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Sequential Caputo versus Nonsequential Caputo Fractional Initial and Boundary Value Problems

Aghalaya S. Vatsala
University of Louisiana at Lafayette
Department of Mathematics
Lafayette, 70504, USA
vatsala@louisiana.edu

Bhuvaneswari Sambandham
Dixie State University
Department of Mathematics
Saint George, 84790, USA
buna.sambandham@dixie.edu

Abstract

This is a survey article on the authors' work on a sequential fractional differential equation with initial conditions and boundary conditions. It is known that the dynamic equation of the integer order derivatives are always sequential. In the literature, the majority of the work done on Caputo fractional differential equations, the order of the fractional derivative is seldom studied as sequential. In the Caputo differential equation of order q, when $0 < q < 1$, we can show how the value of q can be chosen to improve the model from the data. For a nonsequential Caputo differential equation the basis solution is that of the corresponding integer order. In this work, we demonstrate that for sequential Caputo differential equation, the basis solution depends on q. In short, the initial and boundary conditions also depends on the value of q. Thus the value of q can be used as a parameter when we are comparing its solution with the corresponding integer order differential equation. The advantage of using the Caputo fractional derivative is that it is global in nature.

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1 Introduction

It is known that in [23] the half order fractional diffusion model has been established to be useful and also economical. The theory of derivatives of noninteger order dates back to Leibnitz note to L'Hospital (1695) in which the meaning of the derivative of one half is discussed. For three centuries, the theory of fractional derivatives developed mainly as a pure theoretical field and as a useful topic of research for mathematicians. However, in the past four decades the application of fractional dynamic equations was felt in several scientific and engineering areas. See [4, 5, 7, 9, 11, 14–17, 20–24, 28, 33] and the references therein for more details. See [11, 14, 16–18, 23, 24, 29] for fractional initial value problems. See [1–3, 6, 10, 12, 13, 25, 27, 30, 31, 34, 37, 38] for fractional boundary value problems. Also, see [8, 19, 27, 31–33] for some numerical work on fractional dynamic equations.

Although there are many types of fractional derivatives the Caputo derivative is the closest to the integer derivative. In fact, the Caputo derivative reduces to the integer derivative, when the order of the fractional derivative tends to an integer. In particular, if the order of the fractional derivative is q, when $0 < q < 1$, the solution of the initial value problem involves the Mittag–Leffler function which is the generalization of the exponential function. In general, the exponential function is very much used to solve the n^{th} order linear equation with constant coefficients. In contrast, the Mittag–Leffler function of order q when $0 < q < 1$, is seldom used to solve linear nq order Caputo fractional differential equation having lower order fractional derivative terms. One of the main reason for this is that the Caputo fractional derivative is not sequential, where as the integer derivative is sequential. In this work, we will demonstrate the applications of Caputo fractional differential equations when the Caputo fractional derivative involved is sequential. See [4, 7, 8, 15, 26, 29–32, 34–36] for work done on sequential initial value problems and sequential boundary value problems. For nonsequential Caputo fractional dynamic equation of order nq when $(n-1) < nq < n$, the basis solution is assumed to be that of the integer derivative, which is $1, t, \dots, t^{(n-1)}$. In addition, for an initial value problem, the initial conditions are the same as the initial conditions of an n^{th} order differential equation. For a nonsequential boundary value problem of order say, $2q$ when $1 < 2q < 2$, the boundary conditions are the same as that of a second order differential equation. However, if we assume the order of the fractional derivative say nq, is sequential of order q, the basis solution will be $1, t^q, t^{2q}, \dots, t^{(n-1)q}$. The initial condition and the boundary conditions also involve the fractional derivative of lower order terms. In this situation, we can use the Mittag–Leffler function to solve the linear sequential initial value problem, with the initial conditions involving the fractional derivative. We can also use the Laplace transform method since the Caputo fractional derivative by definition is a convolution integral. Similarly, for boundary value problem, the boundary conditions will involve the fractional derivative. In short, when the sequential Caputo fractional derivative tends to an integer, the initial conditions and the boundary conditions also tend to the initial and boundary conditions of the corresponding integer order.

