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Existence Results for Systems of Second-Order Impulsive Differential Equations via Variational Method

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Dedicated to Johnny Henderson on the occasion of his 70th birthday.

Abstract

In this paper, we establish an existence and uniqueness result for systems of second-order impulsive differential equations with boundary conditions. The functional energy and Nash-type equilibrium are employed.

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1 Introduction

Many problems arising from diverse areas of natural science, when modeled from a mathematical point of view, involve the study of solutions of nonlinear differential equations, and can be written as a fixed point equation such as

$$u = N(u), \tag{1.1}$$

associated to some operator N. In many cases, the equation also has a variational form, i.e., it is equivalent to an equation of the type

$$E'(u) = 0, \tag{1.2}$$

where E is the energy functional and E' is its derivative. Thus, the fixed points of the operator N appear as critical points of the functional E. The critical points could be minima, maxima, or saddle points, conferring to the fixed points a variational property. Thus, it makes sense to ask whether a fixed point of N is a minimum, a maximum, or a saddle point of E. The problem is even more interesting in the case of a system in which one can associate an energy functional to each of the equations. In recent years, many authors have studied the existence of weak solutions for impulsive differential equations via variational methods; see for example [1, 8, 16–19] and the references therein.

More exactly, we shall consider the following boundary value problem for a system of impulsive differential equations

$$\begin{cases} -\ddot{u} + m^2 u & = & f(t,u,v), & t \neq t_k, k = 1, \dots, p, \ t \in J, \\ -\ddot{v} + m^2 v & = & g(t,u,v), & t \neq t_k, k = 1, \dots, p, \ t \in J, \\ \dot{u}(t_k^+) - \dot{u}(t_k^-) & = & I_k(u(t_k)), \ k = 1, \dots, p, & \\ \dot{v}(t_k^+) - \dot{v}(t_k^-) & = & \overline{I}_k(u(t_k)), \ k = 1, \dots, p, & \\ u(0) = u(b) & = & v(0) = v(b) = 0, & \end{cases} \tag{1.3}$$

where $J := [0,b], \ m \neq 0, f, g : J \times \mathbb{R}^2 \to \mathbb{R}$ are two functions, $\dot{u}(t_k^+)$ and $\dot{u}(t_k^+)$ denote the right and the left limits, respectively, of $\dot{u}$ at t_k for $0 \leq k \leq p, \ 0 = t_0 < t_1 \dots, t_k < t_p < b, \ p \in \mathbb{N}$.

Coupled systems arise from mathematical modeling of many processes from ecology, chemistry, biology, and physics. In some classes, the above system was used to analyze initial value problems and boundary value problems for nonlinear competitive or cooperative differential systems from mathematical biology [9] and mathematical economics [6], which can be set in operator form (1.1). Recently, Precup [12] proved the role of matrix convergence and vector metric in the study of semi-linear operator systems. In recent years, many authors studied the existence of solutions for systems of differential equations, with or without impulsive effect, by using the vector version of a fixed point theorem; see [2, 4, 5, 7, 10, 11, 14, 15] and the references therein. The goal of this paper is to solve a class of boundary value problem for a system of impulsive differential equations by using critical point theory in generalized Banach spaces.

This paper is organized as follows. In Section 2, we introduce all the background material used in this paper such as some properties of Sobolev Banach spaces. In Sections 3 and 4, we state and prove our main results by using a Nash-type equilibrium method in vector Banach spaces.

2 Preliminaries

Let $(X_i, |\cdot|_i)$, $i = 1, 2$ be Hilbert spaces identified to their duals, and let $X = X_1 \times X_2$. Consider the system

$$\begin{cases} u = N_1(u, v) \\ v = N_2(u, v), \end{cases}$$

where $(u, v) \in X$. Assume that each equation of the system has a variational form, i.e., that there exist continuous functionals $E_1, E_2 : X \to \mathbb{R}$ such that $E_1(\cdot, v)$ is Fréchet differentiable for every $v \in X_2$, $E_2(u, \cdot)$ is Fréchet differentiable for every $u \in X_1$, and

$$\begin{cases} E_{11}(u, v) = u - N_1(u, v) \\ E_{22}(u, v) = v - N_2(u, v). \end{cases} \tag{2.1}$$

Here, by $E_{11}(u, v), E_{22}(u, v)$ we mean the Fréchet derivative of $E_1(\cdot, v)$ and $E_2(u, \cdot)$, respectively. The following theorem gives the existence of the solution for impulsive boundary value problems by using variational methods and critical point theory.

