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Positive Solutions for Second-Order BVPs with an Unintegrable Weight and a Singular Nonlinearity

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Dedicated to Professor J. Henderson on the occasion of his 70th birthday

Abstract

By means of Guo–Krasnoselskii's fixed point theorem, we prove an existence result for positive solutions to a certain second-order boundary value problem.

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1 Introduction

Existence theory for boundary value problems (bvp for short) associated with nonlinear ordinary differential equations play an important role in both theory and application and still attract a great deal of interest. For recent developments in the theory, we refer the reader to the monographs [1–3, 11, 12]. Singular bvps arise in the study of many real world problems such as radial solution of nonlinear elliptic equations and in the modelling of many physical phenomena where only positive solutions are meaningful. A nice bibliography on the subject is found in each of the monographs [2, 3, 11], one can see also the papers [4, 5, 10, 13].

This article deals with existence of positive solutions to the second-order boundary value problem (bvp in short)

$$\begin{cases} -u''(t) + q(t)u(t) = \phi(t)\, f(t, u(t))\ t \in (0,1)\,, \\ \lim\limits_{t\to 0} u(t) = \lim\limits_{t\to 1} u(t) = 0, \end{cases} \tag{1.1}$$

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where $q, \phi : (0,1) \to \mathbb{R}^+$ and $f : (0,1) \times (0,+\infty) \to \mathbb{R}^+$ are continuous functions.

In all this work, we assume that the function q satisfies the following hypothesis:

$$\begin{cases} \liminf_{s\to 0} q(s) > 0, \ \liminf_{s\to 1} q(s) > 0, \\ \int_0^{1/2} q(s)ds = \int_{1/2}^1 q(s)ds = +\infty \text{ and} \\ \int_0^1 s(1-s)q(s)ds < \infty. \end{cases} \tag{1.2}$$

Our approach in this work is based on a fixed point formulation and since the nonlinearity in (1.1) is supposed to be nonnegative, we will use the Guo–Krasnoselskii's version of expansion and compression of a cone principal to prove our main existence result. In the reminder of this section, we recall this powerful theorem and the necessary theoretical background to its statement.

Let $(E, ||.||)$ be a real Banach space. A nonempty closed convex subset C of E is said to be a cone in E if $C \cap (-C) = \{0_E\}$ and $tC \subset C$ for all $t \geq 0$.

Let Ω be a nonempty subset in E. A mapping $A : \Omega \to E$ is said to be compact if it is continuous and $A(\Omega)$ is relatively compact in E.

The Guo–Krasnoselskii's version of expansion and compression of a cone principal in a Banach space consists in the following theorem.

Theorem 1.1. *Let P be a cone in E and let Ω_1, Ω_2 be bounded open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. If $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a compact mapping such that either:*

1. *$\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$, or*
2. *$\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_2$,*

Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

This work is motivated by those in [6, 7], where authors studied existence of nodal solutions for the case where the ordinary differential equation (ode for short) in bvp (1.1) is posed respectively on the half-line and on the real line and subject to Dirichlet boundary conditions. The main hypothesis therein is $\inf_{|t|\geq T} q(t) > 0$ for some T large and notice that satisfying such a condition, the weight q is unintegrable. This situation makes the arguments evoked in [8] for the construction of the Green's function unusable. The second difficulty encountered in this work consists in the fact that the realization of the inequality $\|Tu\| \geq \|u\|$ requires to Green's function to have the property of rising above its maximum (see Assertion 2 in Lemma 2.4). We will see in the following section how the condition of integrability imposed on q in (1.2) works to provide nice framework in which all the difficulties mentioned above are overcome. In the last section is devoted to the main result of this paper and its proof and it is ended by a corollary, deduced from the main theorem, for the particular case of the bvp (1.1) where $f(t,u) = u^\mu$ with $\mu \neq 1$.

2 Fixed Point Formulation

We begin this section by a characterization of the solution to the ode

$$u'' = qu, \tag{2.1}$$

which will be helpful for the construction of the Green's function associated with the bvp (1.1).