The advantage of assuming sequential Caputo derivative in our dynamic equation is the fractional derivative is global in nature where as the integer derivative is local in nature. Thus we can use the value of q as a parameter to enhance our mathematical model. In this work, we will recall some salient features of the work done on sequential initial and boundary value problems from [29, 35].

2 Preliminaries

In this section, we recall some basic definitions and some known results which are useful in establishing our results for sequential initial value problems and sequential boundary value problems.

Definition 2.1. Let $q > 0$, and $u : (0, \infty) \to \mathbb{R}$. Then the Caputo derivative of order q is given by

$$^cD^q u(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{q-n+1}} ds,$$

where $n \in \mathbb{N}$ such that $(n-1) < q < n$. In particular, if $q = n$, an integer, then $^cD^q u = u^{(n)}(t)$ and $^cD^q u = u^{'}(x)$ if $q = 1$.

In this work, we choose the value q such that it is replaced by nq such that $n - 1 < nq < n$. In short if $q = 1$, then we have the nth order derivative.

Definition 2.2. Let $q > 0$, and $u : (0, \infty) \to \mathbb{R}$. Then the Riemann–Liouville derivative of order q is given by

$$D^q u(t) = \frac{1}{\Gamma(n-q)} (\frac{d}{dt})^n \int_0^t \frac{u(s)}{(t-s)^{q-n+1}} ds,$$

where $n \in \mathbb{N}$ such that $(n-1) < q < n$.

Next we define the Riemann–Liouville integral of order q for $0 < q < 1$.

Definition 2.3. The Riemann–Liouville fractional integral of arbitrary order q is defined by

$$D^{-q} u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{(q-1)} u(s) ds, \tag{2.1}$$

where $0 < q \leq 1$.

Note that the definition of Riemann–Liouville integral of order q for $0 < q < 1$, is the same as the Caputo integral of order q. The next definition is that of Mittag–Leffler function. It plays the same role as that of the exponential function for the integer derivative dynamic equations, specially when $0 < q < 1$.

Definition 2.4. Mittag–Leffler function of two parameters q, r is given by

$$E_{q,r}(\lambda(t-t_0)^q) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^q)^k}{\Gamma(qk+r)},$$

where $q, r > 0$. Also, for $t_0 = 0$ and $r = 1$, we get

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)}, \tag{2.2}$$

where $q > 0$.

If $q = 1$ then $E_{1,1}(\lambda t) = e^{\lambda t}$. See [11, 14, 16, 24] for more details. Reference [11] is exclusively for the study and application of Mittag–Leffler function. Note that when $q = 1$ in Equation (2.2) is the special case of integer derivative and it is the usual exponential function. Consider the linear Caputo fractional differential equation of order q when $n-1 < q < n$,

$$^cD^q u(t) = \lambda u + f(t),\ t > 0, \tag{2.3}$$

$$u^k(0) = b_k\ (b_k \in \mathbb{R};\ k = 0, 1, \ldots, n-1), \tag{2.4}$$

where $f(t)$ is continuous on $[0, \infty)$. Using the Laplace transform method the solution of (2.3) and (2.4) can be obtained as

$$u(t) = \sum_{j=0}^{n-1} b_j t^j E_{q,j+1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s) ds. \tag{2.5}$$

In particular, for $0 < q < 1$, the analytical solution of (2.3) and (2.4) is given by

$$u(t) = b_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s) ds. \tag{2.6}$$

Note that the analytical solution of the form can also be obtained by writing the equivalent of (2.3) and (2.4) in the space of $n-1$ continuously differentiable functions on $[0, \infty)$ in the Volterra integral equation of the second kind. See Section 4.1.3 of [14] monograph for more details. If $q = n$, then, equation (2.3) is an n^{th} order linear differential equations which is sequential. The standard method to solve n^{th} order linear homogeneous differential equations is assuming the solution to be of the form e^{rt}, where r, would be the n roots of the n^{th} degree polynomial $r^n = \lambda$. In order to apply a similar method, we will denote the q in (2.3), by nq, where now $(n-1) < nq < n$. Then we can seek solution of the form $E_{q,1}(rt^q)$, where now $\frac{(n-1)}{n} < q < 1$. This will be possible, only if $({}^cD^{nq}_{0+})u(t)$ is sequential of order q. We next define this precisely.