Theorem 2.1 (See [13]). *Assume that the above conditions are satisfied. In addition, assume that $E_1(\cdot, v)$ and $E_2(u, \cdot)$ are bounded from below for every $u \in X_1, v \in X_2$, and the following boundedness condition holds: there are $R, a > 0$ such that*

$$\textit{either} \quad E_1(u, v) \;\geq\; \inf_{X_1} E_1(\cdot, v) + a \textit{ for } |u|_1 \geq R \textit{ and all } v \in X_2, \tag{2.2}$$

$$\textit{or } E_2(u, v) \;\geq\; \inf_{X_2} E_2(u, \cdot) + a \textit{ for} |v|_2 \geq R \textit{ and all } u \in X_1. \tag{2.3}$$

Then, the unique fixed point (u^*, v^*) of (N_1, N_2) is a Nash-type equilibrium of the pair of functionals (E_1, E_2), i.e.,

$$\begin{aligned} E_1(u^*, v^*) &= \inf_{X_1} E_1(\cdot, v^*) \\ E_2(u^*, v^*) &= \inf_{X_2} E_2(u^*, \cdot). \end{aligned}$$

In order to define the weak solutions for problem (1.3), we need to define derivatives in a distributional sense (or weak derivative).

Definition 2.2. A function $f \in L^1(J)$ is said to be weakly differentiable if there exists $g \in L^1(J)$ such that

$$\int_J \phi(s)g(s)ds = -\int_J f(s)\phi'(s)ds, \quad \forall \phi \in C_0^\infty(J),$$

where $C_0^\infty(J)$ is the space of smooth functions with compact support.

Remark 2.3. There are functions which are weakly differentiable, but not differentiable in the classical sense.

Define the Sobolev space

$$H_0^1(J) = \{u \in L^2(J) : \dot{u} \in L^2(J), u(0) = u(b) = 0\}.$$

We endow $H_0^1(J)$ with the scalar product

$$\langle u, v\rangle_{H_0^1(J)} = \int_0^b (\dot{u}\dot{v} + m^2 uv)dt = m^2\langle u, v\rangle_{L^2(J)} + \langle \dot{u}, \dot{v}\rangle_{L^2(J)},$$

and the corresponding equivalent norm

$$\|u\|_{H_0^1(J)} = (m^2\|u\|^2_{L^2(J)} + \|\dot{u}\|^2_{L^2(J)})^{1/2}. \tag{2.4}$$

Let $H^{-1}(J)$ be the dual of $H_0^1(J)$. If we identify $L^2(J)$ to its dual, then we may write

$$C_0^\infty(J) \subset H_0^1(J) \subset H^1(J) \subset L^2(J) \subset H^{-1}(J).$$

We define the following isometry operator

$$L : H^{-1}(J) \to H_0^1(J), \quad h \mapsto Lh := u_h,$$

where u_h is the unique element of $H_0^1(J)$ guaranteed by Riesz's representation theorem, satisfying the identity

$$\langle u_h, v\rangle_{H_0^1(J)} = \langle h, v\rangle, \quad v \in H_0^1(J). \tag{2.5}$$

Here, by $\langle h, v\rangle$, we mean the value at v of the functional h from (2.5). One then has

$$\|u_h\|^2_{H_0^1(J)} = \langle u_h, u_h\rangle_{H_0^1(J)} = \langle h, u_h\rangle \leq \|h\|_{H^{-1}(J)}\|u_h\|_{H_0^1(J)},$$

whence

$$\|u_h\|_{H_0^1(J)} \leq \|h\|_{H^{-1}(J)}.$$

On the other hand,

$$\|h\|_{H^{-1}(J)} = \sup_{v\neq 0} \frac{|\langle h, v\rangle|}{\|v\|_{H_0^1(J)}} = \sup_{v\neq 0} \frac{|\langle u_h, v\rangle_{H_0^1(J)}|}{\|v\|_{H_0^1(J)}}$$

$$\leq \sup_{v\neq 0} \frac{\|u_h\|_{H_0^1(J)}\|v\|_{H_0^1(J)}}{\|v\|_{H_0^1(J)}}.$$

These two inequalities show that L is an isometry between $H^{-1}(J)$ and $H_0^1(J)$. The advantage of using the norm (2.4) on $H_0^1(J)$ is that a Poincaré-type inequality holds in connection to the embedding $H_0^1(J) \subset L^2(J)$, namely, the obvious relation

$$\|u\|_{L^2(J)} \leq \frac{1}{m}\|u\|_{H_0^1(J)}, \qquad u \in H_0^1(J). \tag{2.6}$$