Lemma 2.1. *Assume that Hypothesis* (1.2) *holds and let* v *be a nontrivial solution to the ode* (2.1). *Then* v *has at most one zeros and the limits* $\lim_{t\to 0} v(t), \lim_{t\to 1} v(t)$ *exist and are finite.*

Proof. To the contrary, suppose that $v(x_1) = v(x_2) = 0$ with $0 \le x_1 < x_2 \le 1$ and $v > 0$ in (x_1, x_2). In one hand, there exist y_1, y_2 in (x_1, x_2) such that $y_1 < y_2$ and $v'(y_1) > 0 > v'(y_2)$ and in the other hand, we have $v'' = qv \ge 0$ in (x_1, x_2) leading to v' is nondecreasing in (x_1, x_2),then to the contradiction $v'(y_1) \le v'(y_2)$.

Now, let us prove that $\lim_{t\to 1} v(t)$ exist and is finite ($\lim_{t\to 0} v(t)$ is checked similarly). Since v admits at most one zero in $(0, 1)$ one can suppose that $v > 0$ in $(a, 1)$ for some $a \in (0, 1)$ and in such a situation v' is increasing in $(a, 1)$ and does not vanish in $(b, 1)$ with $b \ge a$. Therefore, $\lim_{t\to 1} v(t)$ exist and we distinguish the following two cases:

i) $v' < 0$ in $(b, 1)$, in this case we have $\lim_{t\to 1} v(t) < v(b)$.

ii) $v' > 0$ in $(b, 1)$, in this case for all $s \in (a, 1)$ we have

$$\begin{aligned} v'(s) &= \left(v'(a) + \int_a^s v''(\tau)\, d\tau\right) = \left(v'(a) + \int_a^s q(\tau) v(\tau)\, d\tau\right) \\ &\le \left(v'(a) + v(s) \int_a^s q(\tau)\, d\tau\right) \end{aligned}$$

leading to

$$\frac{v'(s)}{v(s)} \le \frac{v'(a)}{v(s)} + \int_a^s q(\tau)\, d\tau \le \frac{v'(a)}{v(a)} + \int_a^s q(\tau)\, d\tau.$$

Integrating on (a, t), we obtain

$$\ln\left(\frac{v(t)}{v(a)}\right) \le \frac{v'(a)}{v(a)} + \int_a^t \int_a^s q(\tau)\, d\tau ds \le \frac{v'(a)}{v(a)} + \int_a^1 (1-s) q(s)\, ds,$$

and so,

$$v(t) \le v(a) \exp\left(\frac{v'(a)}{v(a)} + \int_a^1 (1-s) q(s)\, ds\right).$$

This ends the proof. □

Hereafter, for any solution φ to the ode (2.1) we set $\varphi(0) = \lim_{t\to 0}\varphi(t)$ and $\varphi(1) = \lim_{t\to 1}\varphi(t)$.

Let φ_1 and φ_2 be the solutions to the ode (2.1) satisfying

$$\varphi_1\left(\frac{1}{2}\right) = 1,\ \varphi_1'\left(\frac{1}{2}\right) = 0,\ \varphi_2\left(\frac{1}{2}\right) = 1 \text{ and } \varphi_2'\left(\frac{1}{2}\right) = -1.$$

It is easy to see that $\varphi_1 \geq 1$ and the Wronksian $W(\varphi_1, \varphi_2) = -1$. Thus, any solution to the ode (2.1) takes the form $u = a\varphi_1 + b\varphi_2$ with $a, b \in \mathbb{R}$.

Since

$$\left(\frac{\varphi_2}{\varphi_1}\right)' = \frac{\varphi_2'\varphi_1 - \varphi_2\varphi_1'}{(\varphi_1)^2} = \frac{1}{(\varphi_1)^2} > 0 \text{ in } (0,1)$$

we have $\frac{\varphi_2(0)}{\varphi_1(0)} < \frac{\varphi_2(1)}{\varphi_1(1)}$ and the system

$$\begin{cases} x\varphi_1(0) + y\varphi_2(0) = 1 \\ x\varphi_1(1) + y\varphi_2(1) = 0 \end{cases}$$

has a unique solution (c_1, c_2).

