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On a Fractional Integro-Differential System Involving Riemann–Liouville and Caputo Derivatives with Coupled Multi-Point Boundary Conditions

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Abstract

We introduce a new class of coupled sequential fractional differential equations involving Riemann–Liouville and Caputo derivatives, integral and nonintegral type nonlinearities and equipped with coupled multi-point boundary conditions. Existence results for the given problem are derived by means of Leray–Schauder nonlinear alternative and Krasnosel'skiĭ's fixed point theorem, while the uniqueness of solutions is established via contraction mapping principle. Examples illustrating the main results are presented.

AMS Subject Classifications: Riemann–Liouville fractional derivative, Caputo fractional derivative, system, nonlocal boundary conditions, existence, fixed point theorem.
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1 Introduction

The tools of fractional calculus are found to be of great utility in improving the mathematical modeling of many real world processes. Examples include disease models [8, 11], ecological models [16], economic models [23], fractional neural networks [6, 29], chaotic synchronization [27, 28], etc. The interest in fractional calculus owes to the nonlocal nature of fractional-order differential and integral operators, which takes into account the past history of the phenomenon under investigation [10, 17].

Boundary value problems involving different kinds of fractional-order operators such as Riemann–Liouville, Caputo, Hadamard, etc., received much attention in recent years. For some recent works on fractional order boundary value problems with a variety of boundary conditions, see [1, 2, 7, 9, 15, 19] and the references cited therein. In a recent paper [4], the authors discussed the existence of solutions for a nonlinear fractional integro-differential equation involving two Caputo fractional derivatives of different orders and a Riemann–Liouville integral, equipped with dual anti-periodic boundary conditions. There has also been shown a great interest in the study of coupled systems of fractional differential equations in view of their applications in many physical situations [3, 12–14, 20, 21, 24–26].

In this paper, we investigate the existence and uniqueness of solutions for a nonlinear coupled system of sequential fractional differential equations involving Riemann–Liouville and Caputo derivatives, and integral (in the sense of Riemann–Liouville) and nonintegral type nonlinearities on an arbitrary domain:

$$\begin{cases} {}^{RL}D^{q_1}\Big[({}^cD^{p_1}+\kappa_1)x(t)+\lambda_1 I^{\gamma_1}h(t,x(t),y(t))\Big]=\phi(t,x(t),y(t)),\ t\in[a,b],\\ {}^{RL}D^{q_2}\Big[({}^cD^{p_2}+\kappa_2)y(t)+\lambda_2 I^{\gamma_2}u(t,x(t),y(t))\Big]=\psi(t,x(t),y(t)),\ t\in[a,b], \end{cases} \tag{1.1}$$

complemented with coupled (nonconjugate type) multi-point boundary conditions:

$$\begin{cases} x(a)=\sum_{i=1}^{n-2}\alpha_i y(\xi_i),\ x'(a)=0,\ x(b)=0,\ x'(b)=0,\\ y(a)=\sum_{i=1}^{n-2}\beta_i x(\eta_i),\ y'(a)=0,\ y(b)=0,\ y'(b)=0, \end{cases} \tag{1.2}$$

where $1<p_1,q_1\leq 2$, $1<p_2,q_2\leq 2$, ${}^cD^{\vartheta}$ denotes the Caputo fractional differential operator of order ϑ ($\vartheta=p_1,p_2$), ${}^{Rl}D^{\varrho}$ denotes the Riemann–Liouville fractional differential operator of order ϱ ($\varrho=q_1,q_2$), with $p_1+q_1>3,\ p_2+q_2>3,\ I^{\gamma_1},\ I^{\gamma_2}$ are Riemann–Liouville fractional integrals of order $\gamma_1,\ \gamma_2>1$, $\kappa_i,\ \lambda_i\in\mathbb{R},\ i=1,2$, $h,\phi,u,\psi:[a,b]\times\mathbb{R}^2\to\mathbb{R}$ are given continuous functions, $a<\xi_1<\xi_2<\cdots<\xi_{n-2}<b$, $a<\eta_1<\eta_2<\cdots<\eta_{n-2}<b$, $\alpha_j,\ \beta_j\in\mathbb{R},\ j=1,2,\cdots,n-2$.

Here we emphasize that the present work is motivated by a recent work [5] in which existence and uniqueness results for a mixed fractional order coupled system supplemented with nonlocal multi-point and Riemann–Stieltjes integral-multi-strip conditions were obtained.

We organize the rest of the paper as follows. In Section 2, we recall some basic definitions of fractional calculus and present an auxiliary lemma, which plays a key role in obtaining the desired results. In Section 3, we discuss the existence of solutions for the given problem while the uniqueness result is presented in the last section.

2 Preliminary Material

We begin this section with basic definitions of fractional calculus [10, 17].

Definition 2.1. The Riemann–Liouville fractional integral of order $\beta > 0$ for $\varphi \in L_1[a,b]$, existing almost everywhere on $[a,b]$, is defined by

$$I^{\beta}\varphi(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\varphi(s)ds,$$

where Γ denotes the Euler gamma function.

Definition 2.2. The Riemann–Liouville and Caputo fractional derivatives of order $\beta \in (n-1,n], n \in \mathbb{N}$, for $\varphi \in AC^n[a,b]$, existing almost everywhere on $[a,b]$, are respectively defined by

$${}^{RL}D^{\beta}\varphi(t) = \frac{d^n}{dt^n}\int_a^t \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)}\varphi(s)ds,$$

and

$${}^{c}D^{\beta}\varphi(t) = \int_a^t \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)}\varphi^{(n)}(s)ds.$$

Lemma 2.3. *The general solution of the fractional differential equation* ${}^{c}D^{\beta}\varphi(t) = 0,\ m-1 < \beta \le m,\ t \in [a,b]$, *is*

$$\varphi(t) = \gamma_0 + \gamma_1(t-a) + \gamma_2(t-a)^2 + \ldots + \gamma_{m-1}(t-a)^{m-1},$$

where $\gamma_i \in \mathbb{R},\ i = 0,1,\ldots,m-1$. *Furthermore,*

$$(I^{\beta}\, {}^{c}D^{\beta}\varphi)(t) = \varphi(t) + \sum_{i=0}^{m-1}\gamma_i(t-a)^i.$$

Lemma 2.4 (See [17]). *For* $\beta > 0$ *and* $\varphi \in C[a,b] \cap L[a,b]$, *the general solution of the equation* $({}^{RL}D^{\beta}\varphi)(t) = 0$ *is*

$$\varphi(t) = \sum_{j=1}^{m}\sigma_j(t-a)^{\beta-j},$$

where $\sigma_j \in \mathbb{R},\ j = 1, 2, \ldots, m$. *Moreover,*

$$(I^{\beta\ RL}D^{\beta}\varphi)(t) = \varphi(t) + \sum_{j=1}^{m} \sigma_j (t-a)^{\beta-j},\ ({}^{RL}D^{\beta} I^{\beta}\varphi)(t) = \varphi(t).$$

Lemma 2.5. *Assume* $1 < p_1, q_1 \le 2$, *and* $1 < p_2, q_2 \le 2$. *For* $H, \Phi, U, \Psi \in C[a,b] \cap L[a,b]$, *the solution of the linear system of fractional differential equations:*

$$\begin{cases} {}^{RL}D^{q_1}\Big[({}^{c}D^{p_1} + \kappa_1)x(t) + \lambda_1 I^{\gamma_1} H(t)\Big] = \Phi(t),\ t \in [a,b], \\ {}^{RL}D^{q_2}\Big[({}^{c}D^{p_2} + \kappa_2)y(t) + \lambda_2 I^{\gamma_2} U(t)\Big] = \Psi(t),\ t \in [a,b], \end{cases} \tag{2.1}$$

equipped with the boundary conditions (1.2) *is equivalent to the system of integral equations:*

$$\begin{aligned} x(t) =& -\kappa_1 \int_a^t \frac{(t-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds - \lambda_1 \int_a^t \frac{(t-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} H(s)ds \\ &+ \int_a^t \frac{(t-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \Phi(s)ds + f_1(t)\Big[\kappa_1 \int_a^b \frac{(b-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds \\ &+ \lambda_1 \int_a^b \frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} H(s)ds - \int_a^b \frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \Phi(s)ds\Big] \\ &+ f_6(t)\Big[\kappa_1 \int_a^b \frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)} x(s)ds + \lambda_1 \int_a^b \frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)} H(s)ds \\ &- \int_a^b \frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)} \Phi(s)ds\Big] + f_2(t)\Big[-\kappa_1 \sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds \\ &- \lambda_1 \sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} H(s)ds + \sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \Phi(s)ds\Big] \\ &+ f_3(t)\Big[\kappa_2 \int_a^b \frac{(b-s)^{p_2-1}}{\Gamma(p_2)} y(s)ds + \lambda_2 \int_a^b \frac{(b-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)} U(s)ds \\ &- \int_a^b \frac{(b-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)} \Psi(s)ds\Big] + f_4(t)\Big[\kappa_2 \int_a^b \frac{(b-s)^{p_2-2}}{\Gamma(p_2-1)} y(s)ds \\ &+ \lambda_2 \int_a^b \frac{(b-s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2+p_2-1)} U(s)ds - \int_a^b \frac{(b-s)^{p_2+q_2-2}}{\Gamma(p_2+q_2-1)} \Psi(s)ds\Big] \\ &+ f_5(t)\Big[-\kappa_2 \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{p_2-1}}{\Gamma(p_2)} y(s)ds \\ &- \lambda_2 \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)} U(s)ds + \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)} \Psi(s)ds\Big], \end{aligned} \tag{2.2}$$

