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Linear Fractional-Order h-Difference Equations

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Abstract

In this paper, we study linear fractional order h-difference equations, where the order of the equation is any noninteger positive real number. The nabla fractional operators are used in the sense of Riemann-Liouville definition. We obtain the general solution of the fractional order equation by means of Mittag–Leffler type functions. Several properties of the Mittag–Leffler type functions are obtained. As an application, an eigenvalue problem with Dirichlet boundary condition is considered. We give a method for explicit calculation of the eigenvalues of the boundary value problem.

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1 Introduction

Let $a \in \mathbb{R}$ and $h \in \mathbb{R}^+$, where $\mathbb{R}$ is the set of real numbers. We define

$$h\mathbb{N}_a = \{a,\ a+h,\ a+2h, \ldots\}.$$

This set of numbers is a time scale which is a nonempty closed subset of $\mathbb{R}$. In modeling perspectives of real life phenomenon, this set can be considered as a bridge between the set of natural numbers ($\mathbb{N}$), where $h = a = 1$, and an interval $[a, \infty)$ of real numbers, where $h \to 0$. Several problems in ordinary differential equations and partial differential equations can be analyzed by approximating the equations on the set $h\mathbb{N}_a$, which the method is called the finite difference method. Such a feature of this set motivates us to study further on this subject.

Integer order dynamic equations on time scales which include $h\mathbb{N}_a$ were studied extensively in the literature. We refer the reader to two books by Bohner and Peterson which explore many results and reference several published papers in this area [5, 6].

The study on noninteger order discrete equations is an ongoing research and is in rapid development in recent years. The book by Goodrich and Peterson collects many results under its umbrella and references several published papers in this area [8]. We also bring the following papers to the reader's attention. In [10], Suwan et al. studied monotonicity properties of fractional h-discrete operators. In the papers [4, 7], Torres et al. addressed some fractional variational problems on the set $h\mathbb{N}_a$. In [9], Liu et al. obtained Gronwall's like inequalities for fractional h-sum equations.

The plan of the paper is as follows: in Section 2, we give some preliminaries such as basic definitions and lemmas related to the nabla h-difference and the nabla fractional h-difference operators. We state and prove the power rules. In Section 3, we introduce Mittag–Leffler type functions and prove their properties. Then we study a fractional order linear h-difference equation and obtain the general solution. In Section 4, an eigenvalue problem with the Dirichlet boundary conditions is considered. After step by step calculation of the Green's function for the associated boundary value problem (BVP), we obtain a result which states that the eigenvalues of the BVP can be determined using the eigenvalues of a square matrix where the entries are the values of the Green's function.

2 Preliminaries

Definition 2.1. Let $a \in \mathbb{R}$, $h \in \mathbb{R}^+$ and $f : h\mathbb{N}_a \to \mathbb{R}$ where

$$h\mathbb{N}_a = \{a,\ a+h,\ a+2h, \ldots\}.$$

The first order backward h-difference operator for a function f is defined by

$$\nabla_h f(t) = \frac{f(t) - f(t-h)}{h}, \quad t = a+h, a+2h, \ldots,$$

and the nth-order backward h-difference operator for f is defined recursively by

$$\nabla_h^n f(t) = \nabla_h \nabla_h^{n-1} f(t), \quad t = a + nh, a + (n+1)h, \ldots.$$

We note that if $h = 1$, we have the first order backward h-difference operator, or nabla operator (∇)

$$(\nabla f)(t) = f(t) - f(t-1), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.2. For any $t, r \in \mathbb{R}$ and $h > 0$, the h-rising factorial function is defined by

$$t_h^{\overline{r}} = h^r \frac{\Gamma(\frac{t}{h} + r)}{\Gamma(\frac{t}{h})},$$

where the quotient is well-defined. Here $\Gamma(\cdot)$ denotes the Euler gamma function.

Definition 2.3. Let $\alpha > 0$ and a be two real numbers. For a function $f : h\mathbb{N}_a \to \mathbb{R}$, the nabla h-fractional sum with order α is defined by

$$\nabla_{h,a}^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a/h}^{t/h} \big(t - \rho(sh)\big)_h^{\overline{\alpha-1}} f(sh)h, \quad t \in h\mathbb{N}_a,$$

where $h > 0$ and $\rho(t) = t - h$.

The nabla h-fractional sum operator in Definition 2.3 generalizes the nabla fractional sum operator when $h = 1$. In this regard, it can be considered as a new definition since it is different than the one defined in [10]. We note that if $h = 1$, then we have the αth order fractional sum operator [3]

$$\nabla_{1,a}^{-\alpha} f(t) = \sum_{s=a}^{t} \frac{(t - \rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(s), \quad t \in \mathbb{N}_a. \tag{2.1}$$

In the rest of the paper, we write $\nabla_a^{-\alpha}$ for $\nabla_{1,a}^{-\alpha}$.

Definition 2.4. The nabla h-fractional difference of order α in the sense of Riemann-Liouville is defined by

$$\nabla_{h,a}^{\alpha} f(t) := \nabla_h^n \nabla_{h,a}^{-(n-\alpha)} f(t), \quad t \in h\mathbb{N}_{a+nh},$$

where $a, \alpha \in \mathbb{R}$, $n - 1 < \alpha < n$, and n is a positive integer.

The next lemma outlines some relations between nabla operators in $\mathbb{N}_a$ and nabla operators in $h\mathbb{N}_a$.

Lemma 2.5. *Let a be any real number and $n-1<\alpha<n$, where n is a positive integer. Let function $g:\mathbb{N}_{a/h}\to h\mathbb{N}_a$ be defined by*

$$g(u):=uh$$

and $y:h\mathbb{N}_a\to\mathbb{R}$. Then the following hold:

1. $\nabla_h^n y(uh)=\dfrac{\nabla^n(y\circ g)(u)}{h^n}$,
2. $\nabla_{h,a}^{\alpha} y(uh)=h^{-\alpha}\nabla_{a/h}^{\alpha}(y\circ g)(u)$,

where $u\in\mathbb{N}_{a/h+n}$.

Proof. We omit the proof of (1) since it is straightforward. To prove (2), we claim that $\nabla_{h,a}^{-\alpha}y(uh)=h^{\alpha}\nabla_{a/h}^{-\alpha}(y\circ g)(u)$ for $u\in\mathbb{N}_{a/h}$. Indeed, we have

$$\begin{aligned}
\nabla_{h,a}^{-\alpha}y(uh)&=\frac{1}{\Gamma(\alpha)}\sum_{s=a/h}^{u}\big(uh-\rho(sh)\big)_h^{\overline{\alpha-1}}y(sh)h\\
&=\frac{1}{\Gamma(\alpha)}\sum_{s=a/h}^{u}\frac{\Gamma(\frac{uh-sh+h}{h}+\alpha-1)}{\Gamma(\frac{uh-sh+h}{h})}h^{\alpha}y(sh)\\
&=\frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{s=a/h}^{u}\frac{\Gamma(u-s+\alpha)}{\Gamma(u-s+1)}y(sh)\\
&=\frac{h^{\alpha}}{\Gamma(\alpha)}\sum_{s=a/h}^{u}(u-s+1)^{\overline{\alpha-1}}y(sh)\\
&=h^{\alpha}\sum_{s=a/h}^{u}\frac{(u-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(y\circ g)(s)\\
&=h^{\alpha}\nabla_{a/h}^{-\alpha}(y\circ g)(u)
\end{aligned}$$

where $u\in\mathbb{N}_{a/h}$.