A similar Poincaré inequality holds for the inclusion $L^2(J) \subset H^{-1}(J)$. Indeed, if $h \in L^2(J)$, then using (2.5), (2.6), and the above isometry, we obtain

$$\begin{aligned}
\|h\|^2_{H^{-1}(J)} &= \|u_h\|^2_{H_0^1(J)} = \langle h, u_h\rangle = \langle h, u_h\rangle_{L^2(J)} \\
&\leq \|h\|_{L^2(J)}\|u_h\|_{L^2(J)} \leq \frac{1}{m}\|u_h\|_{H_0^1(J)}\|h\|_{L^2(J)} \\
&= \frac{1}{m}\|h\|_{H^{-1}(J)}\|h\|_{L^2(J)},
\end{aligned}$$

which gives

$$\|h\|_{H^{-1}(J)} \leq \frac{1}{m}\|h\|_{L^2(J)}. \tag{2.7}$$

Lemma 2.4 (See [3]). *There exists $c > 0$ such that, if $u \in H^{1,p}_{per}(J,\mathbb{R}),\ \ 1 < p < \infty$, then*

$$\|u\|_\infty \leq c\|u\|_{H^{1,p}_{per}}.$$

Moreover, if $\int_0^b u(t)dt = 0$, then

$$\|u\|_\infty \leq c\|u'\|_{L^p},$$

where

$$H^{1,p}_{per}(J,\mathbb{R}) = \{u \in H^{1,p}(J,\mathbb{R}) : u(0) = u(b),\ u'(0) = u'(b)\}.$$

Lemma 2.5 (See [3]). *If $u \in H^{1,p}_{per}(J,\mathbb{R})$ ($p \in (1,\infty)$) and $\int_0^b u(t)dt = 0$, then*

$$\|u\|_\infty \leq b^{\frac{1}{p'}}\|u'\|_{L^p}, \text{ with } \frac{1}{p} + \frac{1}{p'} = 1.$$

3 Main Results

Lemma 3.1. *The function $E = (E_1, E_2) : H_0^1(J) \times H_0^1(J) \to \mathbb{R}$ defined by*

$$E_1(u,v) = \int_0^b \left[\frac{1}{2}(\dot{u}^2 + m^2u^2) - F(t,u,v)\right]dt + \sum_{k=1}^{p}\int_0^{u(t_k)} I_k(s)ds,$$

$$E_2(u,v) = \int_0^b \left[\frac{1}{2}(\dot{v}^2 + m^2v^2) - G(t,u,v)\right] dt + \sum_{k=1}^{p} \int_0^{v(t_k)} \overline{I}_k(s)ds,$$

where

$$F(t,u,v) = \int_0^u f(t,s,v)ds$$

and

$$G(t,u,v) = \int_0^v g(t,u,s)ds,$$

is the functional energy of the system (1.3).

Proof. Let $w \in C_0^\infty(J)$. For $t \in [0,t_1]$,

$$\int_0^{t_1} -\ddot{u}w + \int_0^{t_1} m^2uw = \int_0^{t_1} wf(t,u,v).$$

By integration of the above equation, we get

$$(-\dot{u}w)(t_1) + \int_0^{t_1} \dot{u}\dot{w} + \int_0^{t_1} m^2uw = \int_0^{t_1} wf(t,u,v).$$

For $t \in (t_1,t_2]$, we have

$$\int_{t_1}^{t_2} -\ddot{u}w + \int_{t_1}^{t_2} m^2uw = \int_{t_1}^{t_2} wf(t,u,v).$$

By integration of the last equation, and using the jump definition of $u'(t_1^+)$, we obtain

$$(-\dot{u}w)(t_2^-) + (wu')(t_1^+) + \int_{t_1}^{t_2} \dot{u}\dot{w} + \int_{t_1}^{t_2} m^2uw = \int_{t_1}^{t_2} wf(t,u,v).$$

For $t \in (t_p,b)$, we continue the same calculus and we find

$$\int_{t_p}^{b} -\ddot{u}w + \int_{t_p}^{b} m^2uw = \int_{t_p}^{b} wf(t,u,v).$$

Then,

$$\begin{aligned}(-\dot{u}w)(b) + (\dot{u}w)(t_p^+) + \int_{t_p}^{b} \dot{u}\dot{w} + \int_{t_p}^{b} m^2uw &= \int_{t_p}^{b} wf(t,u,v)\\ (\dot{u}w)(t_p^-) + (wI_p)(t_p) + \int_{t_p}^{b} \dot{u}\dot{w} + \int_{t_p}^{b} m^2uw &= \int_{t_p}^{b} wf(t,u,v).\end{aligned}$$