$$\begin{aligned}
y(t) &= -\kappa_2 \int_a^t \frac{(t-s)^{p_2-1}}{\Gamma(p_2)} y(s)ds - \lambda_2 \int_a^t \frac{(t-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)} U(s)ds \\
&+ \int_a^t \frac{(t-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)} \Psi(s)ds + g_1(t)\Big[\kappa_1 \int_a^b \frac{(b-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds \\
&+ \lambda_1 \int_a^b \frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} H(s)ds - \int_a^b \frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \Phi(s)ds\Big] \\
&+ g_2(t)\Big[\kappa_1 \int_a^b \frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)} x(s)ds + \lambda_1 \int_a^b \frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)} H(s)ds \\
&- \int_a^b \frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)} \Phi(s)ds\Big] + g_6(t)\Big[-\kappa_1 \sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds \\
&- \lambda_1 \sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} H(s)ds + \sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \Phi(s)ds\Big] \\
&+ g_4(t)\Big[\kappa_2 \int_a^b \frac{(b-s)^{p_2-1}}{\Gamma(p_2)} y(s)ds + \lambda_2 \int_a^b \frac{(b-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)} U(s)ds \qquad (2.3) \\
&- \int_a^b \frac{(b-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)} \Psi(s)ds\Big] + g_5(t)\Big[\kappa_2 \int_a^b \frac{(b-s)^{p_2-2}}{\Gamma(p_2-1)} y(s)ds \\
&+ \lambda_2 \int_a^b \frac{(b-s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2+p_2-1)} U(s)ds - \int_a^b \frac{(b-s)^{p_2+q_2-2}}{\Gamma(p_2+q_2-1)} \Psi(s)ds\Big] \\
&+ g_3(t)\Big[-\kappa_2 \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{p_2-1}}{\Gamma(p_2)} y(s)ds \\
&- \lambda_2 \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)} U(s)ds + \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)} \Psi(s)ds\Big],
\end{aligned}$$

where

$$f_i(t) = \mathcal{V}_1(t)\phi_i + \mathcal{W}_1(t)\delta_i + \nu_i,\ i = 1, \ldots, 6, \qquad (2.4)$$

$$g_j(t) = \mathcal{V}_2(t)\psi_j + \mathcal{W}_2(t)\rho_j + \omega_j,\ j = 1, \ldots, 5, \qquad (2.5)$$

$$g_6(t) = \mathcal{V}_2(t)\psi_6 + \mathcal{W}_2(t)\rho_6 + \nu_5, \qquad (2.6)$$

$$\mathcal{V}_\ell(t) = \frac{(t-a)^{p_\ell+q_\ell-1}\Gamma(q_\ell)}{\Gamma(p_\ell+q_\ell)},\ \mathcal{W}_\ell(t) = \frac{(t-a)^{p_\ell+q_\ell-2}\Gamma(q_\ell-1)}{\Gamma(p_\ell+q_\ell-1)},\ \ell = 1, 2, \qquad (2.7)$$

$$\phi_k = \frac{-A_4\delta_k}{A_3},\ k = 1, 2, 3, 4, 5,\ \phi_6 = \frac{1 - A_4\delta_6}{A_3}, \qquad (2.8)$$

$$\psi_m = \frac{-B_4\rho_m}{B_3},\ m = 1, 2, 3, 4,\ \psi_5 = \frac{1 - B_4\rho_5}{B_3},\ \psi_6 = \frac{-B_4\rho_6}{B_3}, \qquad (2.9)$$

$$\delta_1 = \frac{A_3\nu_1 - A_3}{\epsilon_1},\ \delta_n = \frac{A_3\nu_n}{\epsilon_1},\ n = 2, 3, 4, 5,\ \delta_6 = \frac{A_3\nu_6 + A_1}{\epsilon_1}, \qquad (2.10)$$

$$\rho_r = \frac{B_3\omega_r}{\epsilon_2},\ r=1,2,3,\ \rho_4 = \frac{B_3\omega_4 - B_3}{\epsilon_2},\ \rho_5 = \frac{B_3\omega_5 + B_1}{\epsilon_2},\ \rho_6 = \frac{B_3\nu_5}{\epsilon_2}, \tag{2.11}$$

$$\begin{cases} \nu_1 &= \nu_2\Big(A_7 - \dfrac{A_3\theta_1}{\epsilon_1}\Big),\ \nu_2 = \dfrac{B_3\sigma_1\epsilon_1}{\epsilon}, \\ \nu_3 &= \nu_5\Big(B_7 - \dfrac{B_3\sigma_1}{\epsilon_2}\Big),\ \nu_4 = \nu_5\Big(\sigma_2 + \dfrac{B_1\sigma_1}{\epsilon_2}\Big), \\ \nu_5 &= \dfrac{\epsilon_1\epsilon_2}{\epsilon},\ \nu_6 = \nu_2\Big(\theta_2 + \dfrac{A_1\theta_1}{\epsilon_1}\Big), \end{cases} \tag{2.12}$$

$$\begin{cases} \omega_1 &= \nu_5\Big(A_7 - \dfrac{A_3\theta_1}{\epsilon_1}\Big),\ \omega_2 = \nu_5\Big(\theta_2 + \dfrac{A_1\theta_1}{\epsilon_1}\Big), \\ \omega_3 &= \dfrac{A_3\theta_1\epsilon_2}{\epsilon},\ \omega_4 = \omega_3\Big(B_7 - \dfrac{B_3\sigma_1}{\epsilon_2}\Big), \\ \omega_5 &= \omega_3\Big(\sigma_2 + \dfrac{B_1\sigma_1}{\epsilon_2}\Big), \end{cases} \tag{2.13}$$

$$\theta_1 = \frac{A_7\epsilon_1 - A_4A_5 + A_3A_6}{A_3},\ \theta_2 = \frac{A_5 - A_1A_7}{A_3}, \tag{2.14}$$

$$\sigma_1 = \frac{B_7\epsilon_2 - B_4B_5 + B_3B_6}{B_3},\ \sigma_2 = \frac{B_5 - B_1B_7}{B_3}, \tag{2.15}$$

$$\begin{cases} A_1 &= \dfrac{(b-a)^{p_1+q_1-1}\Gamma(q_1)}{\Gamma(p_1+q_1)},\ A_2 = \dfrac{(b-a)^{p_1+q_1-2}\Gamma(q_1-1)}{\Gamma(p_1+q_1-1)}, \\ A_3 &= \dfrac{(b-a)^{p_1+q_1-2}\Gamma(q_1)}{\Gamma(p_1+q_1-1)},\ A_4 = \dfrac{(b-a)^{p_1+q_1-3}\Gamma(q_1-1)}{\Gamma(p_1+q_1-2)}, \\ A_5 &= \displaystyle\sum_{i=1}^{n-2}\beta_i\frac{(\eta_i-a)^{p_1+q_1-1}\Gamma(q_1)}{\Gamma(p_1+q_1)}, \\ A_6 &= \displaystyle\sum_{i=1}^{n-2}\beta_i\frac{(\eta_i-a)^{p_1+q_1-2}\Gamma(q_1-1)}{\Gamma(p_1+q_1-1)},\ A_7 = \sum_{i=1}^{n-2}\beta_i, \end{cases} \tag{2.16}$$

$$\begin{cases} B_1 &= \dfrac{(b-a)^{p_2+q_2-1}\Gamma(q_2)}{\Gamma(p_2+q_2)},\ B_2 = \dfrac{(b-a)^{p_2+q_2-2}\Gamma(q_2-1)}{\Gamma(p_2+q_2-1)}, \\ B_3 &= \dfrac{(b-a)^{p_2+q_2-2}\Gamma(q_2)}{\Gamma(p_2+q_2-1)},\ B_4 = \dfrac{(b-a)^{p_2+q_2-3}\Gamma(q_2-1)}{\Gamma(p_2+q_2-2)}, \\ B_5 &= \displaystyle\sum_{i=1}^{n-2}\alpha_i\frac{(\xi_i-a)^{p_2+q_2-1}\Gamma(q_2)}{\Gamma(p_2+q_2)}, \\ B_6 &= \displaystyle\sum_{i=1}^{n-2}\alpha_i\frac{(\xi_i-a)^{p_2+q_2-2}\Gamma(q_2-1)}{\Gamma(p_2+q_2-1)},\ B_7 = \sum_{i=1}^{n-2}\alpha_i, \end{cases} \tag{2.17}$$

and it is assumed that

$$\epsilon = \epsilon_1\epsilon_2 - A_3B_3\sigma_1\theta_1 \neq 0,\ \epsilon_1 = A_1A_4 - A_2A_3 \neq 0,\ \epsilon_2 = B_1B_4 - B_2B_3 \neq 0. \tag{2.18}$$