Definition 2.5. The Caputo fractional derivative of order nq , for $(n-1) < nq < n$ is said to be sequential Caputo fractional derivative of order q, if the relation

$$({}^{c}D^{nq}_{0+})u(t) = {}^{c}D^{q}_{0+}({}^{c}D^{(n-1)q}_{0+})u(t), \tag{2.7}$$

holds for $n = 2, 3, \ldots$ etc.

From now on we denote the sequential Caputo derivative of order q as

$$({}^{sc}D^{nq})u(t),$$

where $n \geq 2$ is an integer. Note that the basis solution for $({}^{c}D^{nq})u(t) = 0$ is

$$1, t, t^2, \ldots, t^{n-1},$$

where as the basis solution of $({}^{sc}D^{nq}_{0+})u(t) = 0$, is given by

$$1, t^q, t^{2q}, \ldots, t^{(n-1)q}.$$

Observe that

$$({}^{sc}D^{nq})E_{q,1}(rt^q) = (r)^n E_{q,1}(\lambda t^q).$$

In order to develop our main result, we present some basic auxiliary results of the Laplace transform of Mittag–Leffler functions and other related functions. It is easy to observe that the Laplace transform of t^p for any $p > -1$ is given by

$$\mathfrak{L}(t^p) = \frac{\Gamma(p+1)}{s^{p+1}}.$$

See the Laplace transform tables in [35,36] for some basic fractional functions including the Mittag–Lefller functions. These results yield the integer results as a special case. Also, it is easy to observe, that the initial conditions provided in (2.4) will not be useful to solve for a linear Caputo sequential differential equation. We need to assume, the initial conditions to involve the fractional derivative at time $t = 0$, is essential. In fact the initial condition, should be of the form

$$({}^{c}D^{kq}_{0+})u(t)|_{t=0} = b_k \ (b_k \in \mathbb{R};\ k = 0, 1, \ldots, n-1). \tag{2.8}$$

The next result is related to taking the Laplace transform of Caputo sequential derivative of order nq, which is sequential of order q. This will be very useful in solving the linear Caputo sequential differential equation of order nq, which is sequential of order q, with constant coefficients.

Theorem 2.6. *The Laplace transform of a sequentially Caputo fractional differentiable function $u(t)$ of order q, on $[0, \infty)$ when nq is such that $n-1 < nq < n$, is given by*

$$\begin{aligned}\mathfrak{L}({}^{sc}D^{nq}u(t)) = s^{nq}U(s) - s^{(nq-1)}u(0) - s^{(n-1)q-1}({}^{sc}D^{q}u(t)|_{t=0}) \\ - s^{(n-2)q-1}({}^{sc}D^{2q}u(t)|_{t=0}) \ldots - s^{q-1}({}^{sc}D^{(n-1)q}u(t)|_{t=0}),\end{aligned} \tag{2.9}$$

where $U(s) = \mathfrak{L}u(t)$.

Next we provide the definitions to introduce Caputo sequential boundary value problem. In order the make the distinction between initial value problem and boundary value problems, we will use x, as the independent variable. We recall the following definitions from [14].

Definition 2.7. The Caputo (left-sided) fractional derivative of $u(x)$ of order q, when $n-1 < q < n$, is given by

$$^{c}D_{0^+}^{q}u(x) = \frac{1}{\Gamma(n-q)}\int_0^x (x-s)^{n-q-1}u^{(n)}(s)ds,\ \ x > 0, \tag{2.10}$$

and the (right-sided) fractional derivative is given by

$$^{c}D_{1^-}^{q}u(x) = \frac{(-1)^n}{\Gamma(n-q)}\int_x^1 (s-x)^{n-q-1}u^{(n)}(s)ds,\ \ x < 1, \tag{2.11}$$

where $u^{(n)}(t) = \dfrac{d^n(u)}{dt^n}$.

In particular, if $q = n$, an integer, then $^{c}D^{q}u = u^{(n)}(x)$ and $^{c}D^{q}u = u'(x)$ if $q = 1$.

