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Nonlocal Conditions for Two-Endpoint Problems

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Abstract

We discuss existence and uniqueness results for two-endpoint problems for a class of nonlinear second order differential equations with a nonlocal condition of integral type at the initial point and zero value at the terminal point. Using Green's function, we convert our problem into a nonlinear integral equation. We shall rely on fixed point theorems and the method of lower and upper solutions to prove our main results.

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1 Introduction

We investigate a class of boundary value problems for nonlinear second order differential equations on a finite interval together with a nonlocal condition of integral type at the initial point and the unknown function takes zero value at the terminal point. More specifically, we consider the differential equation

$$u''(t) + p(t)u(t) = f(t, u(t)), \quad t \in [0, 1] \tag{1.1}$$

and the conditions

$$u(0) = \int_0^1 g(u(s))ds, \tag{1.2}$$

and

$$u(1) = 0, \tag{1.3}$$

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where the functions p, f and g satisfy conditions that will be specified later. Condition (1.2) is unusual in the sense that the value at the initial point depends on the unknown function. To have a feeling about condition (1.2) consider the displacement of a particle under the influence of an external force m, during a time interval which we normalize to be $[0, 1]$. If $u(t)$ denotes the displacement at time t, then the motion is governed by the equation

$$u''(t) + p(t)u(t) = m(t), \ \ t \in [0,1] .$$

We complement this equation by conditions at $t = 0$ and $t = 1$. The value $u(0)$ may be imprecise; so we make several measurements at successive times $0 < t_1 < t_2 < \ldots < t_n$ and then specify the value $u(0)$ to be the average value

$$u(0) = \frac{1}{n}\sum_{i=1}^{n} u(t_i), \tag{1.4}$$

or

$$u(0) = \sum_{i=1}^{n} a_i u(t_i), \tag{1.5}$$

where a_i, $i = 1, 2, \ldots, n$ are positive real numbers. Of course the time interval $[0, t_n]$ must be very small compared to the whole interval $[0, 1]$. It is clear that condition (1.2) is more general than (1.5). Problem (1.1), (1.2), (1.3) is called a nonlocal boundary value problem. It was pointed out in [13] that nonlocal boundary conditions were introduced for the first time by Picone in 1908 for linear differential systems. A good account on nonlocal boundary problems for second order differential equations can be found in the survey [12]. One can also consult the papers [2, 4, 8–11]. The papers [1, 3, 19] deal with nonlocal conditions of integral type. The paper is organized as follows. In Section 2, we introduce notations, definitions and results that will be used throughout the paper. Section 3 is devoted to our main results, in fact, we study problem (1.1), (1.2), (1.3) by the method of fixed point theorems. The method of lower and upper solutions will be developed in Section 4. Some concluding remarks are the object of Section 5.

2 Preliminaries

In this section, we introduce notations, definitions and results that will be used in the rest of the paper. Let I denote the compact real interval $[0, 1]$. For $k = 0, 1, \ldots$, $C^k(I)$ is the space of real valued functions which are continuous together with their derivatives up to order k. Let $X := C^0(I)$ be equipped with the sup-norm

$$\|u\|_0 = \max\{|u(t)| : \ t \in I\}.$$

Then X is a Banach space.

For $p, m \in X$ and $A \in \mathbb{R}$, a nonzero constant, consider the linear nonhomogeneous problem

$$u''(t) + p(t)u(t) = m(t),\ t \in I, \tag{2.1}$$

$$u(0) = A, \tag{2.2}$$

$$u(1) = 0. \tag{2.3}$$

The following assumption on the function p will be used throughout the paper and shall not be repeated.

(A_p) $p \in X$, $p(t) \le \pi^2$ for all $t \in I$ and $p(t) \ne \pi^2$ on a subset of I of positive measure.

This is an assumption of nonresonance type.