We use Definition 2.4 and (1) to obtain the desired identity. Indeed, we have

$$\begin{aligned}
\nabla_{h,a}^{\alpha}y(uh)&=\nabla_h^n\nabla_{h,a}^{-(n-\alpha)}y(uh)\\
&=\nabla_h^n(\nabla_{h,a}^{-(n-\alpha)}\circ y)(uh)\\
&=h^{-n}\nabla^n(\nabla_{h,a}^{-(n-\alpha)}\circ y\circ g)(u)\\
&=h^{-n}\nabla^n(\nabla_{h,a}^{-(n-\alpha)}y(uh))
\end{aligned}$$

$$= h^{-n}\nabla^n h^{n-\alpha}\nabla_{a/h}^{-(n-\alpha)}(y\circ g)(u)$$
$$= h^{-\alpha}\nabla_{a/h}^{\alpha}(y\circ g)(u),$$

where $u \in \mathbb{N}_{a/h+n}$. □

Theorem 2.6 (See [5]). *Let a be any real number, $h > 0$, and $t \in h\mathbb{N}_a$. Then for a function $f(t,\cdot) : \mathbb{N}_{\frac{a}{h}} \to \mathbb{R}$ the following identities are true.*

1. $\nabla_h \sum_{s=\frac{a}{h}}^{t/h} f(t,s) = \sum_{s=\frac{a}{h}}^{t/h} \nabla_h f(t,s) + \dfrac{f(t-h,\frac{t}{h})}{h}$.

2. $\nabla_h \sum_{s=\frac{a}{h}}^{t/h-1} f(t,s) = \sum_{s=\frac{a}{h}}^{t/h-2} \nabla_h f(t,s) + \dfrac{f(t,\frac{t}{h}-1)}{h}$.

Definition 2.7. For u, $v : h\mathbb{N}_a \to \mathbb{R}$, we define the nabla convolution product of u and v by

$$(u*v)(t) = \sum_{s=a/h}^{t/h} u(t-sh+a)v(sh)h, \quad t \in h\mathbb{N}_a.$$

Lemma 2.8. *Let $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\dfrac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}$ and $\dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}$ are defined. Then, we have that*

1. $\nabla_{h,a}^{-\alpha}(t-\rho(a))_h^{\overline{\beta}} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-\rho(a))_h^{\overline{\beta+\alpha}}$, $t \in h\mathbb{N}_a$.

2. $\nabla_{h,a}^{\alpha}(t-\rho(a))_h^{\overline{\beta}} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-\rho(a))_h^{\overline{\beta-\alpha}}$, $t \in h\mathbb{N}_a$.

Proof. Consider

$$\nabla_{h,a}^{-\alpha}(t-\rho(a))_h^{\overline{\beta}} = \frac{1}{\Gamma(\alpha)}\sum_{s=a/h}^{t/h}(t-\rho(sh))_h^{\overline{\alpha-1}}(sh-\rho(a))_h^{\overline{\beta}}h$$
$$= \frac{h}{\Gamma(\alpha)}\sum_{s=a/h}^{t/h} h^{\alpha-1}\frac{\Gamma(\frac{t-\rho(sh)}{h}+\alpha-1)}{\Gamma(\frac{t-\rho(sh)}{h})}h^{\beta}\frac{\Gamma(\frac{sh-\rho(a)}{h}+\beta)}{\Gamma(\frac{sh-\rho(a)}{h})}.$$

Taking $r = \dfrac{sh-\rho(a)}{h} - 1$ in the last expression, we have

$$\nabla_{h,a}^{-\alpha}(t-\rho(a))_h^{\overline{\beta}}$$

$$
\begin{aligned}
&= \frac{h^{\beta+\alpha}}{\Gamma(\alpha)} \sum_{r=0}^{\frac{t-a}{h}} \frac{\Gamma(\frac{t-a}{h}-r+\alpha)}{\Gamma(\frac{t-a}{h}-r+1)} \frac{\Gamma(r+1+\beta)}{\Gamma(r+1)} \\
&= \frac{h^{\beta+\alpha}\Gamma(\beta+1)}{\Gamma(\frac{t-a}{h}+1)} \sum_{r=0}^{\frac{t-a}{h}} \frac{\Gamma(\frac{t-a}{h}+1)}{\Gamma(r+1)\Gamma(\frac{t-a}{h}-r+1)} \frac{\Gamma(\alpha+\frac{t-a}{h}-r)}{\Gamma(\alpha)} \frac{\Gamma(\beta+1+r)}{\Gamma(\beta+1)} \\
&= \frac{h^{\beta+\alpha}\Gamma(\beta+1)}{\Gamma(\frac{t-a}{h}+1)} \sum_{r=0}^{\frac{t-a}{h}} \binom{\frac{t-a}{h}}{r} \frac{\Gamma(\alpha+\frac{t-a}{h}-r)}{\Gamma(\alpha)} \frac{\Gamma(\beta+1+r)}{\Gamma(\beta+1)} \\
&= \frac{h^{\beta+\alpha}\Gamma(\beta+1)}{\Gamma(\frac{t-a}{h}+1)} \frac{\Gamma(\beta+\alpha+1+\frac{t-a}{h})}{\Gamma(\beta+\alpha+1)} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t-\rho(a))_h^{\overline{\beta+\alpha}}.
\end{aligned}
$$

The proof of (1) is complete. The proof of (2) is similar to the proof of (1). So, we omit it. □

3 Noninteger Order Nabla h-Discrete Fractional Equations with Delay

Let $\mu, \lambda, t_0 \in \mathbb{R}$ and $\nu > 0$. Define

$$
\tilde{E}^h_{\lambda,\nu,\mu}(t,t_0) = \frac{1}{h^\mu} \frac{(t-t_0+h)_h^{\overline{\mu}}}{\Gamma(\mu+1)} + \frac{1}{h^\mu} \sum_{n=\frac{t_0}{h}+1}^{t/h} \lambda^{n-\frac{t_0}{h}} \frac{(t-\rho(nh))_h^{\overline{\nu(n-\frac{t_0}{h})+\mu}}}{\Gamma(\nu(n-\frac{t_0}{h})+\mu+1)}, \quad t \in h\mathbb{N}_{t_0}. \tag{3.1}
$$

We note that $\tilde{E}^h_{\lambda,\nu,\mu}(t_0,t_0) = 1$. Next, for the simplicity, we take $t_0 = 0$ and observe some properties for $\tilde{E}^h_{\lambda,\nu,\mu}$.

The finite sum in (3.1) appeared in the papers [1, 2] for some special cases. For the reader's convenience, we list two theorems here.