In all what follows, we let $\Phi_q = c_1\varphi_1 + c_2\varphi_2$ which has the following properties.

Lemma 2.2. *Assume that Hypothesis* (1.2) *holds. Then the function* Φ_q *has the following properties:*

i) $\Phi_q(t) > 0,\ \Phi_q'(t) \leq 0$ *and* $\Phi_q''(t) \geq 0$ *for all* $t \in (0,1)$.

ii) *The function* $\frac{\Phi_q}{\Phi_q'}$ *is bounded near* 1.

Proof. Since Φ_q is a solution to the ode (2.1) and $\Phi_q(1) = 0$, we have from Lemma 2.1 that $\Phi_q > 0$ in $(0,1)$. Consequently, $\Phi_q' \leq 0$ and $\Phi_q'' \geq 0$ in $(0,1)$. From Hypohesis (1.2), we conclude that there is $a \in (0,1)$ such that $\alpha = \inf_{t>a} q(t) > 0$. For all $t \geq a$ we have

$$\begin{aligned}\left(-\Phi_q'(t)\right)^2 &= 2\int_t^1 \Phi_q''(s)\left(-\Phi_q'(s)\right)ds = \int_t^1 q(s)\Phi_q(s)\left(-\Phi_q'(s)\right)ds \\ &\geq \alpha\left(\Phi_q(t)\right)^2.\end{aligned}$$

This leads to

$$\left|\frac{\Phi_q(t)}{\Phi_q'(t)}\right|^2 = \left(\frac{\Phi_q(t)}{-\Phi_q'(t)}\right)^2 \leq \frac{1}{\alpha} \text{ for all } t > a.$$

The proof of Lemma 2.2 is complete. □

Let Ψ_q be the function defined by

$$\Psi_q(t) = \Phi_q(t) \int_0^t \frac{ds}{\Phi_q^2(s)} \text{ for } t \in [0,1)\,.$$

Since $\Psi_q(0) = 0$, $\Psi_q' = q\Psi_q$ and $\Phi_q > 0$ in $(0,1)$, we conclude from Lemma 2.1 that $\Psi_q(1) < \infty$ and $\Psi_q > 0$ in $(0,1]$.

Lemma 2.3. *Assume that Hypothesis* (1.2) *holds, then the function* Ψ_q *has the following properties:*

a) $\Psi_q(t) > 0,\ \Psi_q'(t) < 0$ *and* $\Psi_q''(t) \geq 0$ *for all* $t \in (0,1)$,

b) *For all* $t \in (0,1)\,,\ \Phi_q(t)\Psi_q'(t) - \Psi_q(t)\Phi_q'(t) = 1$,

c) *The function* $\dfrac{\Psi_q}{\Psi_q'}$ *is bounded near* 0.

Proof. Assertions a) and b) are easy to check, so let us prove c). The proof of c) is similar to that of Assertion ii) in Lemma 2.2. □

Let $G_q : [0,1] \times [0,1] \to \mathbb{R}^+$ be the function defined by

$$G_q(t,s) = \begin{cases} \Phi_q\,(t)\,\Psi_q\,(s) & \text{if } 0 \leq t \leq s < 1 \\ \Phi_q\,(s)\,\Psi_q\,(t) & \text{if } 0 \leq s \leq t < 1. \end{cases}$$

Lemma 2.4. *Assume that Hypothesis* (1.2) *holds, then*

1. $G_q(t,s) \leq G_q(s,s) \leq \sup\limits_{0<t<1} \Phi_q\,(t)\,\Psi_q(t) < \infty$ *for all* $t, s \in (0,1)$,
2. $G_q(t,s) \geq \gamma_q\,(t)\,G_q(s,s)$ *for all* $t, s \in (0,1)$ *where*

$$\gamma_q\,(t) = \min\left(1, (\Psi_q\,(1))^{-1}\right) \min\left(\Phi_q\,(t)\,, \Psi_q\,(t)\right).$$

