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Boundary Value Problem for Implicit Caputo–Fabrizio Fractional Differential Equations

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Abstract

This article deals with some existence and Ulam stability results for a class of Caputo–Fabrizio implicit fractional differential equations subject to boundary conditions in finite and infinite dimensional Banach spaces. Our results are based on some fixed point theorems and the technique relies on the concept of measure of noncompactness. Two illustrative examples are presented in the last section.

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1 Introduction

Fractional differential equations involves Riemann–Liouville, Caputo, Hadamard and Hilfer fractional differential operators have been applied in various areas of scientific disciplines, and studied by many mathematicians; see the monographs [1–3, 24, 25, 31, 32, 37], the papers [4, 6–8, 11, 26, 27, 36] and the references therein.

In recent years, a new approach of fractional derivative having a kernel with exponential decay is known as the Caputo–Fabrizio operator has been introduced by Caputo and Fabrizio [16]. Several researchers were recently busy in development of Caputo–Fabrizio fractional differential equations, see [17–20, 25, 35] and the references therein.

Considerable attention has been given to the study of the Ulam–Hyers–Rassias stability of all kinds of functional equations, see the monographs [3, 22], and the papers [5, 9, 10]. More details from historical point of view, and developments of such stabilities are reported in [21, 23, 28–30, 34].

In this paper we investigate the existence of solutions and some Ulam stability results for the following class of Caputo–Fabrizio implicit fractional differential equation

$$({}^{CF}D_0^r u)(t) = f(t, u(t), ({}^{CF}D_0^r u)(t)),\ t \in I := [0, T], \tag{1.1}$$

with the boundary conditions

$$au(0) + bu(T) = c, \tag{1.2}$$

where $T > 0,\ f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, a, b and c are real constants with $a + b \neq 0,\ {}^{CF}D_0^r$ is the Caputo–Fabrizio fractional derivative of order $r \in (0, 1)$.

Recently, in [3, 12–15, 33] the authors applied the measure of noncompactness to some classes of functional Riemann–Liouville or Caputo fractional differential equations in Banach spaces. By applying these techniques, we next discuss the existence of solutions for problem (1.1)–(1.2), when $f : I \times E \times E \to E$ is a given continuous function, $c \in E$, and E is a real (or complex) Banach space with a norm $\|\cdot\|$.

2 Preliminaries

Let $\mathcal{C} := C(I, E)$ be the Banach space of all continuous functions from I into E with the norm

$$\|u\|_C = \sup_{t \in I} \|u(t)\|.$$

In the case $E = \mathbb{R}$, we have

$$\|u\|_C = \sup_{t \in I} |u(t)|.$$

By $L^1(I,E)$ we denote the Banach space of measurable function $u : I \to E$ with are Bochner integrable, equipped with the norm

$$\|u\|_{L^1} = \int_0^T \|u(t)\|dt.$$

Let $\mathcal{M}_X$ denote the class of all bounded subsets of a metric space X.

Definition 2.1 (See [14])**.** Let X be a complete metric space. A map $\mu : \mathcal{M}_X \to [0,\infty)$ is called a measure of noncompactness on X if it satisfies the following properties for all $B, B_1, B_2 \in \mathcal{M}_X$.

(a) $\mu(B) = 0$ if and only if B is precompact (Regularity),

(b) $\mu(B) = \mu(\overline{B})$ (Invariance under closure),

(c) $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$ (Semi-additivity).

Definition 2.2 (See [14])**.** Let X be a Banach space and let Ω_X be the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \to [0,\infty)$ defined by

$$\mu(M) = \inf\{\epsilon > 0 : M \subset \cup_{j=1}^{m} M_j, \operatorname{diam}(M_j) \leq \epsilon\},$$

where $M \in \Omega_E$.

The Kuratowski measure of noncompactness satisfies the following properties

(1) $\mu(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact).

(2) $\mu(M) = \mu(\overline{M})$.

(3) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.

(4) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.

(5) $\mu(cM) = |c|\mu(M),\ c \in \mathbb{R}$.

(6) $\mu(\operatorname{conv} M) = \mu(M)$.