Theorem 3.1 (See [2]). *Assume $\lambda \in \mathbb{R}$. The fractional difference equation of order ν where $\nu \in$ (0,1)*

$$
\nabla_0^\nu y(t) = \lambda y(t-1) \quad for \quad t = 1,2,3,..., \tag{3.2}
$$

has the general solution

$$
y(t) = c\tilde{E}^1_{\lambda,\nu,\nu-1}(t,0), \quad t = 0,1,2,..., \tag{3.3}
$$

where c is constant.

Theorem 3.2 (See [1]). *Suppose* $\nu \in (1,2]$ *and* $|\lambda|\frac{1}{(\nu-1)^{1-\nu}}\frac{1}{\nu^{\nu}} < 1$. *The fractional difference equation*

$$-\Delta^{\nu} y(t) + \lambda y(t+\nu-1) = 0, \quad t = 0,1,2,\ldots, \tag{3.4}$$

has the general solution

$$y(t) = A\tilde{E}^{1}_{\lambda,\nu,\nu-1}(t,0) + B\tilde{E}^{1}_{\lambda,\nu,\nu-2}(t,0), \tag{3.5}$$

where A *and* B *are constants.*

In [1], there is a remark which states that the Mittag–Leffler type functions in Theorem 3.2 are in finite sum. For this reason, we remove the restriction on the λ parameter. We also want to point out that fractional order Δ and ∇ operators are related to each other as given in [3]. Hence Theorem 3.2 can be stated as in the following.

Theorem 3.3. *Suppose* $\nu \in (1,2]$. *The fractional difference equation*

$$\nabla^{\nu} y(t) = \lambda y(t-1), \quad t = 0,1,2,\ldots, \tag{3.6}$$

has the general solution

$$y(t) = A\tilde{E}^{1}_{\lambda,\nu,\nu-1}(t,0) + B\tilde{E}^{1}_{\lambda,\nu,\nu-2}(t,0), \tag{3.7}$$

where A *and* B *are constants.*

Using Lemma 2.2 and the techniques in the paper [1], the following theorem can be proven.

Theorem 3.4. *Assume* $k-1 < \nu < k$ *with* $k \in \mathbb{Z}^{+}$. *The general solution of the linear nabla fractional difference equation*

$$\nabla_{0}^{\nu} y(t) = \lambda y(t-1), \quad t \in \mathbb{N}_{k}, \tag{3.8}$$

is given by

$$y(t) = C_1\tilde{E}^{1}_{\lambda,\nu,\nu-1}(t,0) + C_2\tilde{E}^{1}_{\lambda,\nu,\nu-2}(t,0) + \cdots + C_k\tilde{E}^{1}_{\lambda,\nu,\nu-k}(t,0),$$

where $C_1, C_2, \ldots, C_k$ *are constants.*

Next we list some properties of the Mittag–Leffler type functions.

Proposition 3.5. *The following are valid.*

1. *For* $\alpha > 0$, $\nabla_{h,0}^{-\alpha}\tilde{E}^{h}_{\lambda,\nu,\mu}(t,0) = h^{\alpha}\tilde{E}^{h}_{\lambda,\nu,\mu+\alpha}(t,0), \quad t \in h\mathbb{N}_0$.
2. *For* $\alpha > 0$, $\nabla_{h,0}^{\alpha}\tilde{E}^{h}_{\lambda,\nu,\mu}(t,0) = h^{-\alpha}\tilde{E}^{h}_{\lambda,\nu,\mu-\alpha}(t,0), \quad t \in h\mathbb{N}_0$.

3. *For* $\alpha \in \mathbb{R}^+ \backslash \mathbb{N}$ *and* $k \in \mathbb{N}$, $\tilde{E}^h_{\lambda,\alpha,-k}(t,0) = \lambda h^\alpha \tilde{E}^h_{\lambda,\alpha,\alpha-k}(t-h,0), \quad t \in h\mathbb{N}_h$.

4. *For* $k \in \mathbb{N}$ *and* $\alpha \in (0,k)\backslash \mathbb{N}$, $\nabla^\alpha_{h,0} \tilde{E}^h_{\lambda,\alpha,\alpha-k}(t,0) = \lambda \tilde{E}^h_{\lambda,\alpha,\alpha-k}(t-h,0), \quad t \in h\mathbb{N}_h$.

Proof. Consider

$$\begin{aligned}
&\nabla^{-\alpha}_{h,0} \tilde{E}^h_{\lambda,\nu,\mu}(t,0) \\
&= \frac{h}{\Gamma(\alpha)} \sum_{s=0}^{t/h} (t-\rho(sh))_h^{\overline{\alpha-1}} \tilde{E}^h_{\lambda,\nu,\mu}(sh,0) \\
&= \frac{h}{\Gamma(\alpha)} \sum_{s=0}^{t/h} (t-\rho(sh))_h^{\overline{\alpha-1}} \left[\frac{1}{h^\mu} \frac{(sh+h)_h^{\overline{\mu}}}{\Gamma(\mu+1)} + \frac{1}{h^\mu} \sum_{n=1}^{s} \lambda^n \frac{(sh-\rho(nh))_h^{\overline{\nu n+\mu}}}{\Gamma(\nu n+\mu+1)} \right] \\
&= \frac{h}{\Gamma(\alpha)} \sum_{s=0}^{t/h} (t-\rho(sh))_h^{\overline{\alpha-1}} \left[\frac{1}{h^\mu} \frac{(sh-\rho(0))_h^{\overline{\mu}}}{\Gamma(\mu+1)} \right] \\
&\quad + \frac{h}{\Gamma(\alpha)} \sum_{s=0}^{t/h} (t-\rho(sh))_h^{\overline{\alpha-1}} \left[\frac{1}{h^\mu} \sum_{n=1}^{s} \lambda^n \frac{(sh-\rho(nh))_h^{\overline{\nu n+\mu}}}{\Gamma(\nu n+\mu+1)} \right] \\
&= \frac{1}{h^\mu \Gamma(\mu+1)} \frac{h}{\Gamma(\alpha)} \sum_{s=0}^{t/h} (t-\rho(sh))_h^{\overline{\alpha-1}} (sh-\rho(0))_h^{\overline{\mu}} \\
&\quad + \frac{h}{\Gamma(\alpha)} \sum_{s=1}^{t/h} (t-\rho(sh))_h^{\overline{\alpha-1}} \left[\frac{1}{h^\mu} \sum_{n=1}^{s} \lambda^n \frac{(sh-\rho(nh))_h^{\overline{\nu n+\mu}}}{\Gamma(\nu n+\mu+1)} \right]. \qquad (3.9)
\end{aligned}$$