Definition 2.8. The Caputo fractional derivative of order nq, for $(n-1) < nq < n$ is said to be the (left) sequential Caputo fractional derivative of order q, if the relation

$$(^{c}D_{0^+}^{nq})u(x) = \ ^{c}D_{0^+}^{q}(^{c}D_{0^+}^{(n-1)q})u(x), \tag{2.12}$$

holds for $n = 2, 3, \ldots$ etc. Note that one can define the (right) sequential Caputo fractional derivative of order nq in terms of the (right) sequential Caputo fractional derivative of order q.

We denote the (left) sequential Caputo derivative of order q as $(^{sc}D_{0^+}^{nq})u(x)$, where $n \geq 2$ is an integer. Similarly, we will denote the (right) sequential Caputo derivative of order q as $(^{sc}D_{1^-}^{nq})u(x)$, where $n \geq 2$ is an integer. From the paper [30], we consider the linear sequential Caputo fractional boundary value problem

$$\begin{aligned} -^{sc}D_{0^+}^{2q}u + p(x)^{sc}D_{0^+}^{q}u + q(x)u = F(x) \\ \alpha_0 u(0) - \beta_0{}^{c}D_{0^+}^{q}u(0) = b_0, \\ \alpha_1 u(1) + \beta_1{}^{c}D_{1^-}^{q}u(1) = b_1, \end{aligned} \tag{2.13}$$

where $\alpha_0, \alpha_1 \geq 0$ and $\beta_0, \beta_1 > 0$ provided $\alpha_0\beta_1 + \alpha_1\beta_0 \neq 0$. The linear comparison result has been developed in [30] in such a way that the linear sequential Caputo boundary value problem of Equation (2.13) has a unique solution. This has been proved under the assumption that the left sequential Caputo fractional derivative of order nq will be the same as the right sequential Caputo fractional derivative of order nq for $n = 1, 2$ in [30, 31]. We make a similar assumption throughout this paper also. This guarantees the uniqueness of the solution of all the linear sequential Caputo fractional boundary value problems that we discuss in this paper.

3 Sequential Caputo Initial Value Problems

In this section, we will provide a methodology to solve a Caputo sequential linear initial value problem with the initial conditions having fractional derivatives of lower order. Consider the linear Caputo fractional sequential differential equation of order nq, which is sequential of order q, with initial condition of the form,

$$a_n {}^{sc}D^{nq}u + a_{(n-1)} {}^{sc}D^{(n-1)q}u + \dots a_1 {}^{sc}D^{q}u + a_0 u = f(t), \tag{3.1}$$

with initial conditions,

$$({}^{c}D_{0+}^{kq})u(t)|_{t=0} = b_k \ (b_k \in \mathbb{R};\ k = 0, 1, \dots, n-1). \tag{3.2}$$

The above initial value problem can be solved using Theorem 2.6, just as in the integer case except that while taking the inverse Laplace transform, one needs to use the Laplace transform table developed for fractional differential equations. See the Laplace transform tables for fractional differential equation in [35, 36].

Consider the linear Caputo sequential differential equation of order $2q$ of the form

$${}^{sc}D^{2q}u + b {}^{sc}D^{q}u + cu = 0,\ t \in (0, \infty) \tag{3.3}$$

with initial conditions as,

$$u(0) = A,\ \ {}^{sc}D^{q}u(t)|_{t=0} = B. \tag{3.4}$$

Now applying Laplace transform on (3.3) and (3.4) using Theorem 2.6, we get

$$s^{2q}U(s) - s^{(2q-1)}u(0) - s^{(q-1)}({}^{sc}D^{q}u(t)|_{t=0}) + b(s^{q}U(s) - s^{(q-1)}u(0)) + cU(s) = 0, \tag{3.5}$$

where $U(s) = \mathfrak{L}(u(t))$. Now solving for $U(s)$ from equation (3.5) and substituting the initial conditions from equation (3.4) we get

$$U(s) = \frac{As^{(2q-1)} + (B + bA)s^{(q-1)}}{s^{2q} + bs^{q} + c}. \tag{3.6}$$