Definition 2.3 (See [16,25])**.** The Caputo–Fabrizio fractional integral of order $0 < r < 1$ for a function $h \in L^1(I)$ is defined by

$${}^{CF}I^r h(\tau) = \frac{2(1-r)}{M(r)(2-r)} h(\tau) + \frac{2r}{M(r)(2-r)} \int_0^\tau h(x)dx, \ \ \tau \geq 0,$$

where $M(r)$ is normalization constant depending on r.

Definition 2.4 (See [16, 25])**.** The Caputo–Fabrizio fractional derivative for a function $h \in C^1(I)$ of order $0 < r < 1$, is defined by

$$^{CF}D^r h(\tau) = \frac{(2-r)M(r)}{2(1-r)} \int_0^\tau \exp(-\frac{r}{1-r}(\tau - x))h'(x)dx, \ \tau \in I.$$

Note that $(^{CF}D^r)(h) = 0$ if and only if h is a constant function.

Lemma 2.5. *Let $h \in L^1(I, E)$. A function $u \in \mathcal{C}$ is a solution of problem*

$$\begin{cases} (^{CF}D_0^r u)(t) = h(t), \quad t \in I := [0, T] \\ au(0) + bu(T) = c, \end{cases} \tag{2.1}$$

if and only if u satisfies the following integral equation

$$u(t) = C_0 + a_r h(t) + b_r \int_0^t h(s)ds + \frac{bb_r}{a+b} \int_0^T h(s)ds, \tag{2.2}$$

where

$$a_r = \frac{2(1-r)}{(2-r)M(r)}, \ b_r = \frac{2r}{(2-r)M(r)},$$

$$C_0 = \frac{1}{a+b}[c - ba_r(h(T) - h(0))] - a_r h(0).$$

Proof. Suppose that u satisfies (2.1). From [25, Proposition 1], the equation

$$(^{CF}D_0^r u)(t) = h(t),$$

implies that

$$u(t) - u(0) = a_r(h(t) - h(0)) + b_r \int_0^t h(s)ds.$$

Thus,

$$u(T) = u(0) + a_r(h(T) - h(0)) + b_r \int_0^T h(s)ds.$$

From the mixed boundary conditions $au(0) + bu(T) = c$, we get

$$au(0) + b(u(0) + a_r(h(T) - h(0)) + b_r \int_0^T h(s)ds) = c.$$

Hence,

$$u(0) = \frac{c - b(a_r(h(T) - h(0)) - b_r \int_0^T h(s)ds)}{a+b}.$$

So, we get (2.2).

Conversely, if u satisfies (2.2), then

$$(^{CF}D_0^r u)(t) = h(t), \ t \in I,$$

and

$$au(0) + bu(T) = c.$$

□

From Lemma 2.5, we can conclude the following lemma.

Lemma 2.6. *A function u is a solution of problem* (1.1)–(1.2)*, if and only if u satisfies the following integral equation*

$$u(t) = c_0 + a_r g(t) + b_r \int_0^t g(s)ds + \frac{bb_r}{a+b} \int_0^T g(s)ds, \tag{2.3}$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$ and

$$c_0 = \frac{1}{a+b}[c - ba_r(g(T) - g(0))] - a_r g(0).$$

Now, we consider the Ulam stability for the problem (1.1)–(1.2). Let $\epsilon > 0$ and $\Phi : I \to \mathbb{R}_+$ be a continuous function. We consider the following inequalities

$$\|({}^{HF}D_0^r u)(t) - f(t, u(t), ({}^{HF}D_0^r u)(t))\| \le \epsilon, \; t \in I. \tag{2.4}$$

$$\|({}^{HF}D_0^r u)(t) - f(t, u(t), ({}^{HF}D_0^r u)(t))\| \le \Phi(t), \; t \in I. \tag{2.5}$$

$$\|({}^{HF}D_0^r u)(t) - f(t, u(t), ({}^{HF}D_0^r u)(t))\| \le \epsilon\Phi(t), \; t \in I. \tag{2.6}$$