Note that the term corresponding to $s = 0$ in the second expression of RHS of (3.9),

$$\frac{h}{\Gamma(\alpha)} (t-\rho(0))_h^{\overline{\alpha-1}} \left[\frac{1}{h^\mu} \sum_{n=1}^{0} \lambda^n \frac{(0-\rho(nh))_h^{\overline{\nu n+\mu}}}{\Gamma(\nu n+\mu+1)} \right]$$

becomes zero. The first expression of RHS of (3.9) can be simplified as

$$\begin{aligned}
&\frac{1}{h^\mu \Gamma(\mu+1)} \frac{h}{\Gamma(\alpha)} \sum_{s=0}^{t/h} (t-\rho(sh))_h^{\overline{\alpha-1}} (sh-\rho(0))_h^{\overline{\mu}} \\
&= \frac{1}{h^\mu \Gamma(\mu+1)} \nabla^{-\alpha}_{h,0} (t-\rho(0))_h^{\overline{\mu}} \\
&= \frac{1}{h^\mu \Gamma(\mu+1)} \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-\rho(0))_h^{\overline{\mu+\alpha}} \quad \text{(Using Lemma 2.8)} \\
&= \frac{1}{h^\mu \Gamma(\mu+\alpha+1)} (t-\rho(0))_h^{\overline{\mu+\alpha}} \\
&= \frac{1}{h^\mu} \frac{(t+h)_h^{\overline{\mu+\alpha}}}{\Gamma(\mu+\alpha+1)}. \qquad (3.10)
\end{aligned}$$

Interchanging the order of sums in the second expression of RHS of (3.9), we have

$$\begin{aligned}
&\frac{h}{\Gamma(\alpha)}\sum_{s=1}^{t/h}(t-\rho(sh))_h^{\overline{\alpha-1}}\left[\frac{1}{h^{\mu}}\sum_{n=1}^{s}\lambda^n\frac{(sh-\rho(nh))_h^{\overline{\nu n+\mu}}}{\Gamma(\nu n+\mu+1)}\right]\\
&=\frac{1}{h^{\mu}}\sum_{n=1}^{t/h}\lambda^n\left[\frac{h}{\Gamma(\alpha)}\sum_{s=n}^{t/h}(t-\rho(sh))_h^{\overline{\alpha-1}}\frac{(sh-\rho(nh))_h^{\overline{\nu n+\mu}}}{\Gamma(\nu n+\mu+1)}\right]. \qquad (3.11)
\end{aligned}$$

Using the substitution $m=s-n$ in (3.11), we obtain

$$\begin{aligned}
&\frac{h}{\Gamma(\alpha)}\sum_{s=1}^{t/h}(t-\rho(sh))_h^{\overline{\alpha-1}}\left[\frac{1}{h^{\mu}}\sum_{n=1}^{s}\lambda^n\frac{(sh-\rho(nh))_h^{\overline{\nu n+\mu}}}{\Gamma(\nu n+\mu+1)}\right]\\
&=\frac{1}{h^{\mu}}\sum_{n=1}^{t/h}\frac{\lambda^n}{\Gamma(\nu n+\mu+1)}\left[\frac{h}{\Gamma(\alpha)}\sum_{m=0}^{t/h-n}(t-nh-\rho(mh))_h^{\overline{\alpha-1}}(mh+h)_h^{\overline{\nu n+\mu}}\right]\\
&=\frac{1}{h^{\mu}}\sum_{n=1}^{t/h}\frac{\lambda^n}{\Gamma(\nu n+\mu+1)}\left[\frac{h}{\Gamma(\alpha)}\sum_{m=0}^{t/h-n}(t-nh-\rho(mh))_h^{\overline{\alpha-1}}(mh-\rho(0))_h^{\overline{\nu n+\mu}}\right]\\
&=\frac{1}{h^{\mu}}\sum_{n=1}^{t/h}\frac{\lambda^n}{\Gamma(\nu n+\mu+1)}\left[\nabla_{h,0}^{-\alpha}(t-nh-\rho(0))_h^{\overline{\nu n+\mu}}\right]\\
&=\frac{1}{h^{\mu}}\sum_{n=1}^{t/h}\lambda^n\frac{(t-nh-\rho(0))_h^{\overline{\nu n+\mu+\alpha}}}{\Gamma(\nu n+\mu+\alpha+1)} \quad \text{(Using Lemma 2.8)}\\
&=\frac{1}{h^{\mu}}\sum_{n=1}^{t/h}\lambda^n\frac{(t-\rho(nh))_h^{\overline{\nu n+\mu+\alpha}}}{\Gamma(\nu n+\mu+\alpha+1)}. \qquad (3.12)
\end{aligned}$$

Using (3.10) and (3.12) in (3.9), we obtain

$$\begin{aligned}
\nabla_{h,0}^{-\alpha}\tilde{E}^h_{\lambda,\nu,\mu}(t,0)&=\frac{h^{\alpha}}{h^{\mu+\alpha}}\frac{(t+h)_h^{\overline{\mu+\alpha}}}{\Gamma(\mu+\alpha+1)}+\frac{h^{\alpha}}{h^{\mu+\alpha}}\sum_{n=1}^{t/h}\lambda^n\frac{(t-\rho(nh))_h^{\overline{\nu n+\mu+\alpha}}}{\Gamma(\nu n+\mu+\alpha+1)}\\
&=h^{\alpha}\tilde{E}^h_{\lambda,\nu,\mu+\alpha}(t,0). \qquad (3.13)
\end{aligned}$$

The proof of (1) is complete. The proof of (2) is similar to the proof of (1).

Next, the proof of (3) follows from the lines below. For $k\in\mathbb{N}$, we have

$$\begin{aligned}
&\tilde{E}^h_{\lambda,\alpha,-k}(t,0)\\
&=\frac{1}{h^{-k}}\frac{(t+h)_h^{\overline{-k}}}{\Gamma(-k+1)}+\frac{1}{h^{-k}}\sum_{n=1}^{t/h}\lambda^n\frac{(t-\rho(nh))_h^{\overline{\alpha n-k}}}{\Gamma(\alpha n-k+1)}
\end{aligned}$$

$$= 0 + h^k \left[\lambda \frac{(t-\rho(h))_h^{\overline{\alpha-k}}}{\Gamma(\alpha-k+1)} + \sum_{n=2}^{t/h} \lambda^n \frac{(t-\rho(nh))_h^{\overline{\alpha n-k}}}{\Gamma(\alpha n-k+1)} \right]$$
$$= h^k \left[\lambda \frac{(t-h+h)_h^{\overline{\alpha-k}}}{\Gamma(\alpha-k+1)} + \lambda \sum_{n=1}^{t/h-1} \lambda^n \frac{(t-h-\rho(nh))_h^{\overline{\alpha n+\alpha-k}}}{\Gamma(\alpha n+\alpha-k+1)} \right]$$
$$= \lambda h^k h^{\alpha-k} \left[\frac{1}{h^{\alpha-k}} \frac{(t-h+h)_h^{\overline{\alpha-k}}}{\Gamma(\alpha-k+1)} + \frac{1}{h^{\alpha-k}} \sum_{n=1}^{t/h-1} \lambda^n \frac{(t-h-\rho(nh))_h^{\overline{\alpha n+\alpha-k}}}{\Gamma(\alpha n+\alpha-k+1)} \right]$$
$$= \lambda h^\alpha \tilde{E}^h_{\lambda,\alpha,\alpha-k}(t-h,0).$$

For the proof of (4), we have

$$\begin{aligned} \nabla^\alpha_{h,0} \tilde{E}^h_{\lambda,\alpha,\alpha-k}(t,0) &= h^{-\alpha} \tilde{E}^h_{\lambda,\alpha,-k}(t,0) \quad \text{(Using (1))} \\ &= \lambda h^{-\alpha} h^\alpha \tilde{E}^h_{\lambda,\alpha,\alpha-k}(t-h,0) \quad \text{(Using (3))} \\ &= \lambda \tilde{E}^h_{\lambda,\alpha,\alpha-k}(t-h,0). \end{aligned}$$

The proof is complete. □

Now we are in a position to state and prove general solution to a nonhomogeneous fractional h-difference equation.