Lemma 2.1. *If* (A_p) *is satisfied, then the homogeneous problem i.e. with data* $m = 0$, $A = 0$, *corresponding to* (2.1)*,* (2.2)*,* (2.3) *has only the trivial solution.*

Proof. The lemma was proved in [5], but we give it here for the sake of completeness. Assume on the contrary that the homogeneous problem has a nontrivial solution u_0. It follows from (2.1) with $m = 0$ that

$$u_0''(t)u_0(t) + p(t)u_0(t)^2 = 0,\ t \in I. \tag{2.4}$$

Integrating (2.4) on I and using the homogeneous boundary conditions, we obtain

$$\int_0^1 \left(u_0'(t)^2 - p(t)u_0(t)^2\right) dt = 0.$$

The assumption on p implies that

$$0 = \int_0^1 \left(u_0'(t)^2 - p(t)u_0(t)^2\right) dt > \int_0^1 \left(u_0'(t)^2 - \pi^2 u_0(t)^2\right) dt. \tag{2.5}$$

A slight modification of a nice trick in [15, Example 5.17, page 166] shows that

$$\int_0^1 \left(u_0'(t)^2 - \pi^2 u_0(t)^2\right) dt = \int_0^1 \left(u_0'(t) - \pi u_0(t) \cot(\pi t)\right)^2 dt \ge 0. \tag{2.6}$$

It follows that the functional $J : X_0 := \{u \in X; u(0) = u(1) = 0\} \to \mathbb{R}$, defined by $J(y) = \int_0^1 \left(y'(t)^2 - \pi^2 y(t)^2\right) dt$ is nonnegative and achieves its minimum along the curve $y_0(t) = \sin \pi t$ and the minimum value is $J(y_0) = 0$. It follows from (2.5) that

$$0 = \int_0^1 \left(u_0'(t)^2 - p(t)u_0(t)^2\right) dt > \min_{y \in X_0} J(y) = 0.$$

This contradiction proves the lemma. The corresponding Green's function $G(t, s)$, $(t, s) \in I \times I$ exists and is constructed as follows. See [17, Section 3.2] and [20, Chapter

4]. For a different approach see [14,18]. Let $u_1, u_2 \in C^2(I)$ be two linearly independent solutions of $u''(t) + p(t)u(t) = 0$, $t \in I$, with $u_1(0) = 0$ and $u_2(1) = 0$. Then necessarily, $u_1(1) \neq 0$ and $u_2(0) \neq 0$. Moreover u_1 and u_2 are uniformly bounded and their Wronskian $W(u_1, u_2)$ is constant, which we denote by W_0. Then for all $(t,s) \in I \times I$

$$G(t,s) = \begin{cases} W_0^{-1} u_1(t) u_2(s), & t < s \\ W_0^{-1} u_1(s) u_2(t), & s < t. \end{cases}$$

Problem (2.1), (2.2), (2.3) has a unique solution given by

$$u(t) = \frac{u_2(t)}{u_2(0)} A + \int_0^1 G(t,s) m(s) ds, \ \forall t \in I. \tag{2.7}$$

Let

$$c_0 = \max_{t \in I} \left| \frac{u_2(t)}{u_2(0)} \right|$$

and

$$G_0 = \max_{(t,s) \in I \times I} |G(t,s)| .$$

Define a linear operator

$$\Gamma : C^2(I) \times \mathbb{R}^2 \to X \times \mathbb{R}^2$$

by

$$\Gamma u(t) := \left(u''(t) + p(t)u(t),\ u(0),\ u(1) \right), \ \forall t \in I.$$

Then problem (2.1), (2.2), (2.3) is equivalent to the equation

$$\Gamma u(t) = (m(t), A, 0), \quad \forall t \in I.$$

Condition (A_p) implies that Γ is one-to-one and onto. Moreover

$$\Gamma^{-1} (m(\cdot), A, 0)(t) = \frac{u_2(t)}{u_2(0)} A + \int_0^1 G(t,s) m(s) ds, \ \forall t \in I. \tag{2.8}$$

Since A is constant, u_2 is uniformly bounded and $u_2(0) \neq 0$, the properties of Green's function shows that Γ^{-1} is uniformly bounded and equicontinuous. Moreover

$$\left\| \Gamma^{-1} (m(\cdot), A, 0) \right\| \leq c_0 |A| + G_0 \|m\|_0 .$$

The following result is of independent interest. Let g be the function from (2.2). Define a nonlinear functional $G : X \to \mathbb{R}$ by $G(u) = \int_0^1 g(u(s)) ds.$ □

Lemma 2.2. *Assume that there exists* $L_g \in (0,1)$ *such that* $|g(u) - g(v)| \leq L_g |u - v|$, *for all* $u, v \in \mathbb{R}$. *Then* $(I - G(\cdot))^{-1}$ *exists and is Lipschitz with constant* $(1 - L_g)^{-1}$.

