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Alternative Fixed Point Method

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In honor of Distinguished Professor Johnny Henderson's contributions to the mathematics community, and in celebration of his 70th birthday; I have been blessed to have you as a mentor and friend throughout my professional career.

Abstract

A new method for converting a discrete boundary value problem to a fixed point problem is developed. We conclude with an application of the new fixed point technique to bound a solution to a second order discrete right focal boundary value problem known to exist by exponentials.

AMS Subject Classifications: 47H10, 39A10.
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1 Introduction

The standard method to convert a boundary value problem to a fixed point problem is to apply Green's function techniques [8]. We will create a new method to convert a discrete second order focal boundary value problem to a fixed point problem relying only on elementary principles of difference equations. See the book by Peterson and Kelly [6] for a thorough introduction to difference equations, including a discussion of first order linear equations, which is fundamental to our arguments.

Bringing the operator inside of the nonlinear term in boundary value problems is a recent development in converting boundary value problems to fixed point problems [1, 3–5, 7]. Our result relies on similar ideas to introduce a term corresponding to a weighted average of the input into the operator. Under some relatively non restrictive

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conditions we can bound fixed points of this operator by exponential functions noting that we have conditions on solutions of the form

$$x(t+1) \geq (1+\lambda)x(t)$$

over an interval which results in exponentials bounding solutions over this interval.

2 Alternative Inversion Technique

Existence of solutions of the discrete right focal boundary value problem

$$\Delta^2 x(t) + f(x(t)) = 0, \quad t \in \{0, 1, \ldots, N\}, \tag{2.1}$$

$$x(0) = \Delta x(N+1) = 0 \tag{2.2}$$

was studied by Henderson et al. [2], where N is a positive integer. Conditions on the function f were given for the existence of solutions of (2.1), (2.2) by showing that these conditions resulted in fixed points for the operator J defined by

$$Jx(t) = \sum_{s=0}^{N} G(t,s) f(x(s)),$$

where G is the associated Green's function. We will find an alternative operator H such that x is a solution of (2.1), (2.2) if and only if x is a fixed point of the operator H. As an application, we will use H to verify an interesting exponential property of solutions under certain conditions for the function f.

Theorem 2.1. *If $a(t) \neq -1$ for $t \in \{0, 1, \cdots, N, N+1\}$, then x is a solution of* (2.1)*,* (2.2) *if and only if x is a fixed point of the operator H defined by*

$$Hx(t) = \sum_{w=0}^{t-1} \left(\left(\sum_{r=w}^{N} f(x(r)) - a(w)x(w) \right) \prod_{s=w+1}^{t-1} (1 + a(s)) \right).$$

Proof. A function x is a solution of (2.1), (2.2) if and only if x is a solution of

$$\Delta x(t) = \sum_{r=t}^{N} f(x(r)), \quad t \in \{0, 1, \ldots, N+1\}, \tag{2.3}$$

$$x(0) = 0 \tag{2.4}$$

since

$$-\Delta x(t) = \Delta x(N+1) - \Delta x(t) = \sum_{r=t}^{N} \Delta^2 x(r) = \sum_{r=t}^{N} -f(x(r)),$$

where when necessary we apply the standard convention that

$$\sum_{r=N+1}^{N} f(x(r)) = 0$$

because the summation's lower limit is greater than the summation's upper limit. The equation in (2.3) is equivalent to

$$\Delta x(t) - a(t)x(t) = \sum_{r=t}^{N} f(x(r)) - a(t)x(t),$$

and this is equivalent to

$$(x(t+1)-(1+a(t))x(t))\prod_{s=t+1}^{N}(1+a(s)) = \left(\sum_{r=t}^{N} f(x(r))-a(t)x(t)\right)\prod_{s=t+1}^{N}(1+a(s)),$$

which is equivalent to

$$\Delta\left(x(t)\prod_{s=t}^{N}(1+a(s))\right) = \left(\sum_{r=t}^{N} f(x(r)) - a(t)x(t)\right)\prod_{s=t+1}^{N}(1+a(s)).$$

Thus, we have that (2.3), (2.4) is equivalent to

$$x(t) = \sum_{w=0}^{t-1}\left(\left(\sum_{r=w}^{N} f(x(r)) - a(w)x(w)\right)\prod_{s=w+1}^{t-1}(1+a(s))\right) \tag{2.5}$$

for $t \in \{0, 1, \ldots, N+1\}$, since

$$\begin{aligned} x(t)\prod_{s=t}^{N}(1+a(s)) &= \sum_{w=0}^{t-1}\Delta\left(x(w)\prod_{s=w}^{N}(1+a(s))\right) \\ &= \sum_{w=0}^{t-1}\left(\left(\sum_{r=w}^{N} f(x(r)) - a(w)x(w)\right)\prod_{s=w+1}^{N}(1+a(s))\right); \end{aligned}$$

when necessary we apply the standard convention that

$$\prod_{s=t}^{t-1}(1+a(s)) = 1,$$

because the product's lower limit is greater than the product's upper limit. Therefore, if we define the operator H by

$$Hx(t) = \sum_{w=0}^{t-1}\left(\left(\sum_{r=w}^{N} f(x(r)) - a(w)x(w)\right)\prod_{s=w+1}^{t-1}(1+a(s))\right),$$

then we have that x is a solution of (2.1), (2.2) if and only if x is a fixed point of the operator H. This ends the proof. □

Letting $a \equiv \lambda > 0$ in Theorem 2.1 we have the following corollary.