Theorem 3.6. *Let $\lambda \in \mathbb{R}$, $h > 0$, $k \in N$, and $\alpha \in (k-1,k)$. The general solution of the following problem*

$$\nabla^\alpha_{h,0} y(t) = \lambda y(t-h), \quad t \in h\mathbb{N}_{kh}, \tag{3.14}$$

is given by

$$y(t) = C_1 \tilde{E}^h_{\lambda,\alpha,\alpha-1}(t,0) + C_2 \tilde{E}^h_{\lambda,\alpha,\alpha-2}(t,0) + \cdots + C_k \tilde{E}^h_{\lambda,\alpha,\alpha-k}(t,0),$$

where $C_1, C_2, \ldots, C_k$ are constants.

Proof. Replacing t by uh and using Lemma 2.5-(2), we write the fractional h-difference equation in (3.14) as a fractional difference equation

$$\nabla^\alpha_0 (y \circ g)(u) = \lambda h^\alpha (y \circ g)(u-1),$$

for $u \in \mathbb{N}_k$. Then we use Theorem 3.4 to obtain the desired result. □

Theorem 3.7. *Assume $k-1 < \alpha < k$, and $g : h\mathbb{N}_0 \to \mathbb{R}$. The general solution of the linear nonhomogeneous nabla fractional h-difference equation*

$$\nabla^\alpha_{h,0} u(t) = \lambda u(t-h) + g(t-h), \quad t \in h\mathbb{N}_{kh}, \tag{3.15}$$

is given by

$$u(t) = C_1 \tilde{E}^h_{\lambda,\alpha,\alpha-1}(t,0) + C_2 \tilde{E}^h_{\lambda,\alpha,\alpha-2}(t,0) + \cdots + C_k \tilde{E}^h_{\lambda,\alpha,\alpha-k}(t,0)$$

$$+h^{\alpha-1}\left(\tilde{E}^h_{\lambda,\alpha,\alpha-1}(\cdot,0) * g\right)(t-h), \quad t \in h\mathbb{N}_0,$$

where $C_1, C_2, \ldots, C_k$ are constants.

Proof. In view of Theorem 3.6, it suffices to show that $h^{\alpha-1}\left(\tilde{E}^h_{\lambda,\alpha,\alpha-1}(\cdot,0) * g\right)(t-h)$ is a particular solution of (3.15). Denote by $f(t) = h^{\alpha-1}\left(\tilde{E}^h_{\lambda,\alpha,\alpha-1}(\cdot,0) * g\right)(t-h)$. It is enough to show that

$$\left(\nabla^\alpha_{h,0} f\right)(t) = \lambda f(t-h) + g(t-h), \quad t \in h\mathbb{N}_{kh}. \tag{3.16}$$

Consider

$$\begin{aligned}
&\left(\nabla^\alpha_{h,0} f\right)(t) \\
&= \nabla^\alpha_{h,0}\left[h^{\alpha-1}\left(\tilde{E}^h_{\lambda,\alpha,\alpha-1}(\cdot,0) * g\right)(t-h)\right] \\
&= h^{\alpha-1}\nabla^k_h\, \nabla^{-(k-\alpha)}_{h,0}\Big[\sum_{s=0}^{t/h-1} \tilde{E}^h_{\lambda,\alpha,\alpha-1}(t-h-sh,0)g(sh)h\Big] \\
&= h^{\alpha-1}\nabla^k_h\left[\frac{h}{\Gamma(k-\alpha)}\sum_{s=0}^{t/h}(t-\rho(sh))_h^{\overline{k-\alpha-1}}\Big[\sum_{r=0}^{s-1}\tilde{E}^h_{\lambda,\alpha,\alpha-1}(sh-h-rh,0)g(rh)h\Big]\right].
\end{aligned}$$

Interchanging the order of sums in the last expression, we have

$$\left(\nabla^\alpha_{h,0} f\right)(t) =$$

$$h^{\alpha-1}\nabla^k_h\left[\sum_{r=0}^{t/h-1} g(rh)h\Big[\frac{h}{\Gamma(k-\alpha)}\sum_{s=r+1}^{t/h}(t-\rho(sh))_h^{\overline{k-\alpha-1}}\tilde{E}^h_{\lambda,\alpha,\alpha-1}(sh-h-rh,0)\Big]\right].$$

Using the substitution $m = s-1-r$, we obtain

$$\begin{aligned}
\left(\nabla^\alpha_{h,0} f\right)(t) &= h^{\alpha-1}\nabla^k_h\Big[\sum_{r=0}^{t/h-1} g(rh)h\Big[\frac{h}{\Gamma(k-\alpha)} \\
&\qquad \sum_{m=0}^{t/h-1-r}(t-h-rh-\rho(mh))^{\overline{k-\alpha-1}}\tilde{E}^h_{\lambda,\alpha,\alpha-1}(mh,0)\Big]\Big] \\
&= h^{\alpha-1}\nabla^k_h\left[\sum_{r=0}^{t/h-1} g(rh)h\Big[\nabla^{-(k-\alpha)}_{h,0}\tilde{E}^h_{\lambda,\alpha,\alpha-1}(t-h-rh,0)\Big]\right]
\end{aligned}$$

$$= h^{\alpha-1}\nabla_h^k \left[\sum_{r=0}^{t/h-1} h^{k-\alpha}\tilde{E}^h_{\lambda,\alpha,\alpha-1+k-\alpha}(t-h-rh,0)g(rh)h\right]$$

$$= \nabla_h^k \left[\sum_{r=0}^{t/h-1} h^{k}\tilde{E}^h_{\lambda,\alpha,k-1}(t-h-rh,0)g(rh)\right]. \tag{3.17}$$