Definition 2.7 (See [3])**.** The problem (1.1)–(1.2) is Ulam–Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (2.4), there exists a solution $v \in \mathcal{C}$ of (1.1)–(1.2) with

$$\|u(t) - v(t)\| \le \epsilon c_f, \; t \in I.$$

Definition 2.8 (See [3])**.** The problem (1.1)–(1.2) is generalized Ulam–Hyers stable if there exists $c_f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $c_f(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (2.4), there exists a solution $v \in \mathcal{C}$ of (1.1)–(1.2) with

$$\|u(t) - v(t)\| \le c_f(\epsilon), \; t \in I.$$

Definition 2.9 (See [3])**.** The problem (1.1)–(1.2) is Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of the inequality (2.6), there exists a solution $v \in \mathcal{C}$ of (1.1)–(1.2) with

$$\|u(t) - v(t)\| \le \epsilon c_{f,\Phi}\Phi(t), \; t \in I.$$

Definition 2.10 (See [3])**.** The problem (1.1)–(1.2) is generalized Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each solution $u \in \mathcal{C}$ of the inequality (2.5), there exists a solution $v \in \mathcal{C}$ of (1.1)–(1.2) with

$$\|u(t) - v(t)\| \le c_{f,\Phi}\Phi(t), \; t \in I.$$

Remark 2.11. A function $u \in \mathcal{C}$ is a solution of the inequality (2.5) if and only if there exist a function $h \in \mathcal{C}$ (which depend on u) such that

$$\|h(t)\| \leq \Phi(t),$$

$$({}^{HF}D_0^r u)(t) = f(t, u(t), ({}^{HF}D_0^r u)(t)) + h(t),\ for\ t \in I.$$

Lemma 2.12. *If $u \in$ is a solution of the inequality* (2.5)*, then u is a solution of the following integral inequality*

$$\left\| u(t) - c_0 - a_r g(t) - b_r \int_0^t g(s)ds - \frac{bb_r}{a+b} \int_0^T g(s)ds \right\| \leq \left(a_r + Tb_r + T\frac{bb_r}{a+b} \right) \Phi(t),\ if\ t \in I, \tag{2.7}$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$ and

$$c_0 = \frac{1}{a+b}[c - ba_r(g(T) - g(0))] - a_r g(0).$$

Proof. By Remark 2.11, for $t \in I$ we have

$$u(t) = C_0 + a_r[g(t) + h(t)] + b_r \int_0^t [g(s) + h(s)]ds + \frac{bb_r}{a+b} \int_0^T [g(s) + h(s)]ds.$$

Thus, we obtain

$$\begin{aligned} \|u(t) \quad &- \quad C_0 - a_r g(t) - b_r \int_0^t g(s)ds - \frac{bb_r}{a+b} \int_0^T g(s)ds\| \\ &\leq \quad a_r\|h(t)\| + b_r \int_0^t \|h(s)\|ds + \frac{bb_r}{a+b} \int_0^T \|h(s)\|ds \\ &\leq \quad \left(a_r + Tb_r + T\frac{bb_r}{a+b} \right) \Phi(t). \end{aligned}$$

Hence, we get (2.7). □

For our purpose we will need the following fixed point theorems.

Theorem 2.13 (Schauder fixed point theorem [33])**.** *Let X be a Banach space, D be a bounded closed convex subset of X and $T : D \to D$ be a compact and continuous map. Then T has at least one fixed point in D.*

Theorem 2.14 (Mönch's fixed point theorem [27])**.** *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V)\ or\ V = N(V) \cup \{0\} \Rightarrow \overline{V}\ is\ compact, \tag{2.8}$$

holds for every subset V of D, then N has a fixed point.

3 Existence and Ulam Stability Results

In this section, we present some results concerning the existence and Ulam stability of solutions for the problem (1.1)–(1.2),

Definition 3.1. By a solution of problem (1.1)–(1.2), we mean a function $u \in \mathcal{C}$ such that

$$u(t) = c_0 + a_r g(t) + b_r \int_0^t g(s)ds + \frac{bb_r}{a+b}\int_0^T g(s)ds,$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$ and

$$c_0 = \frac{1}{a+b}[c - ba_r(g(T) - g(0))] - a_r g(0).$$

3.1 The Scalar Case

The following hypotheses will be used in the sequel:

(H_1) There exist a nondecreasing continuous function $\psi : \mathbb{R}_+ \to (0, \infty)$ and continuous functions $p, q : I \to \mathbb{R}_+$ such that

$$|f(t, u, v)| \leq p(t)\psi(|u|) + q(t)|v|, \text{ for each } t \in I \; u, v \in \mathbb{R}.$$

(H_2) There exists a constant $R > 0$, such that

$$R \geq |c_0| + \left[a_r + Tb_r + T\frac{bb_r}{a+b}\right]\frac{p^*\psi(R)}{1-q^*}, \tag{3.1}$$

where $p^* = \sup_{t\in I} p(t)$, and $q^* = \sup_{t\in I} q(t)$, with $0 < q^* < 1$.

(H_3) There exist constants $d_1 > 0,\ 0 < d_2 < 1$, such that

$$(1 + |u_1 - u_2|)|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq d_1\Phi(t)|u_1 - u_2| + d_2|v_1 - v_2|,$$

for each $t \in I$ and $u_i, v_i \in \mathbb{R};\ i = 1, 2$.

(H_4) There exists a constant $\lambda_\Phi > 0$, such that for each $t \in I$ we have

$$\int_0^T \Phi(t)dt \leq \lambda_\Phi \Phi(t).$$

Remark 3.2. From (H_3), for each $t \in I$, and $u, v \in \mathbb{R}$, we have that

$$|f(t, u, v)| \leq |f(t, 0, 0)| + d_1\Phi(t)|u| + d_2|v|.$$

So, (H_3) implies (H_1) with

$$\psi(x) = 1 + x,\ p(t) = \max\{d_1\Phi(t), |f(t, 0, 0)|\},\ q(t) = d_2.$$

Now, we prove an existence result for the problem (1.1)–(1.2) based on Schauder's fixed point theorem.

Theorem 3.3. *Assume that the hypotheses* (H_1) *and* (H_2) *hold. Then the problem* (1.1)–(1.2) *has a least one solution defined on* I.

Proof. Consider the operator $N : \mathcal{C} \to \mathcal{C}$ such that,

$$(Nu)(t) = c_0 + a_r g(t) + b_r \int_0^t g(s)ds + \frac{bb_r}{a+b} \int_0^T g(s)ds, \tag{3.2}$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$ and

$$c_0 = \frac{1}{a+b}[c - ba_r(g(T) - g(0))] - a_r g(0).$$

Consider the ball $B_R := \{u \in \mathcal{C} : \|u\|_C \leq R\}$. Let $u \in B_R$ From (H_1), for each $t \in I$, we have

$$\begin{aligned} |g(t)| &= |f(t, u(t), g(t))| \\ &\leq p(t)\psi(\|u\|_C) + q(t)|g(t)| \\ &\leq p^*\psi(R) + q^*\|g\|_C. \end{aligned}$$

Thus, from (H_2) we get

$$\|g\|_C \leq \frac{p^*\psi(R)}{1-q^*}. \tag{3.3}$$

Next, we have

$$\begin{aligned} |(Nu)(t)| &\leq |c_0| + |a_r g(t)| + |b_r \int_0^t g(s)ds| + |\frac{bb_r}{a+b} \int_0^T g(s)ds| \\ &\leq |c_0| + a_r|g(t)| + b_r \int_0^t |g(s)|ds + \frac{bb_r}{a+b} \int_0^T |g(s)|ds \\ &\leq |c_0| + \left[a_r + Tb_r + T\frac{bb_r}{a+b}\right] \frac{p^*\psi(R)}{1-q^*} \\ &\leq R. \end{aligned}$$

Hence

$$\|N(u)\|_C \leq R.$$

This proves that N transforms the ball B_R into itself. We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.13. The proof will be given in two steps.