Observe that

$$\dot{u}(t_p)w(t_p^-) = \sum_{k=1}^{p} w(t_k)I_k(u(t_k),v(t_k)) + \int_0^{t_p} \dot{u}\dot{w} + \int_0^{t_p} m^2uw - \int_0^{t_p} wf(t,u,v).$$

Consequently,

$$\sum_{k=1}^{p} w(t_k)I_k(u(t_k))) + \int_0^b \dot{u}\dot{w} + \int_0^b m^2uw = \int_0^b wf(t,u,v).$$

For $w = u$, we obtain

$$\sum_{k=1}^{p} u(t_k)I_k(u(t_k)) + \int_0^b \dot{u}^2 + \int_0^b m^2u^2 = \int_0^b uf(t,u,v).$$

Finally, we define the energy functional $E = (E_1, E_2)$ via

$$E_1(u,v) = \int_0^b \left[\frac{1}{2}(\dot{u}^2 + m^2u^2) - F(t,u,v)\right] dt + \sum_{k=1}^{p} \int_0^{u(t_k)} I_k(s)ds,$$

$$E_2(u,v) = \int_0^b \left[\frac{1}{2}(\dot{v}^2 + m^2v^2) - G(t,u,v)\right] dt + \sum_{k=1}^{p} \int_0^{v(t_k)} \overline{I}_k(s)ds,$$

where

$$F(t,u,v) = \int_0^u f(t,s,v)ds, \quad G(t,u,v) = \int_0^v g(t,u,s)ds.$$

This ends the proof. □

Now, we define what we mean by a solution of problem (1.3).

Definition 3.2. A pair of functions $(u,v) \in H_0^1(J) \times H_0^1(J)$ is said to be a weak solution of problem (1.3) if

$$\sum_{k=1}^{p} w(t_k)I_k(u(t_k)) + \int_0^b \dot{u}\dot{w} + \int_0^b m^2uw = \int_0^b wf(t,u,v),$$

and

$$\sum_{k=1}^{p} w(t_k)I_k(u(t_k)) + \int_0^b \dot{u}^2 + \int_0^b m^2u^2 = \int_0^b wf(t,u,v),$$

for every $w \in H_0^1(J)$.

We assume that the following conditions are satisfied:

(H_1) $f, g : J \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions.

(H_2) $f(\cdot,0,0), g(\cdot,0,0) \in L^2(J)$, and there exist $m_{ij} \in \mathbb{R}_+$ $(i,j = 1,2)$ such that

$$\begin{aligned} |f(t,u,v) - f(t,\overline{u},\overline{v})| &\le m_{11}|u - \overline{u}| + m_{12}|v - \overline{v}| \\ |g(t,u,v) - g(t,\overline{u},\overline{v})| &\le m_{21}|u - \overline{u}| + m_{22}|v - \overline{v}| \end{aligned}$$

for all $u, \overline{u}, v, \overline{v} \in \mathbb{R}$, a.e. $t \in J$.

(H_3) There exist $\overline{m}_{ij} \in \mathbb{R}_+ (i, j = 1, 2)$ such that

$$|I_k(x) - I_k(y)| \leq \overline{m}_{k_{11}} |x - y|, \quad |\overline{I}_k(x) - \overline{I}_k(y)| \leq \overline{m}_{k_{22}} |v - \overline{v}|, \text{ for all } x, y \in \mathbb{R}.$$

Lemma 3.3. *Assume that conditions (H_1) and (H_2) hold. Then,*

$$\begin{aligned} -\int_0^b |F(s, u, v)| ds \quad \geq \quad & -\frac{m_{11}}{2} \|u\|^2_{L^2(J)} - m_{12} \|v\|_{L^2(J)} \|u\|_{L^2(J)} \\ & -\|f(t, 0, 0)\|_{L^2(J)} \|u\|_{L^2(J)}. \end{aligned} \tag{3.1}$$

Proof. Indeed, from (H_2) we have

$$|f(t, u, v)| \quad \leq \quad m_{11}|u| + m_{12}|v| + |f(t, 0, 0)|, \tag{3.2}$$

since $f(\cdot, 0, 0) \in L^2(J)$, $f(\cdot, u(\cdot), v(\cdot)) \in L^2(J)$ whenever $(u, v) \in L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$. Also, (3.2) gives

$$|F(t, u, v)| \leq \frac{m_{11}}{2} |u|^2 + m_{12}|v||u| + |f(t, 0, 0)||u|.$$

Then,

$$\begin{aligned} -\int_0^b |F(s, u, v)| ds \quad & \geq \quad -\int_0^b \left[\frac{m_{11}}{2} |u(s)|^2 + m_{12}|v(s)||u(s)| + |f(s, 0, 0)||u(s)| \right] ds \\ & \geq \quad -\frac{m_{11}}{2} \|u\|^2_{L^2(J)} - m_{12} \|v\|_{L^2(J)} \|u\|_{L^2(J)} \\ & \qquad -\|f(\cdot, 0, 0)\|_{L^2(J)} \|u\|_{L^2(J)}. \end{aligned}$$