Proof. Let $\Psi(u) = u - G(u)$. The equation $\Psi(u) = y$ has a unique solution if and only if $u = G(u) + y$ has a unique solution. But, this follows from the fact that the map H_0 defined by $H_0(u) = G(u) + y$ is a contraction. Indeed

$$\|H_0(u) - H_0(v)\|_0 = \|G(u) + y - G(v) - y\|_0 \le L_g \|u - v\|_0 .$$

Hence Ψ is one-to-one and onto. Thus Ψ^{-1} exists and satisfies

$$\left\|\Psi^{-1}(x) - \Psi^{-1}(z)\right\|_0 \le \frac{\|x - z\|_0}{1 - L_g}.$$

To see this, let $u = \Psi^{-1}(x)$ and let $v = \Psi^{-1}(z)$ so that $u = H_0(u) + x$ and $v = H_0(v) + z$. This implies that

$$\begin{aligned} \|u - v\|_0 &= \|H_0(u) + x - H_0(v) - z\|_0 \le \|H_0(u) - H_0(v)\|_0 + \|x - z\|_0 \\ &\le L_g \|u - v\|_0 + \|x - z\|_0 . \end{aligned}$$

So that

$$(1 - L_g)\|u - v\|_0 \le \|x - z\|_0 ,$$

from which the result follows. □

3 Main Results

In this section, we shall study the nonlinear problem (1.1), (1.2), (1.3) under some specific conditions on the nonlinearities f and g. It follows from (2.7) that we have

Definition 3.1. $u \in C^2(I)$ is a solution of (1.1), (1.2), (1.3) if and only if $u \in X$ is a solution of the nonlinear integral equation

$$u(t) = \frac{u_2(t)}{u_2(0)} \int_0^1 g(u(s))ds + \int_0^1 G(t,s) f(s,u(s))ds, \ \forall t \in I. \tag{3.1}$$

Let

$$H(u)(t) := \frac{u_2(t)}{u_2(0)} \int_0^1 g(u(s))ds$$

and

$$F(u)(t) := \int_0^1 G(t,s) f(s,u(s))ds,$$

for all $t \in I$. Then (3.1) is equivalent to the abstract equation

$$u = H(u) + F(u). \tag{3.2}$$

This is a fixed point equation. Our next results will be based on Banach contraction principle, Krasnoselskiĭ theorem and Schaefer theorem [16]. We shall start with the following conditions on the data f, g.

(H$_1$) There exists $L_0 > 0$ such that $|g(u) - g(v)| \leq L_0 |u - v|$, for all $u, v \in \mathbb{R}$,

(H$_2$) $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exists $L_f > 0$ such that $|f(t, u) - f(t, v)| \leq L_f |u - v|$, for all $u, v \in \mathbb{R}$, $t \in I$.

(H$_3$) $L_0 c_0 + L_f G_0 < 1$.

Theorem 3.2. *Assume (H$_1$), (H$_2$), (H$_3$) hold. Then problem* (1.1), (1.2), (1.3) *has a unique solution* $u^* \in C^2(I)$.

Proof. The operator $H(\cdot) + F(\cdot) : X \to X$ is a contraction. Indeed, for all $u, v \in X$, we have

$$|(H(u) - H(v))(t)| = \left| \frac{u_2(t)}{u_2(0)} \int_0^1 [g(u(s)) - g(v(s))] ds \right|$$
$$\leq c_0 \int_0^1 |g(u(s)) - g(v(s))| \, ds \leq L_0 c_0 \int_0^1 |u(s) - v(s)| \, ds$$
$$\leq L_0 c_0 \|u - v\|_0 .$$

Hence

$$\|H(u) - H(v)\| \leq L_0 c_0 \|u - v\|_0 . \tag{3.3}$$

Similarly, we prove

$$\|F(u) - F(v)\|_0 \leq L_f G_0 \|u - v\|_0 . \tag{3.4}$$

Adding (3.3) and (3.4), we see that

$$\|(H + F)(u) - (H + F)(v)\|_0 \leq (L_0 c_0 + L_f G_0) \|u - v\|_0 .$$

Condition (H$_3$) implies that the operator $H + F$ is a contraction. By Banach contraction principle it has a unique fixed point $u^* \in X$, which is the unique solution to (3.2). So that $u^* \in X$ is the unique solution of (3.1), which in turn is the unique solution of (1.1), (1.2), (1.3). Moreover $u^* \in C^2(I)$. The proof is complete. □