Step 1

$N : B_R \to B_R$ is continuous. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I$, we have

$$\begin{aligned} |(Nu_n)(t) - (Nu)(t)| \;\leq\; & |a_r(g_n(t) - g(t))| \\ & + |b_r \int_0^t (g_n(s) - g(s)ds| \\ & + \left| \frac{bb_r}{a+b} \int_0^T (g_n(s) - g(s))ds \right|, \end{aligned} \tag{3.4}$$

where $g_n,\ g \in \mathcal{C}$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)) \quad and \quad g(t) = f(t, u(t), g(t)).$$

Since$\|u_n - u\|_C \to 0$ as $n \to \infty$ and f, g and g_n are continuous, then the Lebesgue dominated convergence theorem, implies that

$$\|N(u_n) - N(u)\|_C \to 0 \quad \text{as } n \to \infty.$$

Hence, the operator N is continuous.

Step 2

$N(B_R)$ is bounded and equicontinuous. Since $N(B_R) \subset B_R$ and B_R is bounded, then $N(B_R)$ is bounded. Next, let $t_1, t_2 \in I$, with $0 \leq t_1 \leq t_2 \leq T$, and let $u \in B_R$. Then we have

$$\begin{aligned} |(Nu)(t_2) - (Nu)(t_1)| \;\leq\; & |a_r g(t_2) + b_r \int_0^{t_2} g(s)ds + \frac{bb_r}{a+b} \int_0^T g(s)ds - a_r g(t_1) \\ & - b_r \int_0^{t_1} g(s)ds - \frac{bb_r}{a+b} \int_0^T g(s)ds| \\ \leq\; & |a_r g(t_2) + b_r \int_0^{t_2} g(s)ds - a_r g(t_1) + b_r \int_{t_1}^0 g(s)ds| \\ \leq\; & a_r |g(t_2) - g(t_1)| + b_r \int_{t_1}^{t_2} |g(s)|ds \\ \leq\; & a_r |g(t_2) - g(t_1)| + b_r (t_2 - t_1) \|g\|_C. \end{aligned}$$

Since $\|g\|_C \leq \dfrac{p^*\psi(R)}{1-q^*}$, in view to (3.3), we obtain

$$|(Nu)(t_2) - (Nu)(t_1)| \leq a_r |g(t_2) - g(t_1)| + b_r (t_2 - t_1) \frac{p^*\psi(R)}{1-q^*}.$$

As $t_2 \to t_1$ the continuity of g implies that the right-hand side of the above inequality tends to zero.

As a consequence of the above two steps, together with the Ascoli–Arzelá theorem, we can conclude that $N : B_R \to B_R$ is continuous and compact. From an application of Theorem 2.13, we deduce that N has a fixed point u which is a solution of problem (1.1)–(1.2). □

Now, we are concerned with the generalized Ulam–Hyers–Rassias stability of problem (1.1)–(1.2).

Theorem 3.4. *Assume that the hypotheses* (H_2)–(H_4) *hold. Then the problem* (1.1)–(1.2) *has at least one solution defined on* I *and it is generalized Ulam–Hyers–Rassias stable.*

Proof. From Remark 3.2, there exists a solution v of the problem (1.1)–(1.2). That is

$$v(t) = c_h + a_r g(t) + b_r \int_0^t h(s)ds + \frac{bb_r}{a+b} \int_0^T h(s)ds,$$

where $h \in \mathcal{C}$, with $h(t) = f(t, v(t), h(t))$ and

$$c_h = \frac{1}{a+b}[c - ba_r(h(T) - h(0))] - a_r h(0).$$

Let u be a solution of the inequality (2.5), then from Lemma 2.12, u is a solution of the integral inequality (2.7), that is

$$\begin{aligned} &\left| u(t) - c_g - a_r g(t) - b_r \int_0^t g(s)ds - \frac{bb_r}{a+b} \int_0^T g(s)ds \right| \\ \leq\ & \left(a_r + Tb_r + T\frac{bb_r}{a+b} \right) \Phi(t), \end{aligned}$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$ and

$$c_g = \frac{1}{a+b}[c - ba_r(g(T) - g(0))] - a_r g(0).$$

Thus, for each $t \in I$, we obtain

$$\begin{aligned} |u(t,w) - v(t,w)| \ \leq\ & \left| u(t) - c_g - a_r g(t) - b_r \int_0^t g(s)ds - \frac{bb_r}{a+b} \int_0^T g(s)ds \right| \\ +\ & \left| c_g + a_r g(t) + b_r \int_0^t g(s)ds + \frac{bb_r}{a+b} \int_0^T g(s)ds \right. \\ -\ & \left. c_h - a_r g(t) + b_r \int_0^t h(s)ds - \frac{bb_r}{a+b} \int_0^T h(s)ds \right|. \end{aligned}$$