This ends the proof. □

In all of this section, we assume that the spectral radius of the matrix

$$M \quad = \quad \frac{1}{m^2} \begin{bmatrix} m_{11} + \displaystyle\sum_{k=1}^{p} m^3 \overline{m}_{k_{11}} \sqrt{bt_k} & m_{12} \\ m_{21} & m_{22} + \displaystyle\sum_{k=1}^{p} m^3 \overline{m}_{k_{22}} \sqrt{bt_k} \end{bmatrix} \tag{3.3}$$

is strictly less than one.

Lemma 3.4. *The energy E of the problem has a Fréchet derivative.*

Proof. Direct computation shows that the derivative of E at any u, after the direction $w \in H^1_0(\mathbb{R})$, is given by

$$(E'_1(u, v), w) \quad = \quad \lim_{\lambda \to 0} (E_1(u + \lambda w, v) - E_1(u, v)) \lambda^{-1}$$

$$\begin{aligned}
(E_1'(u,v),w) &= \lim_{\lambda\to 0}\Big[\int_0^b\left[\frac{1}{2}((\dot u+\lambda\dot w)^2+m^2(u+\lambda w)^2)-F(t,u+\lambda w,v)\right]dt \\
&\quad +\sum_{k=1}^p\int_0^{u(t_k)+\lambda w(t_k)} I_k(s)ds-\int_0^b\left[\frac{1}{2}(\dot u^2+m^2u^2)-F(t,u,v)\right]dt \\
&\quad -\sum_{k=1}^p\int_0^{u(t_k)} I_k(s)ds\Big]\lambda^{-1} \\
&= \int_0^b[\dot u\dot w+m^2u^2w^2-f(t,u,v)w]dt+\sum_{k=1}^p w(t_k)I_k(u(t_k)) \\
&= \langle u,w\rangle_{H_0^1(J)}-\langle f(\cdot,u,v),w\rangle_{L^2(J)}+\sum_{k=1}^p w(t_k)I_k(u(t_k)).
\end{aligned}$$

Hence, the Fréchet derivative of E_1 at any $u\in H_0^1(J)$ is given by

$$\begin{aligned}
E_{11}(u,v) &= u-Lf(\cdot,u,v)+\sum_{k=1}^p I_k(u(t_k)) \\
&= u-N_1(u,v)
\end{aligned}$$

and

$$\begin{aligned}
E_{22}(u,v) &= v-Lg(\cdot,u,v)+\sum_{k=1}^p \overline{I}_k(v(t_k)) \\
&= v-N_2(u,v),
\end{aligned}$$

where $N_1,N_2 : H_0^1(J)\times H_0^1(J)\to H_0^1(J)$ are defined by

$$N_1(u,v)=Lf(\cdot,u,v)-\sum_{k=1}^p I_k(u(t_k)),$$

and

$$N_2(u,v)=Lg(\cdot,u,v)-\sum_{k=1}^p \overline{I}_k(v(t_k)).$$

This ends the proof. □

This shows that weak solutions of (1.3) are the critical points of the functional E.

Theorem 3.5. *Assume that the conditions* (H_1)*,* (H_2)*, and* (H_3) *hold. In addition, assume that there exist two functions* $g,g_1 : J\times\mathbb{R}\to\mathbb{R}$ *such that* $g(t,\cdot)$*,* $g_1(t,\cdot)$ *are coercive and satisfy*

$$g_1(t,y)\le G(t,x,y)\le g(t,y),\ \textit{for all } x,y\in\mathbb{R}, \textit{a.e. } t\in J, \tag{3.4}$$

and

$$I_k(x) \geq 0, \quad \overline{I}_k(y) \geq 0, \textit{ for all } x, y \in \mathbb{R}. \tag{3.5}$$