Consider

$$\nabla_h \left[\sum_{r=0}^{t/h-1} h^{k}\tilde{E}^h_{\lambda,\alpha,k-1}(t-h-rh,0)g(rh)\right]$$

$$= \sum_{r=0}^{t/h-2} \nabla_h \left[h^{k}\tilde{E}^h_{\lambda,\alpha,k-1}(t-h-rh,0)g(rh)\right]$$

$$+ \frac{1}{h}\left[h^{k}\tilde{E}^h_{\lambda,\alpha,k-1}(t-h-rh,0)g(rh)\right]_{t\to t, r\to t/h-1} \quad \text{(Using Theorem 2.6)}$$

$$= \sum_{r=0}^{t/h-2} h^{k-1}\tilde{E}^h_{\lambda,\alpha,k-2}(t-h-rh,0)g(rh)$$

$$+ \frac{1}{h}\left[h^{k}\tilde{E}^h_{\lambda,\alpha,k-1}(t-h-t+h,0)g(t-h)\right]$$

$$= \sum_{r=0}^{t/h-2} h^{k-1}\tilde{E}^h_{\lambda,\alpha,k-2}(t-h-rh,0)g(rh) + h^{k-1}\tilde{E}^h_{\lambda,\alpha,k-1}(0,0)g(t-h)$$

$$= \sum_{r=0}^{t/h-2} h^{k-1}\tilde{E}^h_{\lambda,\alpha,k-2}(t-h-rh,0)g(rh) + h^{k-1}g(t-h)$$

$$= \sum_{r=0}^{t/h-2} h^{k-1}\tilde{E}^h_{\lambda,\alpha,k-2}(t-h-rh,0)g(rh) + h^{k-1}\tilde{E}^h_{\lambda,\alpha,k-2}(0,0)g(t-h)$$

$$= \sum_{r=0}^{t/h-2} h^{k-1}\tilde{E}^h_{\lambda,\alpha,k-2}(t-h-rh,0)g(rh) + h^{k-1}\tilde{E}^h_{\lambda,\alpha,k-2}(t-h-t+h,0)g(t-h)$$

$$= \sum_{r=0}^{t/h-1} h^{k-1}\tilde{E}^h_{\lambda,\alpha,k-2}(t-h-rh,0)g(rh). \tag{3.18}$$

Consider

$$\nabla_h^2 \left[\sum_{r=0}^{t/h-1} h^{k}\tilde{E}^h_{\lambda,\alpha,k-1}(t-h-rh,0)g(rh)\right]$$

$$
\begin{aligned}
&= \nabla_h \left(\nabla_h \left[\sum_{r=0}^{t/h-1} h^k \tilde{E}^h_{\lambda,\alpha,k-1}(t-h-rh,0) g(rh) \right] \right) \\
&= \nabla_h \left(\sum_{r=0}^{t/h-1} h^{k-1} \tilde{E}^h_{\lambda,\alpha,k-2}(t-h-rh,0) g(rh) \right) \quad \text{(Using (3.18))} \\
&= \sum_{r=0}^{t/h-2} \nabla_h \left[h^{k-1} \tilde{E}^h_{\lambda,\alpha,k-2}(t-h-rh,0) g(rh) \right] \\
&\quad + \frac{1}{h} \left[h^{k-1} \tilde{E}^h_{\lambda,\alpha,k-2}(t-h-rh,0) g(rh) \right]_{t \to t, r \to t/h-1} \quad \text{(Using Theorem 2.6)} \\
&= \sum_{r=0}^{t/h-2} h^{k-2} \tilde{E}^h_{\lambda,\alpha,k-3}(t-h-rh,0) g(rh) \\
&\quad + \frac{1}{h} \left[h^{k-1} \tilde{E}^h_{\lambda,\alpha,k-2}(t-h-t+h,0) g(t-h) \right] \\
&= \sum_{r=0}^{t/h-2} h^{k-2} \tilde{E}^h_{\lambda,\alpha,k-3}(t-h-rh,0) g(rh) + h^{k-2} \tilde{E}^h_{\lambda,\alpha,k-2}(0,0) g(t-h) \\
&= \sum_{r=0}^{t/h-2} h^{k-2} \tilde{E}^h_{\lambda,\alpha,k-3}(t-h-rh,0) g(rh) + h^{k-2} g(t-h) \\
&= \sum_{r=0}^{t/h-2} h^{k-2} \tilde{E}^h_{\lambda,\alpha,k-3}(t-h-rh,0) g(rh) + h^{k-2} \tilde{E}^h_{\lambda,\alpha,k-3}(0,0) g(t-h) \\
&= \sum_{r=0}^{t/h-1} h^{k-2} \tilde{E}^h_{\lambda,\alpha,k-3}(t-h-rh,0) g(rh). \qquad (3.19)
\end{aligned}
$$

Continuing in a similar way, we obtain

$$
\begin{aligned}
&\nabla_h^k \left[\sum_{r=0}^{t/h-1} h^k \tilde{E}^h_{-\lambda,\alpha,k-1}(t-h-rh,0) g(rh) \right] \\
&= \sum_{r=0}^{t/h-1} h^{k-k} \tilde{E}^h_{\lambda,\alpha,k-(k+1)}(t-h-rh,0) g(rh) \\
&= \sum_{r=0}^{t/h-2} \tilde{E}^h_{\lambda,\alpha,-1}(t-h-rh,0) g(rh) + \tilde{E}^h_{\lambda,\alpha,-1}(0,0) g(t-h) \\
&= \sum_{r=0}^{t/h-2} \tilde{E}^h_{\lambda,\alpha,-1}(t-h-rh,0) g(rh) + g(t-h)
\end{aligned}
$$

$$= \lambda \sum_{r=0}^{t/h-2} h^{\alpha} \tilde{E}^h_{\lambda,\alpha,\alpha-1}(t-2h-rh,0)g(rh) + g(t-h) \quad \text{(Using Proposition 3.5)}$$
$$= \lambda h^{\alpha-1} \sum_{r=0}^{t/h-2} \tilde{E}^h_{\lambda,\alpha,\alpha-1}(t-2h-rh,0)g(rh)h + g(t-h)$$
$$= \lambda h^{\alpha-1} \left(\tilde{E}^h_{\lambda,\alpha,\alpha-1}(\cdot,0) * g\right)(t-2h) + g(t-h)$$
$$= \lambda f(t-h) + g(t-h),$$

implying that (3.16) holds. The proof is complete. □

4 An Eigenvalue Problem

Let $\alpha \in (1,2)$. We consider the following boundary value problem (BVP):

$$\left(\nabla^{\alpha}_{h,0} u\right)(t) = \lambda u(t-h), \quad t \in h\mathbb{N}^L_{2h}, \tag{4.1}$$
$$u(0) = u(L) = 0, \tag{4.2}$$

where $L = kh$ for some positive integer k. The general solution of (4.1) is given by

$$u(t) = C_1 \tilde{E}^h_{\lambda,\alpha,\alpha-1}(t,0) + C_2 \tilde{E}^h_{\lambda,\alpha,\alpha-2}(t,0), \quad t \in h\mathbb{N}_0, \tag{4.3}$$

where C_1 and C_2 are constants. Using $u(0) = 0$ in (4.3), we obtain

$$C_1 \tilde{E}^h_{\lambda,\alpha,\alpha-1}(0,0) + C_2 \tilde{E}^h_{\lambda,\alpha,\alpha-2}(0,0) = 0,$$

implying that

$$C_1 + C_2 = 0.$$

Using $u(L) = 0$ in (4.3), we obtain

$$C_1 \tilde{E}^h_{\lambda,\alpha,\alpha-1}(L,0) + C_2 \tilde{E}^h_{\lambda,\alpha,\alpha-2}(L,0) = 0. \tag{4.4}$$

Since $C_2 = -C_1$, we have

$$C_1 \left[\tilde{E}^h_{\lambda,\alpha,\alpha-1}(L,0) - \tilde{E}^h_{\lambda,\alpha,\alpha-2}(L,0)\right] = 0. \tag{4.5}$$

Using Proposition 3.5, we have

$$\nabla_{h,0} \tilde{E}^h_{\lambda,\alpha,\alpha-1}(L,0) = h^{-1} \tilde{E}^h_{\lambda,\alpha,\alpha-2}(L,0).$$

This implies that

$$C_1 \tilde{E}^h_{\lambda,\alpha,\alpha-1}(L-h) = 0. \tag{4.6}$$

It is clear that for $\lambda \geq 0$, $\tilde{E}^h_{\lambda,\alpha,\alpha-1}(L-h) > 0$. Therefore $C_1 = C_2 = 0$.