Now if we can take the inverse Laplace transform on both sides of equation (3.6), we get the solution of the sequential Caputo initial value problem (3.3) and (3.4). Here we will consider the special case, when $b = 0$, and c as c^2. In this case, we get,

$$U(s) = \frac{As^{(2q-1)}}{s^{2q} + c^2} + \frac{1}{c}\frac{Bcs^{(q-1)}}{s^{2q} + c^2}. \tag{3.7}$$

Taking the inverse Laplace transform, we get,

$$u(t) = A\cos_{q,1}(ct^{q}) + \frac{B}{c}\sin_{q,1}(ct^{q}). \tag{3.8}$$

Here q is such that $1/2 \leq q \leq 1$, when $q = 1$, we obtain the second order result as a special case. Note that, if $q = 1$, we get the integer result as a special case. See the graphs of $\cos_{q,1}(t^q)$ and $\sin_{q,1}(t^q)$ for different values of q. These graphs are from [29]. It is easy to observe from the graphs that as $q \to 1$, the solutions reduces to the usual $\cos t$, $\sin t$, functions. Consider the linear Caputo fractional differential equation of order $2q$, $1 < 2q < 2$

$$^cD^{2q}u(t) + u(t) = 0 \tag{3.9}$$

where $u(0) = 0$, $D^q(u(0)) = 0$ for $0.5 < q < 1$.

Let $u = E_{q,1}(\lambda t^q)$. Then the solution for (3.9) is given by the equation

$$u(t) = \cos_{q,1}(t^q) = \sum_{k=0}^{\infty} \frac{(t^q)^{2k}(-1)^k}{\Gamma(2kq+1)} \tag{3.10}$$

and

$$v(t) = \sin_{q,1}(t^q) = \sum_{k=0}^{\infty} \frac{(t^q)^{(2k+1)}(-1)^k}{\Gamma((2k+1)q+1)} \tag{3.11}$$

where $t \geq 0$, $0.5 < q < 1$. In the graph below when $0.5 < q < 1$, the zeros of $\sin_q(t^q)$ and $\cos_q(t^q)$ are approximately close to $\sin t$ and $\cos t$ graphs. When $q = 0.5$, there is

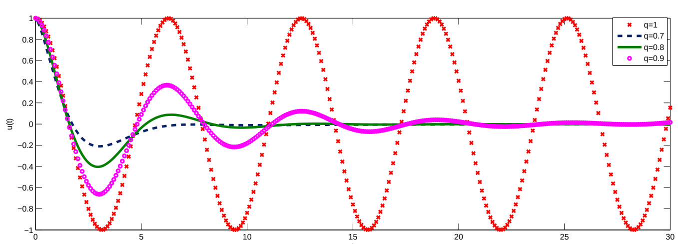

Figure 3.1: $\cos_q(t^q)$ graph

a bifurcation in the $\sin_q(t^q)$ and $\cos_q(t^q)$ graph. That is they no longer are oscillatory solutions. When $0 < q < 0.5$ we have exponentially decaying graph given below.

Remark 3.1. Observe that the graphs of $\sin_q(t^q)$, and $\cos_q(t^q)$ tends to the graph of the trigonometric functions $\sin t$ and $\cos t$ functions. On the other hand, if $q << 1$, the graphs exhibit damping behavior even though there is no damping term in the dynamic equation. This behavior of the fractional trigonometric function is very useful in mechanical systems and asymptotic behavior in the stability study of epidemiological models. (It can be SIR models or SEIR models). Basically, we can enhance the integer model by choosing the value q as a parameter.

The above graphs show that the solution of the $2q$ order differential equation reduces to the first order differential equation when $2q = 1$. In fact $q = 0.5$, is the bifurcation