Corollary 2.2. *The function x is a solution of* (2.1), (2.2) *if and only if x is a fixed point of the operator A defined by*

$$Ax(t) = \sum_{w=0}^{t-1} \left((1+\lambda)^{t-w-1} \sum_{r=w}^{N} f(x(r)) \right) - \sum_{w=0}^{t-1} \lambda x(w) \, (1+\lambda)^{t-w-1} .$$

3 Application

Let E be the Banach space of real valued functions defined on the discrete interval $[0, N+2]$ with the supnorm, and let P be the set of nonnegative functions in E. For positive constants b and c and an integer M with $1 \leq M \leq N$, define

$$\overline{P_c} = \{x \in E : \|x\| \leq c\}$$

and

$$\overline{P(\beta, b, \psi, c)} = \{x \in E : b \leq x(t) \leq c \text{ for } t = M, M+1, \cdots, N+2\}.$$

While it is unknown how useful the operator H will be in existence of solutions arguments, the properties of the operator H can be used to prove interesting exponential properties of solutions for the boundary value problem (2.1), (2.2) with relatively nonrestrictive conditions on f, since being a solution is equivalent to being a fixed point of the operator H (and A).

Theorem 3.1. *If $f : [0, \infty) \to [0, \infty)$ and $f(x) \geq \lambda x$ on $[b, c]$, then*

$$b(1+\lambda)^{t-M} \leq x^*(t) \leq \frac{c}{(1+\lambda)^{N+1-t}}$$

for $t \in \{M, M+1, \cdots, N+1\}$ for any solution $x^* \in \overline{P(\beta, b, \psi, c)}$.

Proof. Let $f : [0, \infty) \to [0, \infty)$, $f(x) \geq \lambda$ on $[b, c]$, and suppose $x^* \in \overline{P(\beta, b, \psi, c)}$ is a solution of (2.1), (2.2). Thus, by Theorem 2.2 we have that x^* is a fixed point of the operator A. Letting $t \in \{M, M+1, \cdots, N\}$, we have

$$\begin{aligned} x^*(t+1) - (1+\lambda)x^*(t) &= Ax^*(t+1) - (1+\lambda)Ax^*(t) \\ &= \sum_{w=0}^{t} (1+\lambda)^{t-w} \sum_{r=w}^{N} f(x^*(r)) - \sum_{w=0}^{t} (\lambda x^*(w)) \, (1+\lambda)^{t-w} \\ &\quad - \sum_{w=0}^{t-1} (1+\lambda)^{t-w} \sum_{r=w}^{N} f(x^*(r)) + \sum_{w=0}^{t-1} (\lambda x^*(w)) \, (1+\lambda)^{t-w} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{r=t}^{N} f(x^*(r)) - \lambda x^*(t) \\
&\geq f(x^*(t)) - \lambda x^*(t) \geq 0,
\end{aligned}
$$

since $f : [0, \infty) \to [0, \infty)$ and $x^*(t) \in [b, c]$ so $f(x^*(t)) \geq \lambda x^*(t)$. Hence,

$$x^*(t+1) \geq (1+\lambda)x^*(t)$$

for $t \in \{M, M+1, \cdots, N\}$. Thus, by finite induction,

$$x^*(t) \geq x^*(M)(1+\lambda)^{t-M}$$

and

$$\frac{x^*(N+1)}{(1+\lambda)^{N+1-t}} \geq x^*(t)$$

for $t \in \{M, M+1, \cdots, N, N+1\}$. Therefore,

$$\frac{c}{(1+\lambda)^{N+1-t}} \geq x^*(t) \geq b(1+\lambda)^{t-M}$$

for $t \in \{M, M+1, \cdots, N, N+1\}$, since $b \leq x^*(M)$ and $x^*(N+1) \leq c$. □

When $b = 0$, so $f(x) \geq \lambda x$ on $[0, c]$, we have upper and lower bounds on solutions by exponentials in $\overline{P_c}$ over the entire domain instead of bounds on an interval in the domain.

Corollary 3.2. *If $f : [0, \infty) \to [0, \infty)$ and $f(x) \geq \lambda x$ on $[0, c]$, then*

$$x^*(1)(1+\lambda)^{t-1} \leq x^*(t) \leq \frac{c}{(1+\lambda)^{N+1-t}}$$

for $t \in \{1, 2, \cdots, N+1\}$ for any solution $x^* \in \overline{P_c}$.