Summarizing the above discussion, we have the following result.

Theorem 4.1. *If $\lambda \in \mathbb{R}$ is an eigenvalue of* (4.1)–(4.2)*, then λ must be negative.*

Consider the following boundary value problem:

$$\left(\nabla_{h,0}^{\alpha} u\right)(t) = m(t-h), \quad t \in h\mathbb{N}_{2h}^{L}, \tag{4.7}$$

$$u(0) = u(L) = 0. \tag{4.8}$$

Theorem 4.2. *The solution of the BVP* (4.7)–(4.8) *is*

$$u(t) = \sum_{s=0}^{L/h-1} G(t, sh) m(sh) h,$$

where

$$G(t, sh) = \begin{cases} G_1(t, sh), & s \in \mathbb{N}_0^{\frac{t}{h}-1} \\ G_2(t, sh), & s \in \mathbb{N}_{\frac{t}{h}}^{\frac{L}{h}-1}. \end{cases} \tag{4.9}$$

$$G_1(t, sh) = \frac{1}{\Gamma(\alpha)} \left[-\frac{(t)_h^{\overline{\alpha-1}}}{(L)_h^{\overline{\alpha-1}}} (L-h-\rho(sh))_h^{\overline{\alpha-1}} + (t-h-\rho(sh))_h^{\overline{\alpha-1}} \right]$$

and

$$G_2(t, sh) = \frac{1}{\Gamma(\alpha)} \left[-\frac{(t)_h^{\overline{\alpha-1}}}{(L)_h^{\overline{\alpha-1}}} (L-h-\rho(sh))_h^{\overline{\alpha-1}} \right].$$

Proof. Using Theorem 3.7, the general solution of (4.7) is given by

$$u(t) = C_1 \tilde{E}_{0,\alpha,\alpha-1}^{h}(t,0) + C_2 \tilde{E}_{0,\alpha,\alpha-2}^{h}(t,0) + h^{\alpha-1} \left(\tilde{E}_{0,\alpha,\alpha-1}^{h}(\cdot, 0) * m \right)(t-h), \tag{4.10}$$

for $t \in h\mathbb{N}_0$, where C_1, C_2 are constants. We have

$$\tilde{E}_{0,\alpha,\alpha-1}^{h}(t,0) = \frac{1}{h^{\alpha-1}} \frac{(t+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}, \tag{4.11}$$

$$\tilde{E}_{0,\alpha,\alpha-2}^{h}(t,0) = \frac{1}{h^{\alpha-2}} \frac{(t+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}, \tag{4.12}$$

and

$$\begin{aligned} h^{\alpha-1} \left(\tilde{E}_{0,\alpha,\alpha-1}^{h}(\cdot, 0) * m \right)(t-h) &= h^{\alpha-1} \sum_{s=0}^{t/h-1} \tilde{E}_{0,\alpha,\alpha-1}^{h}(t-h-sh, 0) m(sh) h \\ &= h^{\alpha-1} \sum_{s=0}^{t/h-1} \left[\frac{1}{h^{\alpha-1}} \frac{(t-h-sh+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} \right] m(sh) h \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t/h-1} (t-h-\rho(sh))_h^{\overline{\alpha-1}} m(sh)h$$
$$= \nabla_{h,0}^{-\alpha} m(t-h). \tag{4.13}$$

Using (4.11)–(4.13) in (4.10), we obtain

$$u(t) = \frac{C_1}{h^{\alpha-1}} \frac{(t+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} + \frac{C_2}{h^{\alpha-2}} \frac{(t+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t/h-1} (t-h-\rho(sh))_h^{\overline{\alpha-1}} m(sh)h, \tag{4.14}$$

for $t \in h\mathbb{N}_0$. Using $u(0) = 0$ in (4.14), we have

$$\frac{C_1}{h^{\alpha-1}} \frac{(h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} + \frac{C_2}{h^{\alpha-2}} \frac{(h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} + 0 = 0,$$

implying that

$$C_1 + C_2 = 0.$$

Using $u(L) = 0$ in (4.14), we obtain

$$\frac{C_1}{h^{\alpha-1}} \frac{(L+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} + \frac{C_2}{h^{\alpha-2}} \frac{(L+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{L/h-1} (L-h-\rho(sh))_h^{\overline{\alpha-1}} m(sh)h = 0. \tag{4.15}$$

Since $C_2 = -C_1$, we have

$$C_1 \left[\frac{1}{h^{\alpha-1}} \frac{(L+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} - \frac{1}{h^{\alpha-2}} \frac{(L+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} \right] + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{L/h-1} (L-h-\rho(sh))_h^{\overline{\alpha-1}} m(sh)h = 0. \tag{4.16}$$

Now, consider

$$\frac{1}{h^{\alpha-1}} \frac{(L+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} - \frac{1}{h^{\alpha-2}} \frac{(L+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}$$
$$= \frac{1}{h^{\alpha-1}} h^{\alpha-1} \frac{\Gamma(\frac{L+h}{h} + \alpha - 1)}{\Gamma(\alpha)\Gamma(\frac{L+h}{h})} - \frac{1}{h^{\alpha-2}} h^{\alpha-2} \frac{\Gamma(\frac{L+h}{h} + \alpha - 2)}{\Gamma(\alpha-1)\Gamma(\frac{L+h}{h})}$$
$$= \frac{\Gamma(\frac{L}{h} + \alpha)}{\Gamma(\alpha)\Gamma(\frac{L}{h} + 1)} - \frac{\Gamma(\frac{L}{h} + \alpha - 1)}{\Gamma(\alpha-1)\Gamma(\frac{L}{h} + 1)}$$
$$= \frac{\Gamma(\frac{L}{h} + \alpha - 1)}{\Gamma(\alpha-1)\Gamma(\frac{L}{h} + 1)} \left[\frac{\frac{L}{h} + \alpha - 1}{\alpha - 1} - 1 \right]$$

$$= \frac{\Gamma(\frac{L}{h}+\alpha-1)}{\Gamma(\alpha-1)\Gamma(\frac{L}{h}+1)}\left[\frac{\frac{L}{h}}{\alpha-1}\right]$$
$$= \frac{\Gamma(\frac{L}{h}+\alpha-1)}{\Gamma(\alpha)\Gamma(\frac{L}{h})}$$
$$= \frac{1}{h^{\alpha-1}}\frac{(L)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}. \tag{4.17}$$