Proof. Assertion 1 is obtained from the monotonicity of the functions Φ_q and Ψ_q. Let us prove Assertion 2. For all $t, s \in (0,1)$, we have

$$\frac{G_q(t,s)}{G_q(s,s)} = \begin{cases} \dfrac{\Psi_q\,(t)}{\Psi_q\,(s)} \geq \dfrac{\Psi_q\,(t)}{\Psi_q\,(1)} \text{ if } t \leq s \\ \dfrac{\Phi_q\,(t)}{\Phi_q\,(s)} \geq \Phi_q\,(t) \text{ if } s \leq t \end{cases} \geq \gamma_q\,(t)\,.$$

This ends the proof. □

In all of this paper, we let E be the linear space defined by

$$E = \left\{ u \in C\left((0,1), \mathbb{R}\right) : \lim_{t\to 0} u(t) = \lim_{t\to 1} u(t) = 0 \right\}.$$

Equipped with the sup-norm denoted by $\|\cdot\|$, E becomes a Banach space. By P we denote the cone of E defined by

$$P = \{u \in E : u(t) \geq \gamma_q(t) \|u\| \text{ for all } t \in (0,1)\}.$$

The following lemma provide a fixed point formulation to the bvp (1.1).

Lemma 2.5. *Assume that Hypothesis* (1.2) *holds and*

$$\begin{cases} \text{for all } \rho > 0 \text{ there exists a nonincreasing function} \\ \Lambda_\rho : (0,+\infty) \to (0,+\infty) \text{ such that} \\ f(t,w) \leq \Lambda_\rho(w) \text{ for all } t \in (0,1) \text{ and all } w \in (0,\rho], \\ \lim_{t\to 0} G(t,t)\phi(t)\Lambda_\rho(r\gamma_q(t)) = \lim_{t\to 1} G(t,t)\phi(t)\Lambda_\rho(r\gamma_q(t)) = 0 \text{ and} \\ \int_0^1 G(s,s)\phi(s)\Lambda_\rho(r\gamma_q(s))\,ds < \infty \text{ for all } r \in (0,\rho]. \end{cases} \tag{2.2}$$

Then there exists a continuous operator $T : P \setminus \{0\} \to P$ such that for all r, R with $0 < r < R$, $T(P \cap (B(0,R) \setminus B(0,r)))$ is relatively compact in E and fixed points of T are positive solutions to the bvp (1.1).

Proof. The is divided into two steps.

Step 1. In this step we prove the existence of the operator T. To this aim let $u \in P \setminus \{0\}$. By means of Hypothesis (2.2) with $R = \|u\|$, for all $t \in (0,1)$ we have from Assertion 1 in Lemma 2.4 and Hypothesis (2.2),

$$\int_0^1 G_q(t,s)\phi(s) f(s,u(s))ds \leq \int_0^1 G_q(s,s)\phi(s)\Lambda_R(R\gamma_q(s))\,ds < \infty.$$

Thus, let v be the function defined by

$$v(t) = \int_0^1 G_q(t,s)\phi(s) f(s,u(s))ds.$$

Clearly, v is continuous on $(0,1)$ and $v(t) > 0$ for all $t \in (0,1)$. Moreover, taking in account the limit in Hypothesis (2.2), Assertion ii) in Lemma 2.2 and Assertion c) in Lemma 2.3, we obtain by means of L'Hôpital's rule

$$\begin{aligned} \lim_{t\to 0} v(t) &\leq \lim_{t\to 0} \Psi_q(t) \int_t^1 \Phi_q(s)\phi(s)\Lambda_R(R\gamma_q(s))\,ds \\ &= \lim_{t\to 0} \left(\frac{\Psi_q(t)}{\Psi_q'(t)} G_q(t,t)\phi(t)\Lambda_R(R\gamma_q(t)) \right) = 0 \end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 1} v(t) &\leq \lim_{t\to 1} \Phi_q(t) \int_0^t \Psi_q(s)\phi(s)\,\Lambda_R(R\gamma_q(s))\,ds \\ &= \lim_{t\to 0}\left(\frac{\Phi_q(t)}{\Phi'_q(t)} G_q(t,t)\phi(t)\,\Lambda_R(R\gamma_q(t))\right) = 0.\end{aligned}$$