Then the system (1.3) *has a unique solution* $(u^*, v^*) \in H_0^1(J) \times H_0^1(J)$*, which is a Nash-type equilibrium of the pair of functionals* (E_1, E_2) *associated to the system, i.e.,*

$$\begin{aligned} E_1(u^*, v^*) &= \inf_{H_0^1(J)} E_1(\cdot, v^*) \\ E_2(u^*, v^*) &= \inf_{H_0^1(J)} E_2(u^*, \cdot). \end{aligned}$$

Proof. We shall apply Theorem 2.1. First, using the Lipschitz conditions in (H_2), we can obtain that $E_1(\cdot, v)$, $E_2(u, .)$ are bounded for each $u, v \in H_0^1(J)$, and

$$\begin{aligned} E_1(u,v) &= \int_0^b \left[\frac{1}{2}(\dot{u}^2 + m^2 u^2) - F(t,u,v)\right] dt + \sum_{k=1}^p \int_0^{u(t_k)} I_k(s)ds \\ &= \frac{1}{2}\|u\|^2_{H_0^1(J)} - \int_0^b F(s,u,v)ds + \sum_{k=1}^p \int_0^{u(t_k)} I_k(s)ds. \end{aligned}$$

Using inequalities (3.1) and (3.5), we obtain

$$\begin{aligned} E_1(u,v) \geq\ & \frac{1}{2}\|u\|^2_{H_0^1(J)} - \frac{m_{11}}{2}\|u\|^2_{L^2(J)} - m_{12}\|v\|_{L^2(J)}\|u\|_{L^2(J)} \\ & -\|f(\cdot,0,0)\|_{L^2(J)}\|u\|_{L^2(J)}. \end{aligned}$$

By the Poincaré inequality, we get

$$\begin{aligned} E_1(u,v) &\geq \frac{1}{2}\|u\|^2_{H_0^1(J)} - \frac{m_{11}}{2m^2}\|u\|^2_{H_0^1(J)} - \frac{m_{12}}{m}\|v\|_{L^2(J)}\|u\|_{H_0^1(J)} \\ &\quad -\frac{1}{m}\|f(\cdot,0,0)\|_{L^2(J)}\|u\|_{H_0^1(J)} \\ &\geq \frac{1}{2}\left(1 - \frac{m_{11}}{2m^2}\right)\|u\|^2_{H_0^1(J)} - \frac{m_{12}}{m}\|v\|_{L^2(J)}\|u\|_{H_0^1(J)} \\ &\quad -\frac{1}{m}\|f(\cdot,0,0)\|_{L^2(J)}\|u\|_{H_0^1(J)}. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} E_2(u,v) \geq\ & \frac{1}{2}\left(1 - \frac{m_{22}}{2m^2}\right)\|v\|^2_{H_0^1(J)} - \frac{m_{12}}{m}\|u\|_{L^2(J)}\|v\|_{H_0^1(J)} \\ & -\frac{1}{m}\|g(\cdot,0,0)\|_{L^2(J)}\|v\|_{H_0^1(J)}. \end{aligned}$$

Consequently, the functionals $E_1(\cdot, v)$ and $E_2(u, \cdot)$ are bounded from below for each $u, v \in H_0^1(J)$. In addition, we use the inequality from (3.4) to obtain

$$E_2(u,v) = \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b G(t,u(t),v(t))dt + \sum_{k=1}^p \int_0^{v(t_k)} \overline{I}_k(s)ds$$

$$\geq \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b g(t, v(t))dt,$$

and we use the inequality from (3.5) to obtain

$$E_2(u, v) \geq \phi(v), \quad \text{for all } v \in H_0^1(J), \tag{3.6}$$

where

$$\phi(v) = \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b g(t, v(t))dt.$$

Since g is a coercive function, ϕ is bounded from below and thus $E_2(u, \cdot)$ is bounded from below uniformly with respect to u. Next,

$$\begin{aligned} E_2(u, v) &= \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b G(t, u(t), v(t))dt + \sum_{k=1}^{p} \int_0^{v(t_k)} \overline{I}_k(s)ds \\ &\leq \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b g_1(t, v(t))dt + \sum_{k=1}^{p} \int_0^{v(t_k)} \overline{I}_k(s)ds. \end{aligned}$$

Then,

$$E_2(u, v) \leq \phi_1(v), \ \text{ for all } v \in H_0^1(J), \tag{3.7}$$

where

$$\phi_1(v) = \frac{1}{2}\|v\|^2_{H_0^1(J)} - \int_0^b g_1(t, v(t))dt + \sum_{k=1}^{p} \int_0^{v(t_k)} \overline{I}_k(s)ds.$$

From (3.6) and (3.7),

$$\phi(v) \leq E_2(u, v) \leq \phi_1(v), \ \text{ for all } u, v \in H_0^1(J). \tag{3.8}$$