Proof. Solving the fractional differential equations (2.1) in a standard manner by using Lemmas 2.3 and 2.4, we get

$$\begin{aligned} x(t) &= -\kappa_1 \int_a^t \frac{(t-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds - \lambda_1 \int_a^t \frac{(t-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} H(s)ds \\ &\quad + \int_a^t \frac{(t-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \Phi(s)ds + c_1 \frac{(t-a)^{p_1+q_1-1}\Gamma(q_1)}{\Gamma(p_1+q_1)} \\ &\quad + c_2 \frac{(t-a)^{p_1+q_1-2}\Gamma(q_1-1)}{\Gamma(p_1+q_1-1)} + c_3 + c_4(t-a), \end{aligned} \tag{2.19}$$

$$\begin{aligned} y(t) &= -\kappa_2 \int_a^t \frac{(t-s)^{p_2-1}}{\Gamma(p_2)} y(s)ds - \lambda_2 \int_a^t \frac{(t-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)} U(s)ds \\ &\quad + \int_a^t \frac{(t-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)} \Psi(s)ds + b_1 \frac{(t-a)^{p_2+q_2-1}\Gamma(q_2)}{\Gamma(p_2+q_2)} \\ &\quad + b_2 \frac{(t-a)^{p_2+q_2-2}\Gamma(q_2-1)}{\Gamma(p_2+q_2-1)} + b_3 + b_4(t-a), \end{aligned} \tag{2.20}$$

where $c_i, b_i \in \mathbb{R},\ i = 1,2,3,4$, are unknown arbitrary constants. Using the boundary conditions (1.2) in equations (2.19)–(2.20), together with notation (2.16) and (2.17), we obtain $c_4 = 0, b_4 = 0$, and

$$A_1c_1 + A_2c_2 + c_3 = I_1, \tag{2.21}$$

$$B_1b_1 + B_2b_2 + b_3 = E_1, \tag{2.22}$$

$$A_3c_1 + A_4c_2 = I_2, \tag{2.23}$$

$$B_3b_1 + B_4b_2 = E_2, \tag{2.24}$$

$$c_3 - B_5b_1 - B_6b_2 - B_7b_3 = E_3, \tag{2.25}$$

$$b_3 - A_5c_1 - A_6c_2 - A_7c_3 = I_3, \tag{2.26}$$

where A_i and B_i, $i = 1, \ldots, 7$, are respectively given by (2.16) and (2.17), and I_i, E_i, $i = 1,2,3$, are defined by

$$\begin{aligned} I_1 &= \kappa_1 \int_a^b \frac{(b-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds + \lambda_1 \int_a^b \frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} H(s)ds \\ &\quad - \int_a^b \frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \Phi(s)ds, \\ I_2 &= \kappa_1 \int_a^b \frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)} x(s)ds + \lambda_1 \int_a^b \frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)} H(s)ds \\ &\quad - \int_a^b \frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)} \Phi(s)ds, \end{aligned}$$

$$
\begin{aligned}
I_3 &= -\kappa_1 \sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i - s)^{p_1-1}}{\Gamma(p_1)} x(s) ds - \lambda_1 \sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i - s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1 + p_1)} H(s) ds \\
&\quad + \sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i - s)^{p_1+q_1-1}}{\Gamma(p_1 + q_1)} \Phi(s) ds, \\
E_1 &= \kappa_2 \int_a^b \frac{(b - s)^{p_2-1}}{\Gamma(p_2)} y(s) ds + \lambda_2 \int_a^b \frac{(b - s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2 + p_2)} U(s) ds \\
&\quad - \int_a^b \frac{(b - s)^{p_2+q_2-1}}{\Gamma(p_2 + q_2)} \Psi(s) ds, \\
E_2 &= \kappa_2 \int_a^b \frac{(b - s)^{p_2-2}}{\Gamma(p_2 - 1)} y(s) ds + \lambda_2 \int_a^b \frac{(b - s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2 + p_2 - 1)} U(s) ds \\
&\quad - \int_a^b \frac{(b - s)^{p_2+q_2-2}}{\Gamma(p_2 + q_2 - 1)} \Psi(s) ds, \\
E_3 &= -\kappa_2 \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i - s)^{p_2-1}}{\Gamma(p_2)} y(s) ds - \lambda_2 \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i - s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2 + p_2)} U(s) ds \\
&\quad + \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i - s)^{p_2+q_2-1}}{\Gamma(p_2 + q_2)} \Psi(s) ds. \qquad (2.27)
\end{aligned}
$$

Solving (2.21) and (2.23), we get

$$
\begin{aligned}
c_3 &= \frac{\epsilon_1}{A_3} c_2 + I_1 - \frac{A_1}{A_3} I_2, \qquad (2.28) \\
c_2 &= \frac{A_3}{\epsilon_1} c_3 - \frac{A_3}{\epsilon_1} I_1 + \frac{A_1}{\epsilon_1} I_2. \qquad (2.29)
\end{aligned}
$$

On the other hand, from (2.22) and (2.24), we obtain

$$
\begin{aligned}
b_3 &= \frac{\epsilon_2}{B_3} b_2 + E_1 - \frac{B_1}{B_3} E_2, \qquad (2.30) \\
b_2 &= \frac{B_3}{\epsilon_2} b_3 - \frac{B_3}{\epsilon_2} E_1 + \frac{B_1}{\epsilon_2} E_2, \qquad (2.31)
\end{aligned}
$$

where $\epsilon_i, i = 1, 2$, are given by (2.18)). Substituting the values of b_1 from (2.24) and b_3 from (2.30) in (2.25), we get

$$c_3 = \sigma_1 b_2 + B_7 E_1 + \sigma_2 E_2 + E_3. \qquad (2.32)$$

Finding the values of c_1 and c_3 respectively from (2.23) and (2.28) and inserting in (2.26), we obtain

$$b_3 = \theta_1 c_2 + A_7 I_1 + \theta_2 I_2 + I_3, \qquad (2.33)$$

where $\theta_i,\ \sigma_i,\ i = 1, 2$, are respectively given by (2.14) and (2.15). Using (2.31) in (2.32) and (2.29) in (2.33) leads to

$$c_3 = \frac{B_3\sigma_1}{\epsilon_2} b_3 + \Big(B_7 - \frac{B_3\sigma_1}{\epsilon_2}\Big)E_1 + \Big(\sigma_2 + \frac{B_1\sigma_1}{\epsilon_2}\Big)E_2 + E_3, \tag{2.34}$$

$$b_3 = \frac{A_3\theta_1}{\epsilon_1} c_3 + \Big(A_7 - \frac{A_3\theta_1}{\epsilon_1}\Big)I_1 + \Big(\theta_2 + \frac{A_1\theta_1}{\epsilon_1}\Big)I_2 + I_3. \tag{2.35}$$

Solving (2.34) and (2.35) for c_3 and b_3, we get

$$c_3 = \nu_1 I_1 + \nu_6 I_2 + \nu_2 I_3 + \nu_3 E_1 + \nu_4 E_2 + \nu_5 E_3, \tag{2.36}$$

$$b_3 = \omega_1 I_1 + \omega_2 I_2 + \nu_5 I_3 + \omega_4 E_1 + \omega_5 E_2 + \omega_3 E_3, \tag{2.37}$$

where $\nu_i,\ i = 1, \ldots, 6$, and $\omega_j,\ j = 1, \ldots, 5$ are defined by (2.12) and (2.13) respectively. Substituting (2.36) in (2.29) and (2.37) in (2.31), we find that

$$c_2 = \delta_1 I_1 + \delta_6 I_2 + \delta_2 I_3 + \delta_3 E_1 + \delta_4 E_2 + \delta_5 E_3, \tag{2.38}$$

$$b_2 = \rho_1 I_1 + \rho_2 I_2 + \rho_6 I_3 + \rho_4 E_1 + \rho_5 E_2 + \rho_3 E_3, \tag{2.39}$$

where δ_i and ρ_i $(i = 1, \ldots, 6)$ are given by (2.10) and (2.11) respectively. Using (2.38) in (2.23) and (2.39) in (2.24), we get

$$c_1 = \phi_1 I_1 + \phi_6 I_2 + \phi_2 I_3 + \phi_3 E_1 + \phi_4 E_2 + \phi_5 E_3, \tag{2.40}$$

$$b_1 = \psi_1 I_1 + \psi_2 I_2 + \psi_6 I_3 + \psi_4 E_1 + \psi_5 E_2 + \psi_3 E_3, \tag{2.41}$$

where ϕ_i and $\psi_i,\ i = 1, \ldots, 6$, are respectively given by (2.8) and (2.9). Inserting $c_4 = 0, b_4 = 0$, and the values of c_k and b_k, $k = 1, 2, 3$, from (2.36)–(2.41) in (2.19) and (2.20) leads to the solution (2.2) and (2.3). The converse follows by direct computation. This completes the proof. □

3 Existence Results

Let $\mathcal{X} = \{x(t)|x(t) \in C([a, b], \mathbb{R})\}$ denote the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}$ equipped with the norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$. Obviously $(\mathcal{X}, \|\cdot\|)$ is a Banach space and consequently, the product space $(\mathcal{X} \times \mathcal{X}, \|\cdot\|)$ is a Banach space with the norm $\|(x, y)\| = \|x\| + \|y\|$ for $(x, y) \in \mathcal{X} \times \mathcal{X}$.

