{\frac{f(t,x)}{x}, x \neq 0, t \geq 0\right\}$, $m = \inf\left\{\frac{f(t,x)}{x}, x \neq 0, t \geq 0\right\}$.

4 Discussion

A first concern in this section is to clarify the meaning of the notion of "ε-nearly a fixed point" appearing in Section 2. A resume of the results in that section might be stated as "*a fixed point of P is ε-nearly a fixed point of Q*" and "*a fixed poind of Q is ε-nearly a fixed point of P*". In view of the notation in Section 3, we may restate Theorem 2.2 as:

Let an arbitrary $\varepsilon > 0$ be given. If f is bounded by $M > 0$ and ϕ is a fixed point of the mapping Q_J with $J > \frac{M}{\varepsilon}$, then

$$|P_J(\phi)(t) - \phi(t)| < \varepsilon, \quad t \geq 0,$$

which implies that

$$\|P_J(\phi) - \phi\| < \varepsilon, \quad J > \frac{M}{\varepsilon},$$

thus justifying the expression "ϕ is ε-nearly a fixed point of P_J": ϕ misses to be a fixed point of P_J by the (small) quantity ε. Moreover, not only $\|P_J(\phi) - \phi\|$ can be arbitrarily small, but once an $\varepsilon > 0$ is given, then there are infinitely many mappings P_J with this property: "ϕ is ε-nearly a fixed point" of P_J for any $J > \frac{M}{\varepsilon}$. In turn, one might say that "ϕ is ε-nearly a fixed point" of the family of mappings $\{P_J\}$ if ϕ is "ε-nearly a fixed point" for all but a finite number of elements in the family. However, one may note that the above relations do not imply that this ϕ is really a fixed point of any of the mappings P_J. No matter how close the fixed point ϕ of Q and $P_J(\phi)$ may be, the unique fixed point of all these mappings P_J is the function α, and, $\alpha \not\equiv \phi$, since we have assumed that (3.7) holds true.

In a similar manner, *as a is the (unique) fixed point of any element of the class $\{P_J\}_{J>0}$, then for any arbitrary $\varepsilon > 0$ it holds*

$$\|Q_J(a) - a\| < \varepsilon, \quad J > \frac{M}{\varepsilon},$$

so

$$\lim_{J\to\infty} \|Q_J(a) - a\| = 0. \tag{4.1}$$

Of course, (4.1) does not yield that there exists a $J > 0$ such that a is a fixed point of Q_J. Regarding fixed points ϕ of the mappings Q_J and in view of the definition of R_J

and Q_J, we may easily see that ϕ is a fixed point of Q_J for some $J > 0$ if and only if it is a fixed point of the natural mapping induced by the right-hand-side of (3.1), i.e., of the mapping

$$Q^*(\phi)(t) := a(t) - \int_0^t A(t-s) f(s, \phi(s))\, ds, \quad t \geq 0,$$

and, equivalently, ϕ satisfies the equation (3.1) if and only if ϕ is a fixed point of Q_J for any $J > 0$, i.e.,

$$Q_J(\phi)(t) = \phi(t), \quad \text{for any } J > 0.$$

It should be noted that, though the mappings Q_J share fixed points, in fact these mappings differ from each other as verified by their definitions and in view of their images at the constant function $\phi_0 = 0$.

So all the elements in the collection $\{Q_J\}_{J>0}$ share fixed points, and, their images at the function a satisfy (4.1), that is, as J tends to infinity, these images come arbitrarily close to a. One would expect that such a behaviour would be a hint that a fixed point of some Q_J may be close as J grows larger; or even that the function a lives inside, or, at least, close to a fixed point region. However, in general, this is not true. As it will be apparent by the help of the very simple equation in Example (4.1) below, a function which behaves as "ε-nearly a fixed point" may be very far away from a real fixed point, even the difference between the values of these two functions may tend to infinity as t grows.

Before citing Example 4.1, we offer an easy way to visualize such a behaviour by looking at a "parallel" behaviour observed at a family of continuous functions defined on the interval $[0, 1]$. To be more specific, an analogous situation may appear if, in place of the mappings $Q^*, \{Q_J\}_{J>0}$ defined on C_T we consider the constant function $q^*(t) = 1, t \in [0, 1]$ and the family of functions $\{q_J(t), t \in [0, 1]\}_{J>0}$ defined by