Assertion 2 in Lemma 2.4 leads to

$$\begin{aligned}v(t) &= \int_0^1 G_q(t,s)\phi(s)\, f(s,u(s))ds \\ &\geq \gamma_q(t) \int_0^1 G_q(s,s)\phi(s)\, f(s,u(s))ds \geq \gamma_q(t)\,\|u\|\,,\end{aligned}$$

proving that $v \in P$ and the operator $T : P \setminus \{0\} \to P$ where for $u \in P \setminus \{0\}$

$$Tu(t) = \int_0^1 G_q(t,s)\phi(s)\, f(s,u(s))ds,$$

is well defined.

Step 2. Let $R > r > 0$ and Φ be defined by

$$\Phi(s) = \phi(s)\,\Lambda_R(r\gamma_q(s))$$

where Λ_R is the function given by Hypothesis (2.2). Let us prove that the restriction of T to $\Omega = P \cap (B(0,R) \setminus B(0,r))$ is compact. First, we prove that T is continuous on Ω. Let (u_n) be a sequence in Ω such that $\lim\limits_{n\to\infty} u_n = u$. For all $n \geq 1$ we have

$$\begin{aligned}\|Tu_n - Tu\| &= \sup_{t\in(0,1)} (|Tu_n(t) - Tu(t)|) \\ &\leq \int_0^1 G_q(s,s)\,\phi(s)\,|f(s,u_n(s)) - f((s,u(s))|\,ds.\end{aligned}$$

Because of

$$\begin{aligned}&|f(s,u_n(s)) - f(s,u(s))| \to 0, \text{ as } n \to +\infty, \\ &|f(s,u_n(s)) - f((s,u(s))| \leq 2\Phi(s) \text{ for all } s > 0, \\ &\text{and } \int_0^1 G_q(s,s)\,\Phi(s)\,ds < \infty,\end{aligned}$$

Lebesgue's dominated convergence theorem guarantees $\lim\limits_{n\to\infty} \|Tu_n - Tu\| = 0$. Hence, we have proved that T is continuous.

For all $u \in \Omega$, we have

$$\|Tu\| \leq \int_0^1 G_q(s,s)\,\Phi(s)ds,$$

proving that $T\Omega$ is bounded in E.

Let $0 < \eta \leq t_1 \leq t_2 \leq \zeta < 1$. For all $u \in \Omega$, we have

$$\begin{aligned}|Tu\,(t_2) - Tu\,(t_1)| \leq |\Phi_q(t_2) - \Phi_q(t_1)| \int_0^{\zeta} \Psi_q\,(s)\,\Phi(s)ds \\ + \Phi_q(\eta) \int_{t_1}^{t_2} \Psi_q\,(s)\,\Phi(s)ds \\ + |\Psi_q(t_2) - \Psi_q(t_1)| \int_{\eta}^{1} \Phi_q(s)\Phi(s)ds \\ + \Psi_q\,(\zeta) \int_{t_1}^{t_2} \Phi_q\,(s)\,\Phi(s)ds.\end{aligned}$$

The above estimate proves that $T\Omega$ is equicontinuous on compact intervals of $(0, 1)$.

For all $u \in \Omega$ and $t > 0$, we have

$$Tu(t) \leq \int_0^1 G_q(t, s)\Phi(s)ds = H(t)$$

Arguing as in Step 1, we obtain from Hypothesis (2.2) that

$$\lim_{t\to 0} H(t) = \lim_{t\to 1} H(t) = 0,$$

proving the equiconvergence of $T\Omega$.

In view of Corduneanu's compactness criterion [9, p. 62], $T\Omega$ is relatively compact in E.