In view of Lemma 2.5, we define an operator $\mathcal{T} : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ as

$$\mathcal{T}(x, y)(t) := (\mathcal{T}_1(x, y)(t), \mathcal{T}_2(x, y)(t)), \tag{3.1}$$

where

$\mathcal{T}_1(x, y)(t)$

$$
\begin{aligned}
= \; & -\kappa_1\int_a^t \frac{(t-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds-\lambda_1\int_a^t \frac{(t-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds \\
& +\int_a^t \frac{(t-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\phi(s,x(s),y(s))ds+f_1(t)\Big[\kappa_1\int_a^b \frac{(b-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds \\
& +\lambda_1\int_a^b \frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds-\int_a^b \frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\phi(s,x(s),y(s))ds\Big] \\
& +f_6(t)\Big[\kappa_1\int_a^b \frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)}x(s)ds+\lambda_1\int_a^b \frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)}h(s,x(s),y(s))ds \\
& -\int_a^b \frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)}\phi(s,x(s),y(s))ds\Big]+f_2(t)\Big[-\kappa_1\sum_{i=1}^{n-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds \\
& -\lambda_1\sum_{i=1}^{n-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds \\
& +\sum_{i=1}^{n-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\phi(s,x(s),y(s))ds\Big]+f_3(t)\Big[\kappa_2\int_a^b \frac{(b-s)^{p_2-1}}{\Gamma(p_2)}y(s)ds \\
& +\lambda_2\int_a^b \frac{(b-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}u(s,x(s),y(s))ds-\int_a^b \frac{(b-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\psi(s,x(s),y(s))ds\Big] \\
& +f_4(t)\Big[\kappa_2\int_a^b \frac{(b-s)^{p_2-2}}{\Gamma(p_2-1)}y(s)ds+\lambda_2\int_a^b \frac{(b-s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2+p_2-1)}u(s,x(s),y(s))ds \\
& -\int_a^b \frac{(b-s)^{p_2+q_2-2}}{\Gamma(p_2+q_2-1)}\psi(s,x(s),y(s))ds\Big]+f_5(t)\Big[-\kappa_2\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2-1}}{\Gamma(p_2)}y(s)ds \\
& -\lambda_2\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}u(s,x(s),y(s))ds \\
& +\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\psi(s,x(s),y(s))ds\Big], \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{T}_2(x,y)(t) \\
= \; & -\kappa_2\int_a^t \frac{(t-s)^{p_2-1}}{\Gamma(p_2)}y(s)ds-\lambda_2\int_a^t \frac{(t-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}u(s,x(s),y(s))ds \\
& +\int_a^t \frac{(t-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\psi(s,x(s),y(s))ds+g_1(t)\Big[\kappa_1\int_a^b \frac{(b-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds \\
& +\lambda_1\int_a^b \frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds-\int_a^b \frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\phi(s,x(s),y(s))ds\Big] \\
& +g_2(t)\Big[\kappa_1\int_a^b \frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)}x(s)ds+\lambda_1\int_a^b \frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)}h(s,x(s),y(s))ds \\
& -\int_a^b \frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)}\phi(s,x(s),y(s))ds\Big]+g_6(t)\Big[-\kappa_1\sum_{i=1}^{n-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds
\end{aligned}
$$

$$-\lambda_1 \sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i - s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} h(s,x(s),y(s))ds$$
$$+\sum_{i=1}^{n-2} \beta_i \int_a^{\eta_i} \frac{(\eta_i - s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \phi(s,x(s),y(s))ds\Big] + g_4(t)\Big[\kappa_2 \int_a^b \frac{(b-s)^{p_2-1}}{\Gamma(p_2)} y(s)ds$$
$$+\lambda_2 \int_a^b \frac{(b-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)} u(s,x(s),y(s))ds - \int_a^b \frac{(b-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)} \psi(s,x(s),y(s))ds\Big]$$
$$+g_5(t)\Big[\kappa_2 \int_a^b \frac{(b-s)^{p_2-2}}{\Gamma(p_2-1)} y(s)ds + \lambda_2 \int_a^b \frac{(b-s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2+p_2-1)} u(s,x(s),y(s))ds$$
$$-\int_a^b \frac{(b-s)^{p_2+q_2-2}}{\Gamma(p_2+q_2-1)} \psi(s,x(s),y(s))ds\Big] + g_3(t)\Big[-\kappa_2 \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i - s)^{p_2-1}}{\Gamma(p_2)} y(s)ds$$
$$-\lambda_2 \sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i - s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)} u(s,x(s),y(s))ds$$
$$+\sum_{i=1}^{n-2} \alpha_i \int_a^{\xi_i} \frac{(\xi_i - s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)} \psi(s,x(s),y(s))ds\Big], \tag{3.3}$$

and $f_i(t), i = 1, \ldots, 6$ are given by (2.4), and $g_j(t),\ j = 1, \ldots, 5$ and $g_6(t)$ are respectively defined by (2.5) and (2.6).

In the forthcoming analysis, we assume that $h, \phi, u, \psi : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying the following condition:

($\mathcal{H}_1$) $\forall\ t \in [a,b],\ x, y \in \mathbb{R}$ there exist real constants $\mu_i,\ \varepsilon_i,\ n_i,\ m_i \geq 0,\ i = 1,2,\ \mu_0,\ \varepsilon_0, n_0, m_0 > 0$ such that

$$|h(t,x,y)| \leq \mu_0 + \mu_1|x| + \mu_2|y|,\ \ |\phi(t,x,y)| \leq \varepsilon_0 + \varepsilon_1|x| + \varepsilon_2|y|,$$
$$|u(t,x,y)| \leq n_0 + n_1|x| + n_2|y|,\ \ |\psi(t,x,y)| \leq m_0 + m_1|x| + m_2|y|.$$

Further, we set the following notation:

$$\begin{aligned}
\mathcal{A}_0 &= \frac{(b-a)^{p_1}}{\Gamma(p_1+1)} + F_1\frac{(b-a)^{p_1}}{\Gamma(p_1+1)} + F_6\frac{(b-a)^{p_1-1}}{\Gamma(p_1)} + F_2\sum_{i=1}^{n-2}|\beta_i|\frac{(\eta_i-a)^{p_1}}{\Gamma(p_1+1)},\\
\mathcal{A}_1 &= \frac{(b-a)^{\gamma_1+p_1}}{\Gamma(\gamma_1+p_1+1)} + F_1\frac{(b-a)^{\gamma_1+p_1}}{\Gamma(\gamma_1+p_1+1)} + F_6\frac{(b-a)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\\
&\quad +F_2\sum_{i=1}^{n-2}|\beta_i|\frac{(\eta_i-a)^{\gamma_1+p_1}}{\Gamma(\gamma_1+p_1+1)},\\
\mathcal{A}_2 &= \frac{(b-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)} + F_1\frac{(b-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)} + F_6\frac{(b-a)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\\
&\quad +F_2\sum_{i=1}^{n-2}|\beta_i|\frac{(\eta_i-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)},\\
\mathcal{A}_3 &= F_3\frac{(b-a)^{p_2}}{\Gamma(p_2+1)} + F_4\frac{(b-a)^{p_2-1}}{\Gamma(p_2)} + F_5\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{p_2}}{\Gamma(p_2+1)},\\
\mathcal{A}_4 &= F_3\frac{(b-a)^{\gamma_2+p_2}}{\Gamma(\gamma_2+p_2+1)} + F_4\frac{(b-a)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)} + F_5\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{\gamma_2+p_2}}{\Gamma(\gamma_2+p_2+1)},\\
\mathcal{A}_5 &= F_3\frac{(b-a)^{p_2+q_2}}{\Gamma(p_2+q_2+1)} + F_4\frac{(b-a)^{p_2+q_2-1}}{\Gamma(p_2+q_2)} + F_5\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{p_2+q_2}}{\Gamma(p_2+q_2+1)}, \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_0 &= G_1\frac{(b-a)^{p_1}}{\Gamma(p_1+1)} + G_2\frac{(b-a)^{p_1-1}}{\Gamma(p_1)} + G_6\sum_{i=1}^{n-2}|\beta_i|\frac{(\eta_i-a)^{p_1}}{\Gamma(p_1+1)},\\
\mathcal{B}_1 &= G_1\frac{(b-a)^{\gamma_1+p_1}}{\Gamma(\gamma_1+p_1+1)} + G_2\frac{(b-a)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} + G_6\sum_{i=1}^{n-2}|\beta_i|\frac{(\eta_i-a)^{\gamma_1+p_1}}{\Gamma(\gamma_1+p_1+1)},\\
\mathcal{B}_2 &= G_1\frac{(b-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)} + G_2\frac{(b-a)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} + G_6\sum_{i=1}^{n-2}|\beta_i|\frac{(\eta_i-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)},\\
\mathcal{B}_3 &= \frac{(b-a)^{p_2}}{\Gamma(p_2+1)} + G_4\frac{(b-a)^{p_2}}{\Gamma(p_2+1)} + G_5\frac{(b-a)^{p_2-1}}{\Gamma(p_2)} + G_3\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{p_2}}{\Gamma(p_2+1)},\\
\mathcal{B}_4 &= \frac{(b-a)^{\gamma_2+p_2}}{\Gamma(\gamma_2+p_2+1)} + G_4\frac{(b-a)^{\gamma_2+p_2}}{\Gamma(\gamma_2+p_2+1)} + G_5\frac{(b-a)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}\\
&\quad +G_3\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{\gamma_2+p_2}}{\Gamma(\gamma_2+p_2+1)},\\
\mathcal{B}_5 &= \frac{(b-a)^{p_2+q_2}}{\Gamma(p_2+q_2+1)} + G_4\frac{(b-a)^{p_2+q_2}}{\Gamma(p_2+q_2+1)} + G_5\frac{(b-a)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}
\end{aligned}$$