Using (4.17) in (4.16), we get

$$C_1\left[\frac{1}{h^{\alpha-1}}\frac{(L)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right]+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{L/h-1}(L-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h=0, \tag{4.18}$$

implying that

$$C_1=-\frac{h^{\alpha-1}}{(L)_h^{\overline{\alpha-1}}}\sum_{s=0}^{L/h-1}(L-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h. \tag{4.19}$$

Since $C_2=-C_1$, we have

$$C_2=\frac{h^{\alpha-1}}{(L)_h^{\overline{\alpha-1}}}\sum_{s=0}^{L/h-1}(L-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h. \tag{4.20}$$

It follows from (4.19), (4.20) and (4.14) that

$$u(t)=\frac{C_1}{h^{\alpha-1}}\frac{(t+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}+\frac{C_2}{h^{\alpha-2}}\frac{(t+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t/h-1}(t-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h$$
$$=\left[-\frac{h^{\alpha-1}}{(L)_h^{\overline{\alpha-1}}}\sum_{s=0}^{L/h-1}(L-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h\right]\frac{1}{h^{\alpha-1}}\frac{(t+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}$$
$$+\left[\frac{h^{\alpha-1}}{(L)_h^{\overline{\alpha-1}}}\sum_{s=0}^{L/h-1}(L-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h\right]\frac{1}{h^{\alpha-2}}\frac{(t+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t/h-1}(t-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h$$
$$=\frac{-h^{\alpha-1}}{(L)_h^{\overline{\alpha-1}}}\sum_{s=0}^{L/h-1}(L-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h\left[\frac{(t+h)_h^{\overline{\alpha-1}}}{h^{\alpha-1}\Gamma(\alpha)}-\frac{(t+h)_h^{\overline{\alpha-2}}}{h^{\alpha-2}\Gamma(\alpha-1)}\right]$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t/h-1}(t-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h. \tag{4.21}$$

Now, consider

$$
\begin{aligned}
&\frac{1}{h^{\alpha-1}}\frac{(t+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}-\frac{1}{h^{\alpha-2}}\frac{(t+h)_h^{\overline{\alpha-2}}}{\Gamma(\alpha-1)}\\
&=\frac{1}{h^{\alpha-1}}h^{\alpha-1}\frac{\Gamma(\frac{t+h}{h}+\alpha-1)}{\Gamma(\alpha)\Gamma(\frac{t+h}{h})}-\frac{1}{h^{\alpha-2}}h^{\alpha-2}\frac{\Gamma(\frac{t+h}{h}+\alpha-2)}{\Gamma(\alpha-1)\Gamma(\frac{t+h}{h})}\\
&=\frac{\Gamma(\frac{t}{h}+\alpha)}{\Gamma(\alpha)\Gamma(\frac{t}{h}+1)}-\frac{\Gamma(\frac{t}{h}+\alpha-1)}{\Gamma(\alpha-1)\Gamma(\frac{L}{h}+1)}\\
&=\frac{\Gamma(\frac{t}{h}+\alpha-1)}{\Gamma(\alpha-1)\Gamma(\frac{t}{h}+1)}\left[\frac{\frac{t}{h}+\alpha-1}{\alpha-1}-1\right]\\
&=\frac{\Gamma(\frac{t}{h}+\alpha-1)}{\Gamma(\alpha-1)\Gamma(\frac{t}{h}+1)}\left[\frac{\frac{t}{h}}{\alpha-1}\right]\\
&=\frac{\Gamma(\frac{t}{h}+\alpha-1)}{\Gamma(\alpha)\Gamma(\frac{t}{h})}\\
&=\frac{1}{h^{\alpha-1}}\frac{(t)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}.
\end{aligned}
\tag{4.22}
$$

Using (4.22) in (4.21), we obtain

$$
\begin{aligned}
u(t)&=-\frac{h^{\alpha-1}}{(L)_h^{\overline{\alpha-1}}}\sum_{s=0}^{L/h-1}(L-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h\left[\frac{1}{h^{\alpha-1}}\frac{(t)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right]\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t/h-1}(t-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h\\
&=-\frac{(t)_h^{\overline{\alpha-1}}}{(L)_h^{\overline{\alpha-1}}}\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{L/h-1}(L-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t/h-1}(t-h-\rho(sh))_h^{\overline{\alpha-1}}m(sh)h\\
&=\sum_{s=0}^{t/h-1}G_1(t,sh)m(sh)h+\sum_{s=t/h}^{L/h-1}G_2(t,sh)m(sh)h,
\end{aligned}
\tag{4.23}
$$

where

$$
G_1(t,sh)=\frac{1}{\Gamma(\alpha)}\left[-\frac{(t)_h^{\overline{\alpha-1}}}{(L)_h^{\overline{\alpha-1}}}(L-h-\rho(sh))_h^{\overline{\alpha-1}}+(t-h-\rho(sh))_h^{\overline{\alpha-1}}\right],\tag{4.24}
$$

and

$$
G_2(t,sh)=\frac{1}{\Gamma(\alpha)}\left[-\frac{(t)_h^{\overline{\alpha-1}}}{(L)_h^{\overline{\alpha-1}}}(L-h-\rho(sh))_h^{\overline{\alpha-1}}\right],\tag{4.25}
$$

for $t \in h\mathbb{N}_0^L$. □

Hence Theorem 4.2 implies that the BVP (4.1)–(4.2) can be written as a summation equation

$$u(t) = \lambda \sum_{s=0}^{L/h-1} G(t, sh)u(sh)h, \tag{4.26}$$

for $t \in h\mathbb{N}_0^L$. Further this can be written as a matrix equation such that

$$U(t) = h\lambda \mathbf{G} U(t),$$

for $t \in h\mathbb{N}_0^L$. Here $U(t)$ is $(k+1) \times 1$ vector given by

$$\begin{pmatrix} u(0) \\ u(h) \\ u(2h) \\ \vdots \\ u((k-1)h) \\ u(kh) \end{pmatrix}, \tag{4.27}$$

and $\mathbf{G}$ is $(k+1) \times (k+1)$ matrix. Since the boundary conditions are homogeneous, we consider $U(t) = h\lambda \mathbf{G} U(t)$ on $h\mathbb{N}_h^{L-h}$, where $\mathbf{G}$ is $(k-1) \times (k-1)$ matrix

$$\begin{pmatrix} G_1(h,0) & G_2(h,h) & \cdots & G_2(h, L-2h) & G_2(h, L-h) \\ G_1(2h,0) & G_1(2h,h) & \cdots & G_2(2h, L-2h) & G_2(2h, L-1h) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ G_1(L-2h,0) & G_1(L-2h,h) & \cdots & G_2(L-2h, L-2h) & G_2(L-2h, L-h) \\ G_1(L-h,0) & G_1(L-h,h) & \cdots & G_1(L-h, L-2h) & G_2(L-h, L-h) \end{pmatrix}.$$

The following result can be obtained immediately.

Theorem 4.3. *For nonzero λ, λ is an eigenvalue of the BVP* (4.1)–(4.2) *if and only if $(\lambda h)^{-1}$ is an eigenvalue of the matrix* $\mathbf{G}$.

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