Since ϕ is coercive, for each $\lambda > 0$, there is R_λ such that

$$\phi(v) \geq \lambda \text{ for } \|v\|_{H_0^1(J)} \geq R_\lambda. \tag{3.9}$$

Let $a > 0$ and $\lambda = \inf\limits_{v \in H_0^1(J)} \phi_1(v) + a$ for $\|v\|_{H_0^1(J)} \geq R_\lambda$ and any $u \in H_0^1(J)$. Then, we have

$$E_2(u, v) \geq \phi(v) \geq \inf_{v \in H_0^1(J)} \phi_1(v) + a. \tag{3.10}$$

From the first inequality of (3.8), we have

$$\inf_{v \in H_0^1(J)} E_2(u, v) + a \leq \inf_{v \in H_0^1(J)} \phi_1(v) + a = \lambda. \tag{3.11}$$

But, (3.9) and (3.11) imply that

$$E_2(u, v) \geq \inf_{v \in H_0^1(J)} E_2(u, v) + a \quad \|v\|_{H_0^1(J)} \geq R_\lambda \ \forall u \in H_0^1(J).$$

This shows that E_2 satisfies the condition (2.3). Finally, we prove that $N=(N_1,N_2)$ is a Perov contraction. Indeed, for any $u,v,\overline{u},\overline{v}\in H_0^1(J)$, using the fact that L is an isometry between $H^{-1}(J)$ and $H_0^1(J)$, using the relations (2.6), (2.7), and the Lipschitz condition (H_3), we obtain

$$\begin{aligned}\|N_1(u,v)-N_1(\overline{u},\overline{v})\|_{H_0^1(J)} &\leq \|Lf(\cdot,u,v)-Lf(\cdot,\overline{u},\overline{v})\|_{H_0^1(J)}\\ &\quad+\sum_{k=1}^{p}\|I_k(u(t_k))-I_k(\overline{u}(t_k))\|_{H_0^1(J)}\\ &\leq \|f(\cdot,u,v)-f(\cdot,\overline{u},\overline{v})\|_{H^{-1}(J)}\\ &\quad+\sum_{k=1}^{p}\|I_k(u(t_k))-I_k(\overline{u}(t_k))\|_{H_0^1(J)}\\ &\leq \frac{1}{m}\|f(\cdot,u,v)-f(\cdot,\overline{u},\overline{v})\|_{L^2(J)}\\ &\quad+\sum_{k=1}^{p}\|I_k(u(t_k))-I_k(\overline{u}(t_k))\|_{H_0^1(J)}.\end{aligned}$$

For each $k\in\{1,\ldots,p\}$, we have

$$\begin{aligned}\|I_k(u(t_k))-I_k(\overline{u}(t_k))\|_{H_0^1(J)} &\leq m\left(\int_0^b |I_k(u(t_k))-I_k(\overline{u}(t_k))|^2dt\right)^{\frac{1}{2}}\\ &= m\sqrt{b}|I_k(u(t_k))-I_k(\overline{u}(t_k))|\\ &\leq m\overline{m}_{k_{11}}\sqrt{b}|u(t_k)-\overline{u}(t_k)|\\ &\leq m\overline{m}_{k_{11}}\sqrt{b}\int_0^{t_k}|u'(t)-\overline{u}'(t)|dt\\ &\leq m\overline{m}_{k_{11}}\sqrt{bt_k}\left(\int_0^{t_k}|u'(t)-\overline{u}'(t)|^2dt\right)^{\frac{1}{2}}.\end{aligned}$$

Hence,

$$\|I_k(u(t_k))-I_k(\overline{u}(t_k))\|_{H_0^1(J)}\leq m\overline{m}_{k_{11}}\sqrt{bt_k}\|u-\overline{u}\|_{H_0^1(J)}\quad k=1,\ldots,p.$$

Then,

$$\begin{aligned}\|N_1(u,v)-N_1(\overline{u},\overline{v})\|_{H_0^1(J)} &\leq \frac{m_{11}}{m}\|u-\overline{u}\|_{L^2(J)}+\frac{m_{12}}{m}\|v-\overline{v}\|_{L^2(J)}\\ &\quad+\sum_{k=1}^{p}m\overline{m}_{k_{11}}\sqrt{bt_k}\|u-\overline{u}\|_{H_0^1(J)}.\end{aligned}$$