$$+G_3\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{p_2+q_2}}{\Gamma(p_2+q_2+1)}, \tag{3.5}$$

where $F_i=\sup_{t\in[a,b]}|f_i(t)|,\ i=1,\dots,6$ and $G_j=\sup_{t\in[a,b]}|g_j(t)|,\ j=1,\dots,6,$

$$\begin{aligned} V_0 &= (\mathcal{A}_1+\mathcal{B}_1)|\lambda_1|\mu_0+(\mathcal{A}_2+\mathcal{B}_2)\varepsilon_0+(\mathcal{A}_4+\mathcal{B}_4)|\lambda_2|n_0+(\mathcal{A}_5+\mathcal{B}_5)m_0, & (3.6)\\ V_1 &= (\mathcal{A}_0+\mathcal{B}_0)|\kappa_1|+(\mathcal{A}_1+\mathcal{B}_1)|\lambda_1|\mu_1+(\mathcal{A}_2+\mathcal{B}_2)\varepsilon_1+(\mathcal{A}_4+\mathcal{B}_4)|\lambda_2|n_1\\ &\quad +(\mathcal{A}_5+\mathcal{B}_5)m_1, & (3.7)\\ V_2 &= (\mathcal{A}_1+\mathcal{B}_1)|\lambda_1|\mu_2+(\mathcal{A}_2+\mathcal{B}_2)\varepsilon_2+(\mathcal{A}_3+\mathcal{B}_3)|\kappa_2|+(\mathcal{A}_4+\mathcal{B}_4)|\lambda_2|n_2\\ &\quad +(\mathcal{A}_5+\mathcal{B}_5)m_2, & (3.8)\\ V &= \max\{V_1,V_2\}. & (3.9)\end{aligned}$$

Now we present our main results. The first existence theorem for the system (1.1)-(1.2) relies on Leray–Schauder alternative.

Lemma 3.1 (Leray–Schauder alternative [22])**.** *Let* $\mathrm{M}:\mathcal{Y}\longrightarrow\mathcal{Y}$ *be a completely continuous operator (i.e., a map that restricted to any bounded set in* $\mathcal{Y}$ *is compact). Let* $\mathcal{G}(\mathrm{M})=\{y\in\mathcal{Y}: y=\lambda\mathrm{M}(y)$ *for some* $0<\lambda<1\}$. *Then either the set* $\mathcal{G}(\mathrm{M})$ *is unbounded, or* M *has at lest one fixed point.*

Theorem 3.2. *Assume that* $h,\phi,u,\psi:[a,b]\times\mathbb{R}^2\to\mathbb{R}$ *are continuous functions satisfying the assumption (*$\mathcal{H}_1$*). Then the system* (1.1)–(1.2) *has at least one solution on* $[a,b]$ *if* $V<1$, *where* V *is given by* (3.9).

Proof. First we show that the operator $\mathcal{T}:\mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ is completely continuous. By continuity of functions h,ϕ,u and ψ, it is easy to very that the operator $\mathcal{T}$ is continuous.

Let $B_r\subset\mathcal{X}\times\mathcal{X}$ where $B_r=\{(x,y)\in\mathcal{X}\times\mathcal{X}:\|(x,y)\|\le r\}$. Then there exist positive constants ζ_i $(i=1,\dots,4)$ such that $|h(t,x(t),y(t))|\le\zeta_1,\ |\phi(t,x(t),y(t))|\le\zeta_2,\ |u(t,x(t),y(t))|\le\zeta_3,\ |\psi(t,x(t),y(t))|\le\zeta_4,\ \forall(x,y)\in B_r$. Then, for any $(x,y)\in B_r$, we have

$$\begin{aligned} &|\mathcal{T}_1(x,y)(t)|\\ \le\ & |\kappa_1|\int_a^t\frac{(t-s)^{p_1-1}}{\Gamma(p_1)}|x(s)|ds+|\lambda_1|\int_a^t\frac{(t-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\zeta_1ds\\ &+\int_a^t\frac{(t-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\zeta_2ds+|f_1(t)|\Big[|\kappa_1|\int_a^b\frac{(b-s)^{p_1-1}}{\Gamma(p_1)}|x(s)|ds\\ &+|\lambda_1|\int_a^b\frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\zeta_1ds+\int_a^b\frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\zeta_2ds\Big]\\ &+|f_6(t)|\Big[|\kappa_1|\int_a^b\frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)}|x(s)|ds+|\lambda_1|\int_a^b\frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)}\zeta_1ds\end{aligned}$$

$$
\begin{aligned}
&+\int_a^b \frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)}\zeta_2 ds\Big] + |f_2(t)|\Big[|\kappa_1|\sum_{i=1}^{n-2}|\beta_i|\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1-1}}{\Gamma(p_1)}|x(s)|ds \\
&+|\lambda_1|\sum_{i=1}^{n-2}|\beta_i|\int_a^{\eta_i}\frac{(\eta_i-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\zeta_1 ds + \sum_{i=1}^{n-2}|\beta_i|\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\zeta_2 ds\Big] \\
&+|f_3(t)|\Big[|\kappa_2|\int_a^b\frac{(b-s)^{p_2-1}}{\Gamma(p_2)}|y(s)|ds + |\lambda_2|\int_a^b\frac{(b-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}\zeta_3 ds \\
&+\int_a^b\frac{(b-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\zeta_4 ds\Big] + |f_4(t)|\Big[|\kappa_2|\int_a^b\frac{(b-s)^{p_2-2}}{\Gamma(p_2-1)}|y(s)|ds \\
&+|\lambda_2|\int_a^b\frac{(b-s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2+p_2-1)}\zeta_3 ds + \int_a^b\frac{(b-s)^{p_2+q_2-2}}{\Gamma(p_2+q_2-1)}\zeta_4 ds\Big] \\
&+|f_5(t)|\Big[|\kappa_2|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2-1}}{\Gamma(p_2)}|y(s)|ds \\
&+|\lambda_2|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}\zeta_3 ds + \sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\zeta_4 ds\Big] \\
\leq\ & |\kappa_1|\mathcal{A}_0|x(t)| + |\lambda_1|\mathcal{A}_1\zeta_1 + \mathcal{A}_2\zeta_2 + |\kappa_2|\mathcal{A}_3|y(t)| + |\lambda_2|\mathcal{A}_4\zeta_3 + \mathcal{A}_5\zeta_4,
\end{aligned}
$$

which implies that

$$
\|\mathcal{T}_1(x,y)\| \ \leq\ |\kappa_1|\mathcal{A}_0\|x\| + |\lambda_1|\mathcal{A}_1\zeta_1 + \mathcal{A}_2\zeta_2 + |\kappa_2|\mathcal{A}_3\|y\| + |\lambda_2|\mathcal{A}_4\zeta_3 + \mathcal{A}_5\zeta_4.
$$

Similarly, we can get

$$
\|\mathcal{T}_2(x,y)\| \ \leq\ |\kappa_1|\mathcal{B}_0\|x\| + |\lambda_1|\mathcal{B}_1\zeta_1 + \mathcal{B}_2\zeta_2 + |\kappa_2|\mathcal{B}_3\|y\| + |\lambda_2|\mathcal{B}_4\zeta_3 + \mathcal{B}_5\zeta_4.
$$

From the above inequalities, it follows that the operator $\mathcal{T}$ is uniformly bounded, since $\|\mathcal{T}(x,y)\| \leq |\kappa_1|(\mathcal{A}_0+\mathcal{B}_0)r + |\lambda_1|(\mathcal{A}_1+\mathcal{B}_1)\zeta_1 + (\mathcal{A}_2+\mathcal{B}_2)\zeta_2 + |\kappa_2|(\mathcal{A}_3+\mathcal{B}_3)r + |\lambda_2|(\mathcal{A}_4+\mathcal{B}_4)\zeta_3 + (\mathcal{A}_5+\mathcal{B}_5)\zeta_4$.