This implies that,

$$\begin{aligned}|u(t,w)-v(t,w)| &\leq \left(a_r+Tb_r+T\frac{bb_r}{a+b}\right)\Phi(t)\\ &+ |c_g-c_h|+a_r|g(t)-h(t)|+b_r\int_0^t|g(s)-h(s)|ds\\ &+ \frac{bb_r}{a+b}\int_0^T|g(s)-h(s)|ds.\end{aligned}$$

On the other hand, from (H_3), for each $t\in I$, we have

$$\begin{aligned}|g(t)-h(t)| &= |f(t,u(t),g(t))-f(t,v(t),h(t))|\\ &\leq d_1\Phi(t)+d_2|g(t)-h(t)|,\end{aligned}$$

which gives

$$|g(t)-h(t)|\leq\frac{d_1}{1-d_2}\Phi(t). \tag{3.5}$$

Again,

$$\begin{aligned}|c_g-c_h| &\leq \frac{ba_r}{a+b}(|g(T)-h(T)|+|g(0)-h(0)|)+a_r|g(0)-h(0)|\\ &\leq \left(\frac{2ba_rd_1}{(a+b)(1-d_2)}+\frac{a_rd_1}{1-d_2}\right)\Phi(t).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|u(t,w)-v(t,w)| &\leq \left(a_r+Tb_r+T\frac{bb_r}{a+b}\right)\Phi(t)\\ &+ \left[\left(\frac{2ba_rd_1}{(a+b)(1-d_2)}+\frac{a_rd_1}{1-d_2}\right)+\frac{a_rd_1}{1-d_2}\right]\Phi(t)\\ &+ \frac{b_rd_1}{1-d_2}\int_0^t\Phi(s)ds\\ &+ \frac{bb_rd_1}{(a+b)(1-d_2)}\int_0^T\Phi(s)ds.\end{aligned}$$

Hence, from (H_4), we get

$$\begin{aligned}|u(t,w)-v(t,w)| &\leq \Bigg(a_r+Tb_r+T\frac{bb_r}{a+b}\\ &+ \frac{2ba_rd_1}{(a+b)(1-d_2)}+\frac{a_rd_1}{1-d_2}+\frac{a_rd_1}{1-d_2}\\ &+ \frac{\lambda_\Phi b_rd_1}{1-d_2}+\frac{\lambda_\Phi bb_rd_1}{(a+b)(1-d_2)}\Bigg)\Phi(t)\\ &= c_{f,\Phi}\Phi(t).\end{aligned}$$

This conclude that problem (1.1)–(1.2) is generalized Ulam–Hyers–Rassias stable. □

3.2 Results in Banach Spaces

The following hypotheses will be used in the sequel:

(H_{01}) There exist a nondecreasing continuous function $\Psi : \mathbb{R}_+ \to (0,\infty)$ and continuous functions $\overline{p}, \overline{q} : I \to \mathbb{R}_+$ such that

$$\|f(t,u,v)\| \leq \overline{p}(t)\Psi(\|u\|) + \overline{q}(t)\|v\|, \text{ for each } t \in I\ u, v \in E.$$

(H_{02}) There exists a constant $M > 0$, such that

$$M \geq \|c_0\| + \left[a_r + Tb_r + T\frac{bb_r}{a+b}\right]\frac{\overline{p}^*\Psi(M)}{1-\overline{q}^*}, \tag{3.6}$$

where $\overline{p}^* = \sup_{t\in I}\overline{p}(t)$, and $\overline{q}^* = \sup_{t\in I}\overline{q}(t)$, with $0 < \overline{q}^* < 1$.

(H_{03}) For each bounded sets $\widetilde{K}, \widetilde{L} \subset E$ and each $t \in I$,

$$\mu(f(t,\widetilde{K},\widetilde{L})) \leq \overline{p}(t)\mu(\widetilde{K}) + \overline{q}(t)\mu(\widetilde{L}),$$

where μ is the Kuratowski measure of noncompactness on the space E.