Remark 3.3. It follows from (H$_3$) that $L_0 c_0 < 1$ and $(1 - L_0 c_0)^{-1} L_f G_0 < 1$. It follows from Lemma 2 that $(I - H(\cdot))^{-1}$ exists and is Lipschitz with constant $(1 - L_0 c_0)^{-1}$. Moreover, (3.2) is equivalent to

$$u = (I - H(\cdot))^{-1} F(u). \tag{3.5}$$

The condition $(1 - L_0 c_0)^{-1} L_f G_0 < 1$ implies that the operator $(I - H(\cdot))^{-1} F(\cdot) : X \to X$ is a contraction, which has a unique fixed point in X. Thus (1.1), (1.2), (1.3) has a unique solution.

Next, we assume

(H$_4$) $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $L_1 \in \left(0, c_0^{-1}\right)$ and $g(0) = 0$, (H$_5$) $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exists a function $\psi : \mathbb{R}^2 \to (0, +\infty)$ continuous and nondecreasing with respect to its second argument such that for all $(t, u) \in \mathbb{R}^2$ $|f(t, u)| \le \psi(t, |u|)$,

(H$_6$) there exists $R^* > 0$ such that

$$\frac{G_0}{(1 - L_1 c_0)} \int_0^1 \psi(t, R^*)\, dt < R^*. \tag{3.6}$$

Theorem 3.4. *If* (*H*$_4$)*,* (*H*$_5$)*,* (*H*$_6$) *are satisfied, then* (1.1), (1.2), (1.3) *has at least one solution.*

Proof. With R^* from (H$_6$) let $M := \{u \in X;\ \|u\|_0 \le R^*\}$. Then M is a closed, bounded and convex subset of X. It follows from (H$_4$) that the operator H is a contraction. Moreover, if $u \in M$, then $H(u) \in M$. For

$$|H(u)(t)| = \left| \frac{u_2(t)}{u_2(0)} \int_0^1 g(u(s)) ds \right|$$

$$\le \left| \frac{u_2(t)}{u_2(0)} \right| \int_0^1 |g(u(s))|\, ds \le c_0 L_1 \|u\|_0 < R^*$$

since $c_0 L_1 < 1$. Hence $\|H(u)\|_0 \le R^*$. Next, we show that the operator $F : X \to X$ is compact, i.e. uniformly bounded and equicontinuous. The equicontinuity follows from the compactness of the interval I and the uniform continuity of Green's function $G(t, s)$. Also, if $u \in M$, then

$$|F(u)(t)| = \left| \int_0^1 G(t, s) f(s, u(s)) ds \right| \le G_0 \int_0^1 \psi(t, |u(t))|)\, dt \le G_0 \int_0^1 \psi(t, R^*)\, dt.$$

Thus

$$\|F(u)\|_0 \le K := G_0 \int_0^1 \psi(t, R^*)\, dt.$$

Notice that K is a constant independent of u. To complete the proof let $u, v \in M$, we show that $H(u) + F(v) \in M$. We have

$$\|H(u) + F(v)\|_0 \le \|H(u)\|_0 + \|F(v)\|_0$$

$$\le c_0 L_1 \|u\|_0 + G_0 \int_0^1 \psi(t, R^*)\, dt$$

$$\le c_0 L_1 R^* + G_0 \int_0^1 \psi(t, R^*)\, dt$$

$$\le c_0 L_1 R^* + (1 - c_0 L_1) R^* = R^*.$$

All the conditions of Krasnoselskiĭ fixed point theorem are satisfied. We conclude that the operator $H + F$ has at least a fixed point in X, which means that (1.1), (1.2), (1.3) has at least one solution. □

Our final result in this section will be based on the following assumptions.

(H$_7$) $g : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing,

(H$_8$) $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exists a function $\phi : \mathbb{R}^2 \to (0, +\infty)$ continuous and nondecreasing with respect to its second argument such that for all $(t, u) \in \mathbb{R}^2$ $|f(t, u)| \leq \phi(t, |u|)$,

(H$_9$) $\limsup\limits_{\rho \to \infty} \dfrac{c_0 g(\rho) + G_0}{\rho} \displaystyle\int_0^1 \phi(t, \rho)\, dt > 1.$

Theorem 3.5. *Assume that (H$_7$), (H$_8$), (H$_9$) hold. Then* (1.1), (1.2), (1.3) *has at least one solution.*