Next, we show that the operator $\mathcal{T}$ is equicontinuous. For $t_1, t_2 \in [a,b]$ with $t_1 < t_2$, we obtain

$$
\begin{aligned}
&\Big|\mathcal{T}_1(x,y)(t_2) - \mathcal{T}_1(x,y)(t_1)\Big| \leq \\
&|\kappa_1|\Big[\Big|\int_a^{t_1}\frac{[(t_2-s)^{p_1-1}-(t_1-s)^{p_1-1}]}{\Gamma(p_1)}x(s)ds\Big| + \Big|\int_{t_1}^{t_2}\frac{(t_2-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds\Big|\Big] \\
&+|\lambda_1|\Big[\Big|\int_a^{t_1}\frac{[(t_2-s)^{\gamma_1+p_1-1}-(t_1-s)^{\gamma_1+p_1-1}]}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds\Big| \\
&+\Big|\int_{t_1}^{t_2}\frac{(t_2-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds\Big|\Big] \\
&+\Big|\int_a^{t_1}\frac{[(t_2-s)^{p_1+q_1-1}-(t_1-s)^{p_1+q_1-1}]}{\Gamma(p_1+q_1)}\phi(s,x(s),y(s))ds\Big|
\end{aligned}
$$

$$+ \Big| \int_{t_1}^{t_2} \frac{(t_2 - s)^{p_1+q_1-1}}{\Gamma(p_1 + q_1)} \phi(s, x(s), y(s)) ds \Big|$$
$$+ \big| f_1(t_2) - f_1(t_1) \big| \Big[|\kappa_1| \int_a^b \frac{(b - s)^{p_1-1}}{\Gamma(p_1)} |x(s)| ds$$
$$+ |\lambda_1| \int_a^b \frac{(b - s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1 + p_1)} |h(s, x(s), y(s))| ds$$
$$+ \int_a^b \frac{(b - s)^{p_1+q_1-1}}{\Gamma(p_1 + q_1)} |\phi(s, x(s), y(s))| ds \Big]$$
$$+ \big| f_6(t_2) - f_6(t_1) \big| \Big[|\kappa_1| \int_a^b \frac{(b - s)^{p_1-2}}{\Gamma(p_1 - 1)} |x(s)| ds$$
$$+ |\lambda_1| \int_a^b \frac{(b - s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1 + p_1 - 1)} |h(s, x(s), y(s))| ds$$
$$+ \int_a^b \frac{(b - s)^{p_1+q_1-2}}{\Gamma(p_1 + q_1 - 1)} |\phi(s, x(s), y(s))| ds \Big]$$
$$+ \big| f_2(t_2) - f_2(t_1) \big| \Big[|\kappa_1| \sum_{i=1}^{n-2} |\beta_i| \int_a^{\eta_i} \frac{(\eta_i - s)^{p_1-1}}{\Gamma(p_1)} |x(s)| ds$$
$$+ |\lambda_1| \sum_{i=1}^{n-2} |\beta_i| \int_a^{\eta_i} \frac{(\eta_i - s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1 + p_1)} |h(s, x(s), y(s))| ds$$
$$+ \sum_{i=1}^{n-2} |\beta_i| \int_a^{\eta_i} \frac{(\eta_i - s)^{p_1+q_1-1}}{\Gamma(p_1 + q_1)} |\phi(s, x(s), y(s))| ds \Big]$$
$$+ \big| f_3(t_2) - f_3(t_1) \big| \Big[|\kappa_2| \int_a^b \frac{(b - s)^{p_2-1}}{\Gamma(p_2)} |y(s)| ds$$
$$+ |\lambda_2| \int_a^b \frac{(b - s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2 + p_2)} |u(s, x(s), y(s))| ds$$
$$+ \int_a^b \frac{(b - s)^{p_2+q_2-1}}{\Gamma(p_2 + q_2)} |\psi(s, x(s), y(s))| ds \Big]$$
$$+ \big| f_4(t_2) - f_4(t_1) \big| \Big[|\kappa_2| \int_a^b \frac{(b - s)^{p_2-2}}{\Gamma(p_2 - 1)} |y(s)| ds$$
$$+ |\lambda_2| \int_a^b \frac{(b - s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2 + p_2 - 1)} |u(s, x(s), y(s))| ds$$
$$+ \int_a^b \frac{(b - s)^{p_2+q_2-2}}{\Gamma(p_2 + q_2 - 1)} |\psi(s, x(s), y(s))| ds \Big]$$
$$+ \big| f_5(t_2) - f_5(t_1) \big| \Big[|\kappa_2| \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i - s)^{p_2-1}}{\Gamma(p_2)} |y(s)| ds$$

$$+|\lambda_2|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}|u(s,x(s),y(s))|ds$$
$$+\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}|\psi(s,x(s),y(s))|ds\Big]$$
$$\leq \frac{|\kappa_1|r}{\Gamma(p_1+1)}\Big(|(t_2-a)^{p_1}-(t_1-a)^{p_1}|+2(t_2-t_1)^{p_1}\Big)$$
$$+\frac{|\lambda_1|\zeta_1}{\Gamma(\gamma_1+p_1+1)}\Big(|(t_2-a)^{\gamma_1+p_1}-(t_1-a)^{\gamma_1+p_1}|+2(t_2-t_1)^{\gamma_1+p_1}\Big)$$
$$+\frac{\zeta_2}{\Gamma(p_1+q_1+1)}\Big(|(t_2-a)^{p_1+q_1}-(t_1-a)^{p_1+q_1}|+2(t_2-t_1)^{p_1+q_1}\Big)$$
$$+\big|f_1(t_2)-f_1(t_1)\big|\Big[|\kappa_1|\frac{(b-a)^{p_1}}{\Gamma(p_1+1)}r+|\lambda_1|\zeta_1\frac{(b-a)^{\gamma_1+p_1}}{\Gamma(\gamma_1+p_1+1)}+\frac{(b-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)}\zeta_2\Big]$$
$$+\big|f_6(t_2)-f_6(t_1)\big|\Big[|\kappa_1|\frac{(b-a)^{p_1-1}}{\Gamma(p_1)}r+|\lambda_1|\frac{(b-a)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\zeta_1+\frac{(b-a)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\zeta_2\Big]$$
$$+\big|f_2(t_2)-f_2(t_1)\big|\Big[|\kappa_1|\sum_{i=1}^{n-2}|\beta_i|\frac{(\eta_i-a)^{p_1}}{\Gamma(p_1+1)}r+|\lambda_1|\sum_{i=1}^{n-2}|\beta_i|\frac{(\eta_i-a)^{\gamma_1+p_1}}{\Gamma(\gamma_1+p_1+1)}\zeta_1$$
$$+\sum_{i=1}^{n-2}|\beta_i|\frac{(\eta_i-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)}\zeta_2\Big]+\Big|f_3(t_2)-f_3(t_1)\Big|\Big[|\kappa_2|\frac{(b-a)^{p_2}}{\Gamma(p_2+1)}r$$
$$+|\lambda_2|\frac{(b-a)^{\gamma_2+p_2}}{\Gamma(\gamma_2+p_2+1)}\zeta_3+\frac{(b-a)^{p_2+q_2}}{\Gamma(p_2+q_2+1)}\zeta_4\Big]+\Big|f_4(t_2)-f_4(t_1)\Big|\Big[|\kappa_2|\frac{(b-a)^{p_2-1}}{\Gamma(p_2)}r$$
$$+|\lambda_2|\frac{(b-a)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}\zeta_3+\frac{(b-a)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\zeta_4\Big]$$
$$+\big|f_5(t_2)-f_5(t_1)\big|\Big[|\kappa_2|\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{p_2}}{\Gamma(p_2+1)}r$$
$$+|\lambda_2|\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{\gamma_2+p_2}}{\Gamma(\gamma_2+p_2+1)}\zeta_3+\sum_{i=1}^{n-2}|\alpha_i|\frac{(\xi_i-a)^{p_2+q_2}}{\Gamma(p_2+q_2+1)}\zeta_4\Big].$$

In consequence, $|\mathcal{T}_1(x,y)(t_2)-\mathcal{T}_1(x,y)(t_1)|\to 0$ independent of x and y as $t_2\to t_1$. Analogously, we can obtain that $|\mathcal{T}_2(x,y)(t_2)-\mathcal{T}_2(x,y)(t_1)|\to 0$ independent of x and y as $t_2\to t_1$. Therefore, the operator $\mathcal{T}(x,y)$ is equicontinuous. In view of the foregoing steps, it follows by the Arzelà–Ascoli theorem that the operator $\mathcal{T}$ is completely continuous.