Now, we prove an existence result for the problem (1.1)–(1.2) based on Mönch's fixed point theorem.

Theorem 3.5. *Assume that the hypothesis* (H_{01})–(H_{03}) *hold. If*

$$\rho := \frac{\overline{p}^*}{1-\overline{q}^*}\left(a_r + Tb_r + \frac{Tbb_r}{a+b}\right) < 1, \tag{3.7}$$

then the problem (1.1)–(1.2) *has a least one solution defined on* I.

Proof. Consider the operator $N : \mathcal{C} \to \mathcal{C}$ be the operator defined in (3.2). Define the ball

$$B_M = \{x \in \mathcal{C},\ \|x\|_C \leq M\}.$$

Let $u \in B_M$, from (H_{01}), for each $t \in I$, we have

$$\begin{aligned}
\|g(t)\| &\leq \|f(t,u(t),g(t))\| \\
&\leq \overline{p}(t)\Psi(\|u\|) + \overline{q}(t)\|g(t)\| \\
&\leq \overline{p}^*\Psi(\|u\|_C) + \overline{q}^*\|g\|_C \\
&\leq \overline{p}^*\Psi(\|u\|_C) + \overline{q}^*\|g\|_C.
\end{aligned}$$

This gives

$$\|g\|_C \leq \frac{\overline{p}^*\Psi(M)}{1-\overline{q}^*}.$$

Thus, from (H_{02}), we obtain

$$\begin{aligned} \|(Nu)(t)\| &\leq \|c_0\| + \left[a_r + Tb_r + T\frac{bb_r}{a+b}\right]\frac{\overline{p}^*\Psi(M)}{1-\overline{q}^*}. \\ &\leq M. \end{aligned}$$

Hence

$$\|N(u)\|_C \leq M.$$

This proves that N transforms the ball B_M into itself.

We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.14. We have $N(B_R) \subset B_R$, and as in the proof of Theorem 3.3, we can easily show that $N : B_R \to B_R$ is continuous, and $N(B_R)$ is equicontinuous. Next, we prove that Mönch's condition (2.8) is satisfied. Let V be a subset of B_M such that $V \subset \overline{N(V)} \cup \{0\}$, V is bounded and equicontinuous and therefore the function $t \to v(t) = \mu(V(t))$ is continuous on I. From (H_{03}) and the properties of the measure μ, for each $t \in I$, we have

$$\begin{aligned} v(t) &\leq \mu((NV)(t) \cup \{0\}) \\ &\leq \mu((NV)(t)) \\ &\leq a_r\{\mu(g(t)) : u \in V\} + b_r\int_0^t \{\mu(g(s)) : u \in V\}ds \\ &\quad + \frac{bb_r}{a+b}\int_0^T \{\mu(g(s)) : u \in V\}ds, \end{aligned}$$

where $g \in \mathcal{C}$, with $g(t) = f(t, u(t), g(t))$. However, hypothesis (H_{01}) implies that for each $t \in I$,

$$\begin{aligned} \mu(\{g(t) : u \in V\}) &= \mu(\{f(t, u(t), g(t)) : u \in V\}) \\ &\leq \overline{p}^*\mu(\{u(t) : u \in V\})) + \overline{q}^*\mu(\{g(t) : u \in V\}), \end{aligned}$$

which gives

$$\begin{aligned} \mu(\{g(t) : u \in V\}) &\leq \frac{\overline{p}^*}{1-\overline{q}^*}\mu\{u(s) : u \in V\} \\ &= \frac{\overline{p}^*}{1-\overline{q}^*}\mu(V(t). \end{aligned}$$

Thus, we get

$$\begin{aligned} v(t) &\leq \frac{\overline{p}^*}{1-\overline{q}^*}\left[a_r\mu(V(t) + b_r\int_0^t \mu(V(s)ds + \frac{bb_r}{a+b}\int_0^T \mu(V(s)ds\right] \\ &\leq \frac{\overline{p}^*}{1-\overline{q}^*}\left[a_r\|v\|_C + b_r\int_0^t \|v\|_C ds + \frac{bb_r}{a+b}\int_0^T \|v\|_C ds\right] \\ &\leq \frac{\overline{p}^*}{1-\overline{q}^*}\left(a_r + Tb_r + \frac{Tbb_r}{a+b}\right)\|v\|_C. \end{aligned}$$