Proof. Condition (H$_9$) implies that there is $\rho^* > 0$ such that for all $\rho > \rho^*$, we have

$$\frac{c_0 g(\rho) + G_0}{\rho} \int_0^1 \phi(t, \rho)\, dt > 1. \tag{3.7}$$

Let $\rho_0 = \|u\|_0$. It follows from (3.1) that

$$\rho_0 \leq c_0 g(\rho_0) + G_0 \int_0^1 \phi(t, \rho_0)\, dt.$$

Hence

$$1 \leq \frac{c_0 g(\rho_0) + G_0}{\rho_0} \int_0^1 \phi(t, \rho_0)\, dt. \tag{3.8}$$

Comparing (3.7) and (3.8), we see that $\rho_0 \leq \rho^*$. Let $\Omega := \{u \in X;\ \|u\|_0 \leq \rho^*\}$. Then Ω is a closed, bounded and convex subset of X and the operator $H + F : X \to X$ is continuous and $(H + F)(\Omega)$ is compact. The continuity follows from that of the data f, g. Also, $(H + F)(\Omega)$ is uniformly bounded. For, let $u \in \Omega$, then $\|u\|_0 \leq \rho^*$ and from (3.1)

$$|(H + F)(u)(t)| \leq \left|\frac{u_2(t)}{u_2(0)}\right| \int_0^1 |g(u(s))|\, ds + \left|\int_0^1 G(t, s) f(s, u(s)) ds\right|.$$

The fact that g is nondecreasing and (H$_8$) imply that

$$|(H + F)(u)(t)| \leq c_0 g(\rho^*) + G_0 \int_0^1 \phi(t, \rho^*)\, dt := \ell,$$

so that

$$\|(H + F)(u)\|_0 \leq \ell,\ \forall u \in \Omega. \tag{3.9}$$

Next, we show that $(H + F)$ is equicontinuous. But, this follows from the compactness of the interval I and the uniform continuity of the function u_2 and Green's function $G(\cdot, s)$. Finally, we show that the set of solutions of

$$u = \lambda (H + F)(u), \tag{3.10}$$

is bounded for all $\lambda \in (0,1)$. So, let u be a solution of (3.10). Then

$$\|u\|_0 = \lambda \|(H+F)(u)\|_0 \leq \ell.$$

Therefore all the conditions of Schaefer's theorem are satisfied and consequently the operator $H+F : X \to X$ has at least one fixed point. This shows that (1.1), (1.2), (1.3) has at least one solution. □

4 Lower and Upper Solutions Method

In this section, we shall rely on the method of lower and upper solutions to prove an existence result for problem (1.1), (1.2), (1.3). For more details on the method, one can consult [6, 7].

Definition 4.1. $\alpha \in C^2(I)$ is a lower solution of (1.1), (1.2), (1.3) if

$$\alpha''(t) + p(t)\alpha(t) \geq f(t,\alpha(t)),\ t \in [0,1], \alpha(0) \leq \int_0^1 g(\alpha(s))ds, \alpha(1) \leq 0.$$

Similarly $\beta \in C^2(I)$ is an upper solution of (1.1), (1.2), (1.3) if the above inequalities are reversed when we substitute β for α.

We shall use the following notation for $U,V \in C^2(I)$ $U \leq V$ means $U(t) \leq V(t)$ for all $t \in I$. Also, $[U,V] := \{v \in C^2(I); U \leq v \leq V\}$. To study our problem by this method, we shall assume, in addition to (H_7) the following

(H_{10}) (1.1), (1.2), (1.3) has a lower solution α, and an upper solution β with $\alpha \leq \beta$,

(H_{11}) $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies $f(t,v_1) - f(t,v_2) - \pi^2(v_1 - v_2) > 0$ for all $t \in I$, $v_1 > v_2$.

Define an operator $\Delta : C^2(I) \to [\alpha,\beta]$ by

$$\Delta(y) = \max\{\alpha, \min(y,\beta)\}.$$

Then $\Delta(y) = \alpha$ if $y \leq \alpha$, $\Delta(y) = y$ if $y \in [\alpha,\beta]$ and $\Delta(y) = \beta$ if $y \geq \beta$. Also, Δ is a continuous and bounded operator.

Theorem 4.2. *If* (H_7)*,* (H_{10})*, and* (H_{11}) *hold, then* (1.1)*,* (1.2)*,* (1.3) *has at least one solution.*

Proof. The proof is given in several steps.