By simple computations, we see that fixed points of T are positive solutions to the bvp (1.1), ending the proof of the lemma. □

3 Main Result

The statement of the main result needs to introduce the following notations. Let

$$f^0 = \limsup_{u\to 0} \left(\sup_{t\geq 0} \frac{f(t, u)}{u} \right), \qquad f^{\infty} = \limsup_{w\to +\infty} \left(\sup_{t\geq 0} \frac{f(t, u)}{u} \right),$$
$$f_0\,(\sigma) = \liminf_{w\to 0} \left(\min_{t\in I_\sigma} \frac{f(t, u)}{u} \right) \quad f_{\infty}\,(\sigma) = \liminf_{w\to +\infty} \left(\min_{t\in I_\sigma} \frac{f(t, u)}{u} \right)$$

where for $\sigma \in (0, 1/2)$, $I_\sigma = [\sigma, 1 - \sigma]$. Let also,

$$\Gamma = \sup_{t\in(0,1)} \left(\int_0^1 G_q(t, s)\phi(s)ds \right),$$
$$\Theta(\sigma) = \sup_{t\in(0,1)} \left(\int_{\sigma}^{1-\sigma} G_q(t, s)\phi(s)\gamma_q(s)ds \right).$$

Notice that if Hypothesis (2.2) is fulfilled, then $\Gamma < \infty$. Indeed, since for $\rho = 1$ the function Λ_1 is nonincreasing, we have

$$\begin{aligned}\int_0^1 G_q(t,s)\phi(s)ds &\leq \int_0^1 G_q(s,s)\phi(s)ds = \left(\overline{\Lambda}\right)^{-1}\int_0^1 G\left(s,s\right)\phi\left(s\right)\overline{\Lambda}ds \\ &\leq \left(\overline{\Lambda}\right)^{-1}\int_0^1 G\left(s,s\right)\phi\left(s\right)\Lambda_1\left(\gamma_q(s)\right)ds < \infty,\end{aligned}$$

where $\overline{\Lambda} = \Lambda_1\left(\max_{s\in(0,1)}\gamma_q(s)\right)$.

The main result of this work consists in the following theorem.

Theorem 3.1. *Assume that Hypotheses* (1.2) *and* (2.2) *hold and there exists* $\sigma \in (0, 1/2)$ *such that one of the following situations* (3.1) *and* (3.2) *holds.*

$$f^0\Gamma < 1 < f_\infty\left(\sigma\right)\Theta(\sigma), \tag{3.1}$$

$$f^\infty\Gamma < 1 < f_0\left(\sigma\right)\Theta(\sigma). \tag{3.2}$$

Then the bvp (1.1) *admits at least one positive solution.*

Proof. **Step 1. Existence in the case where** (3.1) **holds**

Let $\epsilon > 0$ be such that $(f^0 + \epsilon)\Gamma < 1$. For such an ϵ, there exists $R_1 > 0$ such that $f(t, w) \leq (f^0 + \epsilon)w$ for all $w \in (0, R_1)$, and let $\Omega_1 = \{u \in E,\ \|u\| < R_1\}$.

Therefore, for all $u \in P \cap \partial\Omega_1$ and all $t \in (0, 1)$, we have

$$\begin{aligned}Tu(t) = \int_0^1 G_q(t,s)\phi(s)f(s,u(s))ds &\leq \int_0^1 G_q(t,s)\phi(s)\left(f^0+\epsilon\right)u(s)ds \\ \leq \|u\|\left(f^0+\epsilon\right)\int_0^1 G_q(t,s)\phi(s)ds &\leq \Gamma\left(f^0+\epsilon\right)\|u\| \leq \|u\|\end{aligned}$$

leading to $\|Tu\| \leq \|u\|$.