$$q_J(t) = t + \frac{1}{J}(1-t), \ t \in [0, 1], \quad J > 0.$$

For this class of continuous function defined on $[0, T]$ we will show that the number $t_0 = 0$ is ε-nearly a fixed point of the family while the number $t_1 = 1$ is the real fixed point of any of the elements of the family. Firstly, note that, since $q^*(1) = 1$, $t_1 = 1$ is the unique fixed point of q^*, and, also, it is the unique fixed point of each function $q_J(t)$, as $q_J(t) = t$ implies that $t = 1$ for any $J > 0$,. Clearly $\{q_J\}_{J>0}$ and q^* share fixed points. Next, note that if $\varepsilon > 0$ is arbitrary, then for any $J > \dfrac{1}{\varepsilon}$ we have

$$|q_J(0) - 0| = \frac{1}{J} < \varepsilon,$$

yet

$$\lim_{J\to\infty} |q_J(0) - 0| = \lim_{J\to\infty} \left| \frac{1}{J} - 0 \right| = 0,$$

thus we may say that $t_0 = 0$ is "ε-nearly a fixed point" of q_J. However, by no means the points t_0 and t_1 can be considered "close" to each other: being the endpoints of the interval $[0,1]$, their distance is the largest possible. Certainly, all but a finite number of values $q_J(t_0)$ may be found arbitrary close to t_0 (corresponding to $J > \frac{1}{\varepsilon}$), but t_0 certainly remains away from the (real) fixed point t_1 as $J \to \infty$. The similarity between the family of functions $\{q^*, q_J, J > 0\}$ and the family of mappings $\{Q^*, Q_J, J > 0\}$ has become apparent.

We now cite the (very simple) example mentioned earlier of an integral equation (3.1) where both, the function f and the kernel A are constant, thus $|f|$ is bounded while $A \notin L^1([0,\infty))$. Though equation (4.2), below, is rather an "extreme" example being a completely trivial equation, (in fact, there is nothing to be "solved" in it), it demonstrates the ideas above in a convincing way.

Example 4.1. Consider equation (3.1) with constant kernel $A = k > 0$ and constant $f = M$, i.e., consider

$$x(t) = a(t) - \int_0^t kM ds, \quad t \geq 0 \tag{4.2}$$

and let $Q^* : C([0,\infty),\mathbb{R}) \to C([0,\infty),\mathbb{R})$ be the natural transformation of the right hand-side defined by

$$Q^*(\phi)(t) := a(t) - \int_0^t kM ds, \quad t \geq 0. \tag{4.3}$$

Clearly A being a positive constant may be regarded as a convolution kernel, yet it satisfies (A-1)–(A-3). It is straightforward that equation (4.2) has the solution

$$x(t) = a(t) - kMt, \quad t \geq 0, \tag{4.4}$$

from which it immediately follows that the difference $|x(t) - a(t)|$ becomes arbitrarily large as t tends to infinity.

Now for an arbitrary $J > 0$ the transformed equation (3.2) is

$$x(t) = a(t) - \int_0^t R_J(t-s)\,a(s) + \int_0^t R_J(t)\left[x(s) - \frac{M}{J}\right] ds, \tag{4.5}$$

and the operators Q_J and P_J are given by

$$P_J(x)(t) := a(t) - \int_0^t R_J(t-s)\,a(s) + \int_0^t R_J(t)\,x(s)\,ds,$$

and

$$Q_J(x)(t) := a(t) - \int_0^t R_J(t-s)\,a(s) + \int_0^t R_J(t)\left[x(s) - \frac{M}{J}\right] ds, \tag{4.6}$$

with the resolvent kernel R_J of $JA = Jk$ being the solution of the equation

$$R_J(t) = Jk - \int_0^t JkR_J(s)\,ds, \quad t \geq 0.$$

Since $R_J(t)$ is continuous for any $J > 0$ (because $A(t)$ is), we may differentiate this resolvent equation and see that R_J is a solution to the initial value problem

$$R'_J(t) = -JkR_J(t), \quad R_J(0) = Jk,$$

and we easily find

$$R_J(t) = Jke^{-Jkt}, \quad t \geq 0, \quad (J > 0).$$

Thus, in the case of equation (4.2), we are able to obtain the analytic expressions of the family of kernels $\{R_J(t), t \geq 0\}_{J>0}$. Note that

$$\int_0^t R_J(s)\,ds = \left[-e^{-Jks}\right]_{s=0}^{s=t} = 1 - e^{-Jkt}, \quad t \geq 0, \ (J > 0). \tag{4.7}$$

To verify the results of Section 2 along with (4.1), we recall that the (unique) fixed point of each of the mappings P_J is the function a. It is straightforward that, for an arbitrary $\varepsilon > 0$, we may take $J > \dfrac{M}{\varepsilon}$ and have

$$|Q_J(a)(t) - a(t)| = \frac{M}{J}\int_0^t R_J(s)\,ds = \frac{M}{J}\left[1 - e^{-Jkt}\right] \leq \frac{M}{J},$$

so

$$\|Q_J(a) - a\| < \varepsilon,$$

i.e., a is ε-nearly a fixed point of Q_J, yet $\lim\limits_{J\to\infty} \|Q_J(a) - a\| = 0$, and (4.1) follows.