Step 1

The modified problem. Let $h(t,u) = f(t,\Delta(u))$ for all $(t,u) \in I \times C^2(I)$ and $\gamma(u) = g(\Delta(u))$ for all $u \in C^2(I)$. Then γ satisfies (H_7) and h satisfies (H_{11}). Moreover h and γ are bounded. Consider the problem

$$u''(t) + p(t)u(t) = h(t,u(t)), \ \forall t \in I, \tag{4.1}$$

$$u(0) = \int_0^1 \gamma(u(s))\, ds, \ \ u(1) = 0. \tag{4.2}$$

Then for any $\sigma \in \mathbb{R}$, the problem (4.1) together with $u(0) = \sigma$ and $u(1) = 0$ has a unique solution. Suppose, on the contrary it has two solutions v_1, v_2 and $v_1(t) \neq v_2(t)$. Then $v_1(0) = \sigma = v_2(0)$ and $v_1(1) = 0 = v_2(1)$. Let $w(t) = v_1(t) - v_2(t)$ for all $t \in I$. Then $w(0) = w(1) = 0$ and there exists $\tau \in I$ such that either $w(\tau) > 0$ or $w(\tau) < 0$. Assume, without loss of generality, that $w(\tau) > 0$. It follows from the continuity of w that there exists $\rho \in I$ such that $w(\rho) := \max_{t\in I} w(t) > 0$. Then $w'(\rho) = 0$ and $w''(\rho) \leq 0$. Thus,

$$0 \geq w''(\rho) = -p(\rho)w(\rho) + h(\rho, v_1(\rho)) - h(\rho, v_2(\rho)), \forall t \in I. \tag{4.3}$$

It follows from (4.3) and (H_{11}) that

$$0 \geq -\pi^2 (v_1(\rho) - v_2(\rho)) + h(\rho, v_1(\rho)) - h(\rho, v_2(\rho)) > 0.$$

This contradiction shows that $w(t) \leq 0$ for all $t \in I$. Similarly, we show that $w(t) \geq 0$ for all $t \in I$. Therefore $w(t) = 0$ for all $t \in I$, so that $v_1(t) = v_2(t)$ for all $t \in I$. From (2.7), we see that (4.1) with $u(0) = \sigma$ and $u(1) = 0$ is equivalent to

$$u(t) = \frac{u_2(t)}{u_2(0)}\sigma + \int_0^1 G(t,s)h(s,u(s))ds, \ \forall t \in I. \tag{4.4}$$

From (2.8), we can write

$$u(t) = \Gamma^{-1}(h(\cdot,u(\cdot)),\sigma,0)(t), \ \ \forall t \in I. \tag{4.5}$$

Define an operator $T : X \to X$ by $(T(u)(t) =$ the right-hand side of (4.4). From the properties of Γ^{-1} it follows that T is uniformly bounded and equicontinuous. Hence T is completely continuous. Now, we show that the set of solutions of

$$u = \lambda T(u), \ \lambda \in (0,1)$$

is bounded. But, this is clear from the boundedness of u_2, the uniform continuity of Green's function and the boundedness of the function h. By Schaefer theorem T has a fixed point v^*, which is solution of the problem (4.1) together with $u(0) = \sigma$ and $u(1) = 0$. Moreover, we have shown that v^* is unique.

Step 2

Let $y_0 = \alpha$. For $k \in \mathbb{N}$ consider the sequence of problems

$$\begin{cases} y_k''(t) + p(t)y_k(t) = h(t, y_k(t)), \quad \forall t \in I, \\ y_k(0) = \displaystyle\int_0^1 \gamma\left(y_{k-1}(s)\right) ds, \\ y_k(1) = 0. \end{cases} \tag{4.6}$$

Notice that $y_k(0) = \sigma_k$ is a constant independent of the unknown function y_k. Hence, for each k, (4.6) has a unique solution y_k. This process generates a sequence (y_k) of functions in X. It follows from (3.1) that

$$y_k(t) = \frac{u_2(t)}{u_2(0)} \int_0^1 \gamma(y_{k-1}(s))ds + \int_0^1 G(t,s)h(s, y_k(s))ds, \ \forall t \in I. \tag{4.7}$$