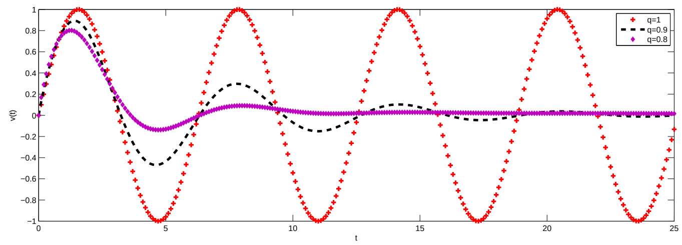

Figure 3.2: $\sin_q(t^q)$ graph

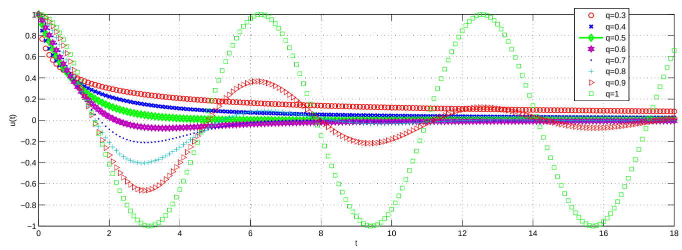

Figure 3.3: $\cos_q(t^q)$ graph

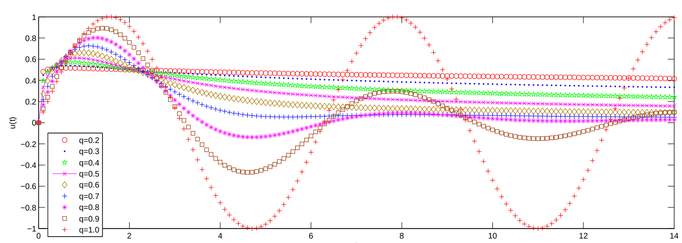

Figure 3.4: $\sin_q(t^q)$ graph

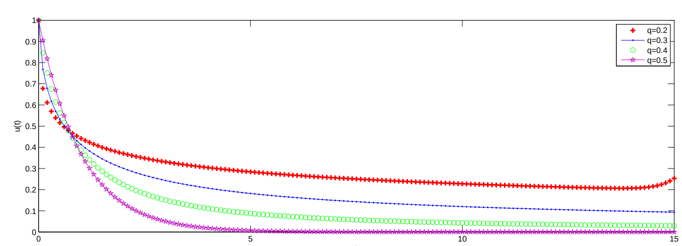

Figure 3.5: $\cos_q(t^q)$ graph

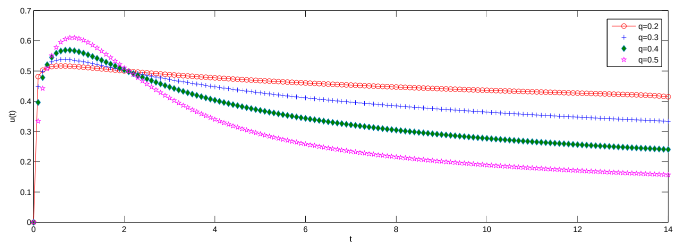

Figure 3.6: $\sin_q(t^q)$ graph

value when the behavior of the solution changes to Caputo fractional differential equation of order $2q = q' < 1$. We conjecture that a similar property will be true for a sequential dynamic equation of order nq where $(n-1) < nq < n$. That is we expect that the $nq = (n-1)$, or $q = \dfrac{(n-1)}{n}$ will be the bifurcation value when the behavior of the solution changes.

4 Sequential Caputo Boundary Value Problems

In this section, we recall some sequential boundary value problem results with a numerical example. For that purpose, consider the sequential Caputo fractional boundary value problem with mixed homogeneous boundary conditions of the form [30]

$$\begin{aligned} -^{sc}D^{2q}u = f(x), \ \ on\ J, \ \ f \in C[J \times \mathbb{R}, \mathbb{R}] \\ \alpha_0 u(a) - \beta_0 {}^cD^q_{a+}u(a) = 0, \\ \alpha_1 u(b) + \beta_1 {}^cD^q_{b-}u(b) = 0, \end{aligned} \tag{4.1}$$