Next, it will be verified that the set $\mathcal{E}=\{(x,y)\in\mathcal{X}\times\mathcal{X}|(x,y)=\sigma\mathcal{T}(x,y), 0\leq\sigma\leq 1\}$ is bounded. Let $(x,y)\in\mathcal{E}$, then $(x,y)=\sigma\mathcal{T}(x,y)$ and for any $t\in[a,b]$, we have $x(t)=\sigma\mathcal{T}_1(x,y)(t),\ y(t)=\sigma\mathcal{T}_2(x,y)(t)$. As before, we can find that

$$\|x\| \leq |\kappa_1|\mathcal{A}_0\|x\|+|\lambda_1|\mathcal{A}_1(\mu_0+\mu_1\|x\|+\mu_2\|y\|)+\mathcal{A}_2(\varepsilon_0+\varepsilon_1\|x\|+\varepsilon_2\|y\|)$$

$$+|\kappa_2|\mathcal{A}_3\|y\| + |\lambda_2|\mathcal{A}_4(n_0 + n_1\|x\| + n_2\|y\|)$$
$$+\mathcal{A}_5(m_0 + m_1\|x\| + m_2\|y\|) \tag{3.10}$$

and

$$\begin{aligned}\|y\| \leq\ & |\kappa_1|\mathcal{B}_0\|x\| + |\lambda_1|\mathcal{B}_1(\mu_0 + \mu_1\|x\| + \mu_2\|y\|) + \mathcal{B}_2(\varepsilon_0 + \varepsilon_1\|x\| + \varepsilon_2\|y\|)\\ &+|\kappa_2|\mathcal{B}_3\|y\| + |\lambda_2|\mathcal{B}_4(n_0 + n_1\|x\| + n_2\|y\|)\\ &+\mathcal{B}_5(m_0 + m_1\|x\| + m_2\|y\|).\end{aligned} \tag{3.11}$$

From (3.10) and (3.11) together with notation (3.6)–3.9, we get

$$\begin{aligned}&\|x\| + \|y\|\\ \leq\ & [(\mathcal{A}_1 + \mathcal{B}_1)|\lambda_1|\mu_0 + (\mathcal{A}_2 + \mathcal{B}_2)\varepsilon_0 + (\mathcal{A}_4 + \mathcal{B}_4)|\lambda_2|n_0 + (\mathcal{A}_5 + \mathcal{B}_5)m_0]\\ &+[(\mathcal{A}_0 + \mathcal{B}_0)|\kappa_1| + (\mathcal{A}_1 + \mathcal{B}_1)|\lambda_1|\mu_1 + (\mathcal{A}_2 + \mathcal{B}_2)\varepsilon_1 + (\mathcal{A}_4 + \mathcal{B}_4)|\lambda_2|n_1\\ &+(\mathcal{A}_5 + \mathcal{B}_5)m_1]\|x\| + [(\mathcal{A}_1 + \mathcal{B}_1)|\lambda_1|\mu_2 + (\mathcal{A}_2 + \mathcal{B}_2)\varepsilon_2 + (\mathcal{A}_3 + \mathcal{B}_3)|\kappa_2|\\ &+(\mathcal{A}_4 + \mathcal{B}_4)|\lambda_2|n_2 + (\mathcal{A}_5 + \mathcal{B}_5)m_2]\|y\|,\end{aligned}$$

which leads to

$$\|(x, y)\| \leq V_0 + \max\{V_1 + V_2\}\|(x, y)\| \leq V_0 + V\|(x, y)\|.$$

From the above inequality, it follows that

$$\|(x, y)\| \leq \frac{V_0}{1 - V}.$$

This shows that the set $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, we deduce that the operator $\mathcal{T}$ has at least one fixed point. Therefore, the system (1.1)–(1.2) has at least one solution on $[a, b]$. □

Example 3.3. Consider the coupled system of fractional differential equations:

$$\begin{cases} {}^{RL}D^{3/2}\Big[({}^{c}D^{9/5} + \frac{1}{16})x(t) + \frac{2}{110}I^{23/5}h(t, x(t), y(t))\Big] = \phi(t, x(t), y(t)),\\ {}^{RL}D^{4/3}\Big[({}^{c}D^{13/7} + \frac{5}{99})x(t) + \frac{3}{707}I^{11/3}u(t, x(t), y(t))\Big] = \psi(t, x(t), y(t)),\end{cases} \tag{3.12}$$

$t \in [0, 1]$, equipped with the boundary conditions:

$$\begin{cases} x'(0) = 0,\ x(1) = 0,\ x'(1) = 0,\ x(0) = \displaystyle\sum_{i=1}^{3} \alpha_i y(\xi_i),\\ y'(0) = 0,\ y(1) = 0,\ y'(1) = 0,\ y(0) = \displaystyle\sum_{i=1}^{3} \beta_i x(\xi_i),\end{cases} \tag{3.13}$$

where $a = 0,\ b = 1,\ q_1 = 3/2,\ p_1 = 9/5,\ q_2 = 4/3,\ p_2 = 13/7,\ \gamma_1 = 23/5,\ \gamma_2 = 11/3,\ \kappa_1 = 1/16,\ \lambda_1 = 2/110,\ \kappa_2 = 5/99,\ \lambda_2 = 3/707,\ \alpha_1 = -1/5,\ \alpha_2 = 1,\ \alpha_3 = 1/2,\ \beta_1 = -1/3,\ \beta_2 = 3/29,\ \beta_3 = 5/12,\ \xi_1 = 1/7,\ \xi_2 = 2/7,\ \xi_3 = 3/7,\ \eta_1 = 1/5,\ \eta_2 = 2/5,\ \eta_3 = 3/5,$

$$\begin{aligned} h(t,x(t),y(t)) &= \frac{x(t)}{15(t+1)^2} + \frac{\sin y(t)}{9} + \frac{3}{2}, \\ \phi(t,x(t),y(t)) &= \frac{x(t)}{30+t^4} + \frac{y(t)}{3\sqrt{t^2+9}} + \frac{x(t)}{24(1+x(t))}, \\ u(t,x(t),y(t)) &= \frac{2x(t)}{\sqrt{t^3+400}} + \frac{\sin(2\pi y(t))}{32\pi} + \frac{1}{2}, \end{aligned}$$

and

$$\psi(t,x(t),y(t)) = \frac{2x(t)|y(t)|}{20(1+|y(t)|)} + \frac{y(t)}{414+t^2} + \frac{16|\tan^{-1} y(t)|}{\pi(t+2)}.$$

Using the given data, we find that

$$A_1 \simeq 0.330259,\ A_2 \simeq 1.51919,\ A_3 \simeq 0.759595,\ A_4 \simeq 1.97495,$$

$$A_5 \simeq 0.043935,\ A_6 \simeq 0.311097,\ A_7 \simeq 0.186782,\ B_1 \simeq 0.371909,$$

$$B_2 \simeq 2.44398,\ B_3 \simeq 0.814659,\ B_4 \simeq 2.90950,\ B_5 \simeq 0.051931,$$

$$B_6 \simeq 0.947497,\ B_7 \simeq 1.30000,\ \epsilon_1 \simeq -0.501725,\ \epsilon_2 \simeq -0.90894,$$

$$\sigma_1 \simeq -0.688414,\ \sigma_2 \simeq -0.529732,\ \theta_1 \simeq 0.073491,\ \theta_2 \simeq -0.023368,$$

$$\nu_1 \simeq 0.172083,\ \nu_2 \simeq 0.577371,\ \nu_3 \simeq 0.639117,\ \nu_4 \simeq -0.232121,$$

$$\nu_5 \simeq 0.935760,\ \nu_6 \simeq -0.041423,\ \omega_1 \simeq 0.278900,\ \omega_2 \simeq -0.067135,$$

$$\omega_3 \simeq -0.104117,\ \omega_4 \simeq -0.071111,\ \omega_5 \simeq 0.025826,\ \delta_1 \simeq 1.25344,$$

$$\delta_2 \simeq -0.874120,\ \delta_3 \simeq -0.967602,\ \delta_4 \simeq 0.351424,\ \delta_5 \simeq -1.41671,$$

$$\delta_6 \simeq -0.595533,\ \rho_1 \simeq -0.249970,\ \rho_2 \simeq 0.060171,\ \rho_3 \simeq 0.093317,$$