Hence

$$\|v\|_C \leq \rho \|v\|_C.$$

From (3.7), we get $\|v\|_C = 0$, that is $v(t) = \mu(V(t)) = 0$, for each $t \in I$, and then $V(t)$ is relatively compact in E. In view of the Ascoli–Arzelà theorem, V is relatively compact in B_M. From Mönch's fixed point Theorem (Theorem 2.14), we conclude that N has a fixed point which is a solution of the problem (1.1)–(1.2). □

As in the proof of Theorem 3.3, we present (without proof) a result about the generalized Ulam–Hyers–Rassias stability.

Theorem 3.6. *Assume that the hypotheses* (H_{02}), (H_{03}), (H_4) *and the following hypothesis holds.*

(H_{04}) *There exist constants* $\overline{d}_1 > 0,\ 0 < \overline{d}_2 < 1$, *such that*

$$(1 + \|u_1 - u_2\|)\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \overline{d}_1 \Phi(t)\|u_1 - u_2\| + \overline{d}_2 \|v_1 - v_2\|,$$

for each $t \in I$ *and* $u_i, v_i \in E;\ i = 1, 2.$

Then the problem (1.1)–(1.2) *has at least one solution defined on* I *and it is generalized Ulam–Hyers–Rassias stable.*

4 Examples

Example 4.1. Consider the Caputo–Fabrizio implicit fractional differential equation

$$({}^{CF}D_0^{\frac{1}{4}} u)(t) = \frac{1 + \ln(1 + t^2)}{10(1 + |u(t)| + |({}^{CF}D_0^{\frac{1}{4}} u)(t)|)},\ t \in [0, 1], \tag{4.1}$$

with the boundary conditions

$$u(0) + 2u(1) = 1. \tag{4.2}$$

Set

$$f(t, u(t), v(t)) = \frac{1 + \ln(1 + t^2)}{10(1 + |u(t)| + |v(t)|)},\ t \in [0, 1].$$

The hypothesis (H_3) is satisfied with

$$d_1 = d_2 = \frac{1 + \ln(2)|}{10}.$$

Simple computations show that all conditions of Theorems 3.3 and 3.4 are satisfied. Hence problem (4.1)–(4.2) has a solution, and it is generalized Ulam–Hyers–Rassias stable.

Example 4.2. Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots), \sum_{n=1}^{\infty} |u_n| < \infty \right\},$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

Consider the Caputo–Fabrizio fractional differential equation

$$({}^{CF}D_0^{\alpha} u)(t) = \frac{c(2^{-n} + u_n(t))}{\exp(t+3)(1 + |u(t)| + |({}^{CF}D_0^{\alpha} u)(t)|)},\ t \in [0,1], \tag{4.3}$$

with the boundary conditions

$$u(0) + u(1) = (2^{-1}, 2^{-2}, \ldots, 2^{-n}, \ldots). \tag{4.4}$$

Set $f = (f_1, f_2, \ldots, f_n, \ldots)$,

$$f_n(t, u(t), v(t)) = \frac{c(2^{-n} + u_n(t))}{\exp(t+3)(1 + |u(t)| + |v(t)|)},\ t \in [0,1].$$

Simple computations with a good choice of the constant c, show that all conditions of Theorem 3.5 are satisfied. Consequently, Theorem 3.5 implies that the problem (4.3)–(4.4) has at least one solution defined on $[0,1]$. Also, hypothesis (H_4) is satisfied with $\lambda_\Phi = e - 1$. Indeed

$$\int_0^T \Phi(t,w)dt = \int_0^T e^{-t}dt = 1 - e^{-1} \le \lambda_\Phi e^{-t} = \lambda_\Phi \Phi(t,w),\ t \in [0,1].$$

Consequently, Theorem 3.6 implies that (4.3)–(4.4) is generalized Ulam–Hyers–Rassias stable.

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