Next, note that in view of (4.2) and (4.3), we have

$$Q^*(\phi)(t) := a(t) - \int_0^t kM ds = a(t) - kMt := x(t),$$

so the function x is a fixed point of Q^*, and, consequently, of Q_J for all $J > 0$; in particular x is a fixed point of Q_n for any $n \in \mathbb{N}$. Indeed, substituting x in the right hand side of the equation (4.6) (with z_J as in (3.3)), we have

$$\begin{aligned}
& a(t) - \int_0^t R_J(t-s)\,a(s)\,ds + \int_0^t R_J(t-s)\left[x(s) - \frac{M}{J}\right] ds \\
= \ & a(t) - \int_0^t R_J(t-s)\,a(s)\,ds + \int_0^t R_J(t-s)\left[a(s) - kMs - \frac{M}{J}\right] ds \\
= \ & a(t) - kM\int_0^t R_J(s)(t-s)\,ds - \frac{M}{J}\int_0^t R_J(s)\,ds
\end{aligned}$$

$$\begin{aligned}
&= a(t) - kM \int_0^t \left[\int_0^s R_J(u)\,du\right] ds - \frac{M}{J}\int_0^t R_J(s)\,ds \\
&= a(t) - kM \int_0^t \left[1 - e^{-kJs}\right] ds - \frac{M}{J}\left(1 - e^{-kJt}\right) ds \\
&= a(t) - kMt + \frac{kM}{Jk}\left(1 - e^{-kMt}\right) - \frac{M}{J}\left(1 - e^{-kMt}\right) ds \\
&= a(t) - kMt = x(t)
\end{aligned}$$

which verifies that the solution of (4.2) also satisfies (4.5) for any $J > 0$.

We may explain the whole situation by firstly looking at (4.1): one expects that if the limit as $J \to \infty$ is achieved, then a fixed point or the limit mapping would obtained. As Q^* along with all members of the family $\{Q_J\}_{J>0}$ share fixed points, this limiting process might lead to a fixed point of Q^*, i.e., a solution of (3.1) could be obtained. However, this procedure does not lead to what expected: as J tends to infinity the fraction $\frac{|f|}{J}$ becomes zero resulting that the integral in the limit mapping, say Q_∞, simply disappears. We can easily see that what remains in Q_∞ is nothing else but a, exactly what is described in (4.1). Therefore, while all members of the family $\{Q_J\}_{J>0}$ do share fixed points, the limit mapping Q_∞ does not share fixed points with any of the members of the family $\{Q_J\}_{J>0}$ Sharing fixed points is not the only property of the family of mappings $\{Q_J\}_{J>0}$ lost at the limit as $J \to \infty$: as it has already been shown in Section 3, continuity of the limiting kernel as $J \to \infty$ is also lost in a severe way since R_∞ blows up at $t = 0$ but vanishes at all positive reals. We may illustrate this situation (along with the results of Section 3) by means of Example 4.1, since the resolvents and the fixed point region of (4.2) may analytically be calculated. We find

$$\lim_{J\to\infty} R_J(t) = \lim_{J\to\infty} Jke^{-Jkt} = 0,\ t > 0, \quad \lim_{J\to\infty} R_J(0) = \lim_{J\to\infty} Jk = \infty,$$

and

$$\lim_{J\to\infty} \int_0^T R_J(s)\,ds = \lim_{J\to\infty} \left[1 - e^{-JkT}\right] = 1,$$

from which we immediately see that the results (i)-(iv) in Proposition (3.5) are verified. (Note that $\lim_{t\to\infty} R_J(t) \int_0^t A(s)\,ds = \lim_{t\to\infty} Jke^{-Jkt}kt = 0$, while $\lim_{t\to\infty} A(t) = k \neq 0$.).