$u \in C^2[J, \mathbb{R}]$ where $\alpha_0, \alpha_1 \geq 0$ and $\beta_0, \beta_1 > 0$, provided $\alpha_0\beta_1 + \alpha_1\beta_0 \neq 0$ where $0.5 < q < 1$. The unique solution of Equation (4.1) in terms of Green's function is given by

$$u(x) = \int_0^1 G(x,s) f(s) ds, \tag{4.2}$$

where $G(x,s)$ is the Green's function that satisfies the homogeneous boundary conditions. The Green's function obtained is of the form

$$G(x,s) = \begin{cases} \dfrac{1}{\Gamma(q+1)}[\dfrac{((x-a)^q + \frac{\beta_0}{\alpha_0}\Gamma(q+1))((b-s)^q - \frac{\beta_1}{\alpha_1}\Gamma(q+1))}{((s-a)^q + (b-s)^q + \frac{\beta_0}{\alpha_0}\Gamma(q+1) - \frac{\beta_1}{\alpha_1}\Gamma(q+1))}] & x < s, \\ \dfrac{1}{\Gamma(q+1)}[\dfrac{((s-a)^q + \frac{\beta_0}{\alpha_0}\Gamma(q+1))((b-x)^q - \frac{\beta_1}{\alpha_1}\Gamma(q+1))}{((s-a)^q + (b-s)^q + \frac{\beta_0}{\alpha_0}\Gamma(q+1) - \frac{\beta_1}{\alpha_1}\Gamma(q+1))}] & x > s. \end{cases} \tag{4.3}$$

Next, we consider the Caputo fractional boundary value problem with mixed non homogeneous boundary conditions. This example is taken from [16].

$$\begin{aligned} -{}^{sc}D^{2q}u = f(x), \ \text{ on } J, \ \ f \in C[J \times \mathbb{R}, \mathbb{R}] \\ \alpha_0 u(a) - \beta_0 {}^cD^q_{a+}u(a) = b_0, \\ \alpha_1 u(b) + \beta_1 {}^cD^q_{b-}u(b) = b_1, \end{aligned} \tag{4.4}$$

$u \in C^2[J, \mathbb{R}]$, $\alpha_0, \alpha_1 \geq 0$ and $\beta_0, \beta_1 > 0$, provided $\alpha_0\beta_1 + \alpha_1\beta_0 \neq 0$ and b_0, b_1 are constants.

The unique solution of Equation (4.4) in terms of Green's function is given by

$$u(x) = C_1(x-a)^q + C_2(b-x)^q + \int_a^b G(x,s)F(s)ds, \tag{4.5}$$

where C_1 and C_2 are constants and can be found by Equation (4.4). It is easy to observe that Green's function is the same as in Equation (4.3); set

$$\Delta = \alpha_0\alpha_1(b-a)^{2q} + 2\beta_0\beta_1(\Gamma(q+1))^2 - (b-a)^q\Gamma(q+1)(\alpha_0\beta_1 + \alpha_1\beta_0),$$

and Equation (4.5) can be written as

$$\begin{aligned} u(x) = & \frac{b_1(\alpha_0(b-a)^q - \beta_0(\Gamma(q+1)) - b_0(\Gamma(q+1)\beta_1)}{\Delta}(x-a)^q \\ & + \frac{b_0(\alpha_1(b-a)^q - \Gamma(q+1)\beta_1) + b_1\beta_0\Gamma(q+1)}{\Delta}(b-x)^q \\ & + \int_a^b F(s)G(x,s)ds. \end{aligned} \tag{4.6}$$

The detailed proof of Green's function is given in [30] .

Remark 4.1. Note that, in the non-sequential Caputo boundary value problem, in order to compute the Green's function, the solution of the operator $-{}^cD^{2q}u = 0$ of the boundary value problem, is $u(x) = Ax + B(1-x)$, which is the same as that of the integer operator, $-u''(x) = 0$. In the sequential case, it will be $u(x) = A(x)^q + B(1-x)^q$. This essentially is useful in choosing the value of q as a parameter, that can be chosen to improve the mathematical model. We also present a numerical example from [31] to demonstrate the use of the value of q as a parameter.

Remark 4.2. It is to be noted that when $q = 1$ in Equations (4.3) and (4.6) the integer result has been a special case of our result including the mathematical formula of the Green's function that has been obtained here. The aim here is twofold. The first one is to develop an efficient numerical scheme that yields the integer result with the least error as the value of q tends to 1. The second one is to use the value of q as a parameter to improve our mathematical model.