Recall from Step 1 that γ and h are bounded. Then there exists $M_\gamma > 0$ and $M_h > 0$ such that

$$|\gamma(y_{k-1}(s))| \leq M_\gamma \text{ and } |h(s, y_k(s))| \leq M_h \text{ for all } k \in \mathbb{N} \text{ and } s \in I.$$

Therefore

$$\|y_k\|_0 \leq c_0 M_\gamma + G_0 M_h, \text{ for all } k \in \mathbb{N}.$$

This shows that the sequence (y_k) is uniformly bounded. Thus, it has a uniformly convergent subsequence $\left(y_{k_j}\right)$. Now, (y_{kj-1}) may not converge to the same function as $\left(y_{k_j}\right)$. Using a diagonalizing process, we may conclude that $\lim\limits_{j\to\infty} y_{k_j} = \lim\limits_{j\to\infty} y_{k_{j-1}} = y^*$. It follows from (4.7) that

$$y^*(t) = \frac{u_2(t)}{u_2(0)} \int_0^1 \gamma(y^*(s))ds + \int_0^1 G(t,s)h(s, y^*(s))ds, \ \forall t \in I.$$

From the definition of γ and h, we have

$$y^*(t) = \frac{u_2(t)}{u_2(0)} \int_0^1 g(\Delta(y^*(s)))ds + \int_0^1 G(t,s)f(s, \Delta(y^*(s)))ds, \ \forall t \in I. \tag{4.8}$$

Step 3

We show that $y^* \in [\alpha, \beta]$ to complete the proof of our result. We only prove $\alpha \leq y^*$. In a similar way, we prove $y^* \leq \beta$. To prove that $\alpha(t) \leq y^*(t)$ for all $t \in I$, we put $\theta(t) = \alpha(t) - y^*(t)$, and show that $\theta(t) \leq 0$. We proceed by contradiction and assume there is $\xi \in I$ such that $\theta(\xi) > 0$. Then there exists $\xi_0 \in I$ such that $\max\limits_{t\in I}\theta(t) = \theta(\xi_0) > 0$. It follows that $\theta'(\xi_0) = 0$ and $\theta''(\xi_0) \leq 0$. Hence

$$0 \geq \theta''(\xi_0) = \left(\alpha''(\xi_0) - y^{*\prime\prime}(\xi_0)\right).$$

From the definition of α and y^* it follows that

$$0 \geq \left(\alpha''(\xi_0) - y^{*\prime\prime}(\xi_0)\right),$$

$$0 \geq -p(\xi_0)\alpha(\xi_0) + f(\xi_0, \alpha(\xi_0)) + p(\xi_0)y^*(\xi_0) - f(\xi_0, y^*(\xi_0)),$$

$$0 \geq -p(\xi_0)\left((\alpha(\xi_0) - y^*(\xi_0))\right) + \left(f(\xi_0, \alpha(\xi_0)) - f(\xi_0, y^*(\xi_0))\right),$$

$$0 \geq -\pi^2\left((\alpha(\xi_0) - y^*(\xi_0))\right) + \left(f(\xi_0, \alpha(\xi_0)) - f(\xi_0, y^*(\xi_0))\right) > 0.$$

We arrive at a contradiction. Hence $\alpha(t) \leq y^*(t)$ for all $t \in I$. Now, if $\xi_0 = 0$, then

$$\alpha(0) - y^*(0) \leq \int_0^1 g(\alpha(s))ds - \int_0^1 g(y^*(s))ds$$

$$= \int_0^1 \left(g(\alpha(s)) - g(y^*(s))\right) ds \leq 0,$$

which follows from (H_7). If $\xi_0 = 1$, then $\alpha(1) - y^*(1) \leq 0$. Similarly, we prove that $y^*(t) \leq \beta(t)$ for all $t \in I$. We infer that $\Delta(y^*(t)) = y^*(t)$, and consequently (4.8) becomes

$$y^*(t) = \frac{u_2(t)}{u_2(0)} \int_0^1 g(y^*(s))ds + \int_0^1 G(t,s)f(s, y^*(s))ds, \ \forall t \in I. \tag{4.9}$$

Comparing (4.9) with (3.1), we conclude that y^* is a solution of (1.1), (1.2), (1.3). The proof of our result in this section is complete. □

5 Concluding Remarks

We could have discussed the same problem with another integral condition at $t = 1$ without any difficulty. Also, one can add a dependence of the data on the first derivative of the unknown function and then we need to assume a Nagumo type condition on the nonlinearity f and some monotony condition on the function g.

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