In turn, we have

$$\begin{aligned}
W_T \quad : &= \inf_{J>0} \frac{M\int_0^T R_J(s)\,ds}{J\left[1 - \int_0^T R_J(s)\,ds\right]} = M \inf_{J>0} \frac{1 - e^{-JkT}}{Je^{-JkT}} \\
&= M \inf_{J>0} \frac{e^{JkT} - 1}{J},
\end{aligned}$$

and we consider the function

$$g(x) := \frac{e^{kTx} - 1}{x}, \quad x > 0$$

for which we find

$$g'(x) = \frac{xkTe^{kTx} - e^{kTx} + 1}{x^2} = \frac{e^{kTx}(xkT - 1) + 1}{x^2}.$$

It is not difficult to see that $g'(x) \geq 0$, $x > 0$, thus g is increasing on $(0, \infty)$ and so does the function $w_T(J)$ is increasing on $(0, \infty)$. It follows that the infimum of $w_T(J)$ is attained as its limit for $J \to 0+$, so

$$W_T = M \inf_{J>0} \frac{e^{JkT} - 1}{J} = M \lim_{J \to 0+} \frac{e^{JkT} - 1}{J} = kTM.$$

Furthermore, for the denominator of $W_T(J)$ it holds

$$\lim_{J \to \infty, 0+} J \left[1 - \int_0^T R_J(s)\, ds \right] = \lim_{J \to \infty, 0+} J e^{-JkT} = 0,$$

and, as a consequence, we take

$$\lim_{J \to \infty} W_T(J) = \lim_{J \to \infty} \frac{M \int_0^T R_J(s)\, ds}{J \left[1 - \int_0^T R_J(s)\, ds \right]} = \infty,$$

which certainly agrees with the fact that for the fixed point x we have $|x(t)| \to \infty$ as $T \to \infty$.

A natural question arising concerns the comparison between the radius W_T given in Theorem 3.3, and, the number r_T calculated directly from (1.1). While in the case of equation (4.2), we have already found that $r_T = MkT = W_T$ (so both procedures give the same result), a comparison between the results given by these two procedures in the general case of A being non-constant is an open question. Further open questions may concern the family of kernels $\{R_J\}_{J>0}$ as well as properties (e.g., continuity, differentiation, behaviour, e.t.c.) of the function W_T.

5 Appendix: The Transformation

We now consider the conditions (A1)–(A3) found in Miller [9, pp. 209–213].

Conditions (A1)–(A3) are defined as follows:

(A1) $A \in C(0, \infty) \cap L^1(0, 1)$.

(A2) $A(t)$ is positive and non-increasing for $t > 0$.

(A3) For each $T > 0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0 < t < \infty$.

In those references above it is shown that the resolvent equation is

$$R(t) = A(t) - \int_0^t A(t-s)R(s)ds$$

and that its solution R is continuous on $(0, \infty)$ and

$$0 < R(t) \le A(t), \qquad \int_0^\infty R(t)dt = 1 \tag{5.1}$$

when the integral of A is infinite. When the integral of A is finite, then the integral of R is less than one.

Notice that if J is a positive constant, then $JA(t)$ still satisfies (A1)–(A3). We started with G a bounded set and asked that our mapping mapped G into itself. A difficulty could occur because the integral $\int_0^t A(t-s)f(s, x(s))ds$ may map bounded sets into unbounded sets. If we could possibly exchange $R(t)$ for $A(t)$ then we could map bounded sets into bounded sets. That is exactly what we do and the transformation can be reversed so that the transformed equation has the same solutions as the original equation.

In a sequence of papers we showed the advantages of transforming the standard integral equation

$$x(t) = a(t) - \int_0^t A(t-s)f(s, x(s))ds \tag{5.2}$$

using a variation of parameters formula of Miller [9, pp. 191–192] into

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s, x(s))}{J}\right]ds, \tag{5.3}$$

with

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds. \tag{5.4}$$

Here are the steps. Start with (5.2) and $a(t)$ continuous on $[0, \infty)$ while A satisfies (A1)–(A3) and J is an arbitrary positive constant. It is only later that we restrict J. We then have

$$\begin{aligned} x(t) &= a(t) - \int_0^t A(t-s)[Jx(s) - Jx(s) + f(s, x(s))ds] \\ &= a(t) - \int_0^t JA(t-s)x(s)ds + \int_0^t JA(t-s)\left[x(s) - \frac{f(s, x(s))}{J}\right]ds. \end{aligned}$$

The linear part is

$$z(t) = a(t) - \int_0^t JA(t-s)z(s)ds \tag{5.5}$$

and the resolvent equation is

$$R(t) = JA(t) - \int_0^t JA(t-s)R(s)ds \tag{5.6}$$

so that by the linear variation-of-parameters formula we have

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds \tag{5.7}$$

and by the non-linear variation of parameters formula [9, pp. 191–193]

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds. \tag{5.8}$$

We will always write this as

$$x(t) = a(t) + \int_0^t R(t-s)\left[x(s) - a(s) - \frac{f(s,x(s))}{J}\right]ds. \tag{5.9}$$

The transformation from (5.2) to (5.9) was first given in [2] for a Caputo equation in which case there are few difficulties. Further discussion of the transformation is found in [1] which allows $a(t)$ to be singular. In that reference the reader can follow from (2.2) on p. 249 to its transformed form on p. 263.

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