Therefore,

$$\|N_1(u,v)-N_1(\overline{u},\overline{v})\|_{H_0^1(J)} \leq \frac{1}{m^2}\left(m_{11}+\sum_{k=0}^{p}m^3\overline{m}_{k_{11}}\sqrt{bt_k}\right)\|u-\overline{u}\|_{H_0^1(J)}$$

$$+\frac{m_{12}}{m^2}\|v-\overline{v}\|_{H_0^1(J)}.$$

Similarly for N_2, we have

$$\begin{aligned}\|N_2(u,v)-N_2(\overline{u},\overline{v})\|_{H_0^1(J)} &\leq \frac{1}{m^2}\left(m_{22}+\sum_{k=0}^{p} m^3\overline{m}_{k22}\sqrt{bt_k}\right)\|v-\overline{v}\|_{H_0^1(J)} \\ &\quad +\frac{m_{21}}{m^2}\|u-\overline{u}\|_{H_0^1(J)}.\end{aligned}$$

Hence, N is a Perov contraction with the Lipschitz matrix M given by (3.3). Therefore, Theorem 2.1 can be applied. □

Remark 3.6. Notice that the theory for systems of two equations can easily be extended to the general case of n-dimensional systems.

4 Example

We conclude this paper with an illustrative example.

Example 4.1. Consider the following system

$$\begin{cases} -\ddot{u}+m^2u &= \alpha_1(t)\cos u(t)+\beta_1(t)\sin u(t)\sin v(t)+\sigma_1(t), \quad t\in[0,b] \\ -\ddot{v}+m^2v &= \alpha_2(t)\sin v(t)+\beta_2(t)\cos u(t)\sin v(t)+\sigma_2(t), \quad t\in[0,b] \\ \dot{u}(t_1^+)-\dot{u}(t_1^-) &= \frac{1}{a_1}|u(t_1)|, \qquad a_1>0, t_1\neq 0, \quad t_1\in(0,b) \\ \dot{v}(t_1^+)-\dot{v}(t_1^-) &= \frac{1}{a_2}|v(t_1)|, \qquad a_2>0 \\ u(0)=u(b) &= v(0)=v(b)=0, \end{cases} \tag{4.1}$$

where $m\neq 0$, $\alpha_i,\beta_i\in C([0,b],\mathbb{R}_+)$, $\sigma_i\in L^2([0,b],\mathbb{R}_+)(i=1,2)$, and

$$f(t,x,y)=\alpha_1(t)\cos x+\beta_1(t)\sin x\sin y+\sigma_1(t),$$

$$G(t,x,y)=\alpha_2(t)\sin y+\beta_2(t)\cos x\sin y+\sigma_2(t).$$

In this case,

$$F(t,x,y)=\alpha_1(t)\sin x+\beta_1(t)(1-\cos x)\sin y+\sigma_1(t)x,$$

$$G(t,x,y)=\alpha_2(t)(1-\cos y)+\beta_2(1-cosy)cosx+\sigma_2(t)y.$$

If the spectral radius of the matrix

$$M=\frac{1}{m^2}\begin{bmatrix} \|\alpha_1\|_\infty+\|\beta_1\|_\infty+\dfrac{m^3\sqrt{bt_1}}{a_1} & \|\beta_1\|_\infty \\ \|\beta_2\|_\infty & \|\alpha_2\|_\infty+\|\beta_2\|_\infty+\dfrac{m^3\sqrt{bt_1}}{a_2} \end{bmatrix} \tag{4.2}$$

is less than one, then the system (4.1) has a unique solution, which is a Nash-type equilibrium of the corresponding pair of energy functionals. In particular, the result holds for the following system on $[0,1]$

$$\begin{cases} -\ddot{u}+u &= \dfrac{2}{15}\cos u(t)+\dfrac{1}{5}\sin u(t)\sin v(t)+\sigma_1(t) \\ -\ddot{v}+v &= \dfrac{1}{6}\sin v(t)+\dfrac{1}{6}\cos u(t)\sin v(t)+\sigma_2(t) \\ \dot{u}(t_1^+)-\dot{u}(t_1^-) &= |u(t_1)|, \quad t_1=\dfrac{1}{9} \\ \dot{v}(t_1^+)-\dot{v}(t_1^-) &= |v(t_1)|, \quad t_1=\dfrac{1}{9} \\ u(0)=u(1) &= v(0)=v(1)=0, \end{cases} \tag{4.3}$$

where $\sigma_i \in L^2([0,1],\mathbb{R}_+)(i=1,2)$. In this case, the matrix M is

$$M = \begin{bmatrix} \frac{2}{3} & \frac{1}{5} \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix} \tag{4.4}$$

and one can easily see that its spectral radius is less than one.

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