We consider the linear sequential Caputo fractional boundary value problem with mixed homogeneous boundary conditions from [31]:

$$\begin{aligned} -{}^{sc}D^{2q}u &= \sin_q(x) \\ u(0) - {}^{c}D^{q}_{0+}u(0) &= 0, \\ u(1) + {}^{c}D^{q}_{1-}u(1) &= 0. \end{aligned} \tag{4.7}$$

The solution of Equation (4.7) is given by

$$u(x) = \int_0^1 G(x,s)(-\sin_q(s))ds, \tag{4.8}$$

where $G(x,s)$ is the Green's function given by

$$G(x,s) = \begin{cases} -\dfrac{(1-s)^q + \Gamma(q+1)}{s^q + (1-s)^q + 2\Gamma(q+1)} \dfrac{x^q + \Gamma(q+1)}{\Gamma(q+1)}, & x < s \\ -\dfrac{(s)^q + \Gamma(q+1)}{s^q + (1-s)^q + 2\Gamma(q+1)} \dfrac{(1-x)^q + \Gamma(q+1)}{\Gamma(q+1)}, & x > s. \end{cases} \tag{4.9}$$

Note that the above formula is a special case of Equation (4.3) with $a = 0,\ b = 1$. Hence

$$\begin{aligned} u(x) = &\int_0^x \left(-\frac{(s)^q + \Gamma(q+1)}{s^q + (1-s)^q + 2\Gamma(q+1)} \frac{(1-x)^q + \Gamma(q+1)}{\Gamma(q+1)} \right) (-\sin_q(s))ds \\ &+ \int_x^1 \left(-\frac{(1-s)^q + \Gamma(q+1)}{s^q + (1-s)^q + 2\Gamma(q+1)} \frac{x^q + \Gamma(q+1)}{\Gamma(q+1)} \right) (-\sin_q(s))ds. \end{aligned} \tag{4.10}$$

Figure 4.1 represents the numerical solution of (4.7), using the integral form of (4.8) for different values of q including $q = 1$.

5 Conclusion

We have discussed the Caputo fractional initial value problem and Caputo fractional boundary value problem, where the initial conditions and the boundary conditions also depend the value q which makes it sequential. We have recalled some numerical examples from [29, 31] which demonstrates that the solutions of the Caputo fractional differential equation tends to the corresponding integer differential equation with initial and boundary value problems. Note that the initial and boundary conditions of the sequential Caputo dynamic equation depends on the value of q where $(n-1) < nq < n$. In this work we have chosen $n = 2$, for both initial and boundary value problem.

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This article is dedicated for the 70th birthday of Professor Johnny Henderson.

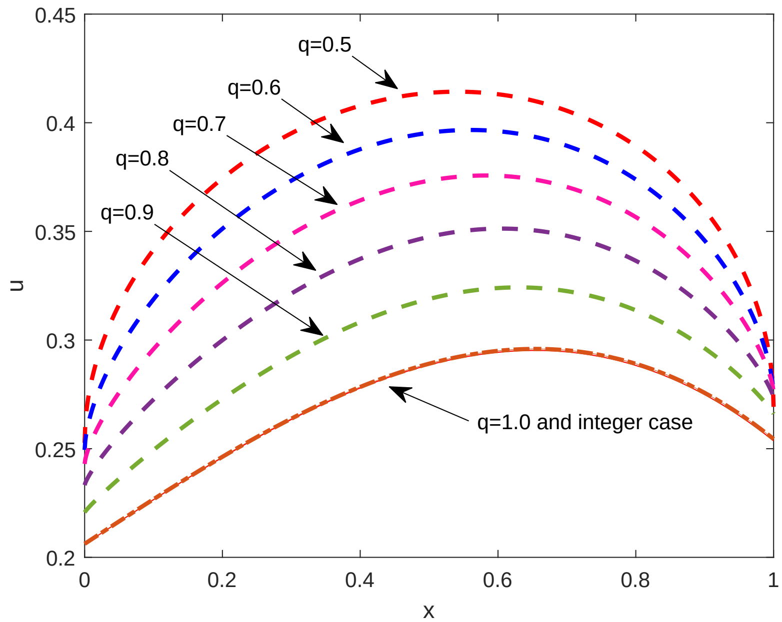

Figure 4.1: $-{}^{sc}D^{2q}u = \sin_q(x)$ when $q = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$.

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