Now, suppose that $f_\infty(\sigma) > \Theta(\sigma)$ for some $\sigma \in (0, 1/2)$ and let $\varepsilon > 0$ be such that

$$(f_\infty\left(\sigma\right) - \varepsilon)\Theta(\sigma) > 1.$$

There exists $R_2 > R_1$ such that $f(t, w) > (f_\infty(\sigma) - \varepsilon)w$ for all $t \in I_\sigma$ and all $w \geq R_2$. Let $\gamma_\sigma = \min\{\gamma_q(s) : s \in I_\sigma\}$, $\widetilde{R}_2 = R_2/\gamma_\sigma$ and $\Omega_2 = \{u \in E : \|u\| < \widetilde{R}_2\}$. For all $u \in P \cap \partial\Omega_2$, and all $t \in (0, 1)$, we have

$$\begin{aligned}\|Tu\| &\geq \sup_{t\in(0,1)}\left(\int_\sigma^{1-\sigma} G_q(t,s)\phi(s)f(s,u(s))ds\right) \\ &\geq \sup_{t\in(0,1)}\left(\int_\sigma^{1-\sigma} G_q(t,s)\phi(s)(f_\infty\left(\sigma\right)-\varepsilon)u(s)ds\right) \\ &\geq \left(f_\infty\left(\sigma\right)-\varepsilon\right)\|u\|\sup_{t\in(0,1)}\left(\int_\sigma^{1-\sigma} G_q(t,s)\phi(s)\gamma_q(s)ds\right) \\ &\geq \|u\|\left(f_\infty\left(\sigma\right)-\varepsilon\right)\Theta(\sigma) \geq \|u\|.\end{aligned}$$

We deduce from Assertion 1 of Theorem 1.1, that T admits a fixed point $u \in P$ with

$$R_1 \leq \|u\| \leq \widetilde{R}_2$$

which is, by Lemma 2.5, a positive solution to the bvp (1.1).

Step 2. Existence in the case where (3.2) **holds**

Let $\varepsilon > 0$ be such that $(f_0(\sigma) - \varepsilon)\Theta(\sigma) > 1$, there exists R_1 such that

$$f(t, w) > (f_0(\theta) - \varepsilon)w$$

for all $t \in I_\sigma$ and all $w \in (0, R_1)$. Let $\Omega_1 = \{u \in E : \|u\| < R_1\}$, for all $u \in P \cap \partial\Omega_1$ and all $t \in (0, 1)$, we have

$$\begin{aligned} \|Tu\| &\geq \sup_{t\in(0,1)} \left(\int_\sigma^{1-\sigma} G_q(t,s)\phi(s)f(s,u(s))ds \right) \\ &\geq \sup_{t\in(0,1)} \left(\int_\sigma^{1-\sigma} G_q(t,s)\phi(s)(f_0(\sigma) - \varepsilon)u(s)ds \right) \\ &\geq (f_0(\sigma) - \varepsilon) \|u\| \sup_{t\in(0,1)} \left(\int_\sigma^{1-\sigma} G_q(t,s)\phi(s)\gamma_q(s)ds \right) \\ &\geq \|u\| (f_0(\sigma) - \varepsilon)\Theta(\sigma) \geq \|u\|. \end{aligned}$$

Let $\epsilon > 0$ be such that $(f^\infty + \epsilon)\Gamma < 1$, there exists $R_\epsilon > 0$ such that

$$f(t, w) \leq (f^\infty + \epsilon)w + \Lambda_{R_\epsilon}(w), \text{ for all } t \in (0,1) \text{ and } w > 0,$$

where Λ_{R_ϵ} is the functions given by Hypothesis 2.2 for $R = R_\epsilon$. Let

$$\begin{aligned} &\Phi_\epsilon(t) = \phi(s)\Lambda_{R_\epsilon}(R_\epsilon\gamma_q(s)) \\ &\widetilde{R}_2 = \frac{\overline{\Phi}_\epsilon\Gamma}{1 - (f^\infty + \epsilon)\Gamma} \\ &\text{with } \overline{\Phi}_\epsilon = \sup_{t\in(0,1)} \left(\int_0^1 G(t,s)\Phi_\epsilon(s)\, ds \right) \end{aligned}$$

and notice that $\Gamma^{-1}(f^\infty + \epsilon)R + \overline{\Phi}_\epsilon \leq R$ for all $R \geq \widetilde{R}_2$.