Proof. We have that

$$x^*(t+1) \geq (1+\lambda)x^*(t)$$

for $t \in \{1, 2, \cdots, N\}$; thus, by finite induction

$$x^*(t) \geq x^*(1)(1+\lambda)^{t-1}$$

and

$$\frac{x^*(N+1)}{(1+\lambda)^{N+1-t}} \geq x^*(t)$$

for $t \in \{1, 2, \cdots, N, N+1\}$. Therefore,

$$\frac{c}{(1+\lambda)^{N+1-t}} \geq x^*(t) \geq x^*(1)(1+\lambda)^{t-1}.$$

This ends the proof. □

The following theorem, which provides conditions on the function f for the existence of a solution for (2.1), (2.2), is from Henderson et al. [2, Theorem 3.1].

Theorem 3.3. *Let $M, \mu, \nu \in \{1, \ldots, N+2\}$, with $0 < M \leq \mu < \nu \leq N+2$, let d and m be positive real numbers with $0 < m \leq \dfrac{d\mu}{N+2}$, and suppose $f : [0,\infty) \to [0,\infty)$ is continuous and satisfies the following conditions:*

(i) $f(w) \geq \dfrac{d}{\nu - M}$, *for* $w \in \left[\dfrac{Md}{N+2}, \dfrac{\nu d}{N+2}\right]$,

(ii) $f(w)$ *is decreasing, for* $w \in [0, m]$, *and* $f(m) \geq f(w)$, *for* $w \in [m, d]$,

(iii) $\displaystyle\sum_{\ell=0}^{\mu} \frac{\ell+1}{N+2} f\left(\frac{m\ell}{\mu}\right) \leq d - \sum_{\ell=\mu+1}^{N} \frac{\ell+1}{N+2} f(m).$

Then, the discrete right focal boundary value problem (2.1), (2.2) *has at least one positive solution*

$$u^* \in P\left(\alpha, \frac{Md}{N+2}, \beta, d\right) = \left\{x \in E : \frac{Md}{N+2} \leq x(t) \leq d \text{ for } t = M, \cdots, N+2\right\}.$$

Given that there is a solution $u^* \in P\left(\alpha, \dfrac{Md}{N+2}, \beta, d\right)$, if we strengthen condition (i) with

$$f(x) \geq \frac{x}{\nu - M} \quad \text{for} \quad w \in \left[\frac{\nu d}{N+2}, d\right],$$

then we can enhance the conclusion of Theorem 3.3 by saying that

$$\left(\frac{Md}{N+2}\right)(1+\lambda)^{t-M} \leq u^*(t) \leq \frac{d}{(1+\lambda)^{N+1-t}}$$

for $t \in \{M, M+1, \cdots, N+1\}$, where $\lambda = \dfrac{1}{\nu - M}$ by Theorem 3.1.

Henderson et al. [2] finished their paper with the following example of Theorem 3.3.

Example 3.4. Let $N = 8$, $M = 1$, $\mu = 2$, $\nu = 10$, $d = 1$, and $m = \dfrac{1}{9}$. Notice that $0 < \tau \leq \mu < \nu \leq 10 = N+2$, and $0 < m = \dfrac{1}{9} \leq \dfrac{1}{5} = \dfrac{d\mu}{N+2}$. We define a continuous $f : [0,\infty) \to [0,\infty)$ by

$$f(w) = \begin{cases} -8w + 1, & 0 \leq w \leq \dfrac{1}{9}, \\[2ex] \dfrac{1}{9}, & w > \dfrac{1}{9}. \end{cases}$$

Then,

(i) $f(w) \geq \frac{1}{9}$, for $w \in \left[\frac{1}{10}, 1\right]$,

(ii) $f(w)$ is decreasing on $\left[0, \frac{1}{9}\right]$, and $f\left(\frac{1}{9}\right) = f(w)$, for $w \in \left[\frac{1}{9}, 1\right]$, and

(iii) $\sum_{\ell=0}^{2} \frac{\ell+1}{10} f\left(\frac{\ell}{18}\right) = \frac{22}{90} \leq \frac{51}{90} = 1 - \sum_{\ell=3}^{10} \frac{\ell+1}{10} f\left(\frac{1}{9}\right)$.

Therefore, by Theorem 3.3, the right focal boundary value problem,

$$\Delta^2 u(k) + f(u(k)) = 0,\ k \in \{0, \dots, 8\},$$

$$u(0) = 0 = \Delta u(9),$$

has at least one positive solution, u^*, with

$$\frac{1}{10} \leq u^*(1) \text{ and } u^*(10) \leq 1.$$

In this example, the conditions of Theorem 3.1 are met with $\lambda = \frac{1}{9}$. Therefore, the conclusion of Example 3.4 can be enhanced, and instead of concluding with the existence of a solution u^* with

$$\frac{1}{10} \leq u^*(1) \text{ and } u^*(10) \leq 1,$$

one can conclude with the existence of a solution u^* satisfying

$$\frac{10^{t-2}}{9^{t-1}} \leq u^*(t) \leq \left(\frac{9}{10}\right)^{9-t}$$

for $t \in \{1, 2, \cdots, 9\}$ (note that $u^*(9) = u^*(10)$, since $\Delta u^*(9) = 0$).

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