Let $R_2 > \max(R_1, \widetilde{R}_2, R_\epsilon)$ and $\Omega_2 = \{u \in E, \|u\| < R_2\}$. For all $u \in P \cap \partial\Omega_2$ and all $t \in (0, 1)$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G_q(t,s)\phi(s)f(s,u(s))ds \\ &\leq \int_0^1 G_q(t,s)\phi(s)\left((f^\infty + \epsilon)u(s) + \Psi_\epsilon(u(s))\right)ds \\ &\leq (f^\infty + \epsilon)\|u\| \int_0^1 G_q(t,s)\phi(s)ds + \overline{\Phi}_\epsilon \\ &\leq (f^\infty + \epsilon)\Gamma\|u\| + \overline{\Phi}_\epsilon \leq \|u\|. \end{aligned}$$

leading to

$$\|Tu\| \leq \|u\|.$$

We deduce from Assertion 2 of Theorem 1.1, that T admits a fixed point $u \in P$ with

$$R_1 \leq \|u\| \leq R_2$$

which is, by Lemma 2.5, a positive solution to the bvp (1.1).

The proof of Theorem 3.1 is complete. □

We obtain from Theorem 3.1 the following existence result for positive solutions to the typical case of the bvp (1.1) where $f(t, u) = m(t) u^{\mu}$ with $\mu \neq 1$.

Corollary 3.2. *Assume that*

$$\begin{cases} f(t,u) = u^{\mu} \textit{ with } \mu \neq 1, \\ \lim\limits_{t \to 0} G_q(t,t)\phi(t) \max\left(1, (\gamma_q(t))^{\mu}\right) = \\ \lim\limits_{t \to 1} G_q(t,t)\phi(t) \max\left(1, (\gamma_q(t))^{\mu}\right) = 0 \\ \textit{and } \displaystyle\int_0^1 G_q(s,s)\phi(s) \max\left(1, (\gamma_q(s))^{\mu}\right) ds < \infty. \end{cases}$$

Then the bvp (1.1) *admits a positive solution.*

Proof. For all $\rho > 0$ and $w \in (0, \rho]$, we have

$$f(t,w) = w^{\mu} \leq \Lambda_{\rho}(w) = \begin{cases} \rho^{\mu} \text{ if } \mu \geq 0, \\ w^{\mu} \text{ if } \mu < 0 \end{cases}$$

and

$$\Lambda_{\rho}(r\gamma_q(t)) = \begin{cases} \rho^{\mu} \text{ if } \mu \geq 0, \\ r^{\mu} (\gamma_q(t))^{\mu} \text{ if } \mu < 0 \end{cases} = \max(\rho^{\mu}, r^{\mu}) \max\left(1, (\gamma_q(t))^{\mu}\right).$$

Thus, we obtain from the above calculation that for $\nu = 0, 1$ we have

$$\begin{aligned} &\lim_{t \to \nu} G_q(t,t)\phi(t) \max\left(1, \Lambda_{\rho}(r\gamma_q(t))\right) \\ &= \max(\rho^{\mu}, r^{\mu}) \lim_{t \to \nu} G_q(t,t)\phi(t) \max\left(1, (\gamma_q(t))^{\mu}\right) = 0. \end{aligned}$$

Moreover, we have

$$\begin{cases} f^0 = 0 \quad \text{and} \quad f_{\infty}(\sigma) = +\infty \text{ for all } \sigma \in (0, 1/2), & \text{if } \mu > 0, \\ f^{\infty} = 0 \quad \text{and} \quad f_0(\sigma) = +\infty \text{ for all } \sigma \in (0, 1/2), & \text{if } \mu \leq 0. \end{cases}$$

Therefore, Theorem 3.1 guarantees existence of a positive solution to such a case of the bvp (1.1). □

Remark 3.3. Theorem 3.1 holds true if we replace Hypothesis (1.2) by one of the following assumptions

$$\liminf_{s \to 0} q(s) > 0, \int_0^{1/2} q(s)ds = +\infty \text{ and } \int_0^1 sq(s)ds < \infty,$$

$$\liminf_{s \to 1} q(s) > 0, \int_{1/2}^1 q(s)ds = +\infty \text{ and } \int_0^1 (1-s)q(s)ds < \infty.$$

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