$$\rho_4 \simeq 0.960007,\ \rho_5 \simeq -0.432316,\ \rho_6 \simeq -0.838697,\ \phi_1 \simeq -3.25895,$$

$$\phi_2 \simeq 2.27271,\ \phi_3 \simeq 2.51577,\ \phi_4 \simeq -0.913704,\ \phi_5 \simeq 3.68345,$$

$$\phi_6 \simeq 2.86488,\ \psi_1 \simeq 0.892751,\ \psi_2 \simeq -0.214900,\ \psi_3 \simeq -0.333277,$$

$$\psi_4 \simeq -3.42860,\ \psi_5 \simeq 2.77149,\ \psi_6 \simeq 2.99535,\ F_1 \simeq 1.00000,$$

$$F_2 \simeq 0.577371,\ F_3 \simeq 0.639117,\ F_4 \simeq 0.232121,\ F_5 \simeq 0.935760,$$

$$F_6 \simeq 0.093971,\ G_1 \simeq 0.278900,\ G_2 \simeq 0.067135,\ G_3 \simeq 0.104117,$$

$$G_4 \simeq 0.999999,\ G_5 \simeq 0.028430,\ G_6 \simeq 0.935760,\ \mathcal{A}_0 = 1.36426,$$

$$\mathcal{A}_1 \simeq 0.001693,\ \mathcal{A}_2 \simeq 0.266340,\ \mathcal{A}_3 \simeq 0.717870,\ \mathcal{A}_4 \simeq 0.006412,$$
$$\mathcal{A}_5 = 0.186490,\ \mathcal{B}_0 = 0.352536,\ \mathcal{B}_1 \simeq 0.000469,\ \mathcal{B}_2 \simeq 0.065378,$$
$$\mathcal{B}_3 \simeq 1.17845,\ \mathcal{B}_4 \simeq 0.007181,\ \mathcal{B}_5 = 0.273629.$$

Moreover, we have $\mu_0 = 3/2,\ \mu_1 = 1/15,\ \mu_2 = 1/9,\ \varepsilon_0 = 1/24,\ \varepsilon_1 = 1/30,\ \varepsilon_2 = 1/9,\ n_0 = 1/2,\ n_1 = 1/10,\ n_2 = 1/16,\ m_0 = 4,\ m_1 = 1/10$, and $m_2 = 1/414$ as

$$|h(t,x(t),y(t))| \leq \frac{3}{2} + \frac{1}{15}\|x\| + \frac{1}{9}\|y\|,\ \ |\phi(t,x(t),y(t))| \leq \frac{1}{24} + \frac{1}{30}\|x\| + \frac{1}{9}\|y\|,$$
$$|u(t,x(t),y(t))| \leq \frac{1}{2} + \frac{1}{10}\|x\| + \frac{1}{16}\|y\|,\ \ |\psi(t,x(t),y(t))| \leq 4 + \frac{1}{10}\|x\| + \frac{1}{414}\|y\|.$$

From (3.7) and (3.8), we get $V_1 \simeq 0.164378$, $V_2 \simeq 0.133751$ and $V = \max\{V_1, V_2\} \simeq 0.164378 < 1$. Therefore, by Theorem 3.2, the problem (3.12)–(3.13) has at least one solution on $[0,1]$.

Our next existence result is based on the following version of Krasnosel'skiĭ's fixed point theorem [18].

Lemma 3.4. *Let $\mathcal{Y}$ be a closed, bounded, convex and nonempty subset of a Banach space $\mathcal{K}$. Let $\mathcal{J}_1, \mathcal{J}_2$ be operators mapping $\mathcal{Y}$ to $\mathcal{K}$ such that*

- (a) *$\mathcal{J}_1 y_1 + \mathcal{J}_2 y_2 \in \mathcal{Y}$ where $y_1, y_2 \in \mathcal{Y}$;*
- (b) *$\mathcal{J}_1$ is compact and continuous;*
- (c) *$\mathcal{J}_2$ is a contraction mapping.*

Then there exists $y \in \mathcal{Y}$ such that $y = \mathcal{J}_1 y + \mathcal{J}_2 y$.

In the sequel, it is assumed that $h, \phi, u, \psi : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying the following condition:

($\mathcal{H}_2$) $\forall t \in [a,b]$ and $x_j, y_j \in \mathbb{R},\ j = 1,2$, there exist $L_i,\ i = 1,\ldots,4$ such that

$$\begin{aligned}
|h(t,x_1,y_1) - h(t,x_2,y_2)| &\leq L_1(|x_1 - x_2| + |y_1 - y_2|),\\
|\phi(t,x_1,y_1) - \phi(t,x_2,y_2)| &\leq L_2(|x_1 - x_2| + |y_1 - y_2|),\\
|u(t,x_1,y_1) - u(t,x_2,y_2)| &\leq L_3(|x_1 - x_2| + |y_1 - y_2|),\\
|\psi(t,x_1,y_1) - \psi(t,x_2,y_2)| &\leq L_4(|x_1 - x_2| + |y_1 - y_2|).
\end{aligned}$$

For computational convenience, we introduce the following notation:

$$\Omega = \Delta_1 + \Delta_2, \tag{3.14}$$
$$\overline{\Omega} = \overline{\Delta}_1 + \overline{\Delta}_2, \tag{3.15}$$
$$\Delta_1 = |\kappa_1|\mathcal{A}_0 + |\lambda_1|L_1\mathcal{A}_1 + L_2\mathcal{A}_2, \tag{3.16}$$

$$\begin{aligned}\Delta_2 &= |\kappa_2|\mathcal{A}_3 + |\lambda_2|L_3\mathcal{A}_4 + L_4\mathcal{A}_5, \quad (3.17)\\ \overline{\Delta}_1 &= |\kappa_1|\mathcal{B}_0 + |\lambda_1|L_1\mathcal{B}_1 + L_2\mathcal{B}_2, \quad (3.18)\\ \overline{\Delta}_2 &= |\kappa_2|\mathcal{B}_3 + |\lambda_2|L_3\mathcal{B}_4 + L_4\mathcal{B}_5, \quad (3.19)\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_0 &= \mathcal{A}_0 - \frac{(b-a)^{p_1}}{\Gamma(p_1+1)},\ \mathcal{Q}_1 = \mathcal{A}_1 - \frac{(b-a)^{\gamma_1+p_1}}{\Gamma(\gamma_1+p_1+1)},\ \mathcal{Q}_2 = \mathcal{A}_2 - \frac{(b-a)^{p_1+q_1}}{\Gamma(p_1+q_1+1)},\\ \mathcal{Q}_3 &= \mathcal{B}_3 - \frac{(b-a)^{p_2}}{\Gamma(p_2+1)},\ \mathcal{Q}_4 = \mathcal{B}_4 - \frac{(b-a)^{\gamma_2+p_2}}{\Gamma(\gamma_2+p_2+1)},\ \mathcal{Q}_5 = \mathcal{B}_5 - \frac{(b-a)^{p_2+q_2}}{\Gamma(p_2+q_2+1)},\end{aligned} \quad (3.20)$$

where $\mathcal{A}_i,\ \mathcal{B}_i\ (i = 0, \dots, 5)$ are respectively given by (3.4) and (3.5).

Theorem 3.5. *Assume that* $h, \phi, u, \psi : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous functions satisfying the condition* $(\mathcal{H}_2)$. *Furthermore, we assume that there exist positive constants* $\mathsf{F}_i\ (i = 1, \dots, 4)$ *such that* $\forall\ t \in [a, b]$ *and* $x, y \in \mathbb{R}$,

$$|h(t,x,y)| \le \mathsf{F}_1,\ |\phi(t,x,y)| \le \mathsf{F}_2,\ |u(t,x,y)| \le \mathsf{F}_3 \ \text{and}\ |\psi(t,x,y)| \le \mathsf{F}_4. \quad (3.21)$$

Then the system (1.1)–(1.2) *has at least one solution on* $[a, b]$, *if*

$$(|\kappa_1|\mathcal{Q}_0 + |\lambda_1|L_1\mathcal{Q}_1 + L_2\mathcal{Q}_2 + \Delta_2) + (\overline{\Delta}_1 + |\kappa_2|\mathcal{Q}_3 + |\lambda_2|L_3\mathcal{Q}_4 + L_4\mathcal{Q}_5) < 1, \quad (3.22)$$

where $\Delta_2,\ \overline{\Delta}_1$ *and* $\mathcal{Q}_i\ (i = 0, \dots, 5)$ *are defined by* (3.17), (3.18) *and* (3.20) *respectively.*

Proof. Consider a closed ball $M_{r^*} = \{(x, y) \in \mathcal{X} \times \mathcal{X} : \|(x, y)\| \le r^*\}$ with

$$r^* \ge \max\{\widehat{r}_1, \widehat{r}_2\}, \quad (3.23)$$

where

$$\begin{aligned}\widehat{r}_1 &= \frac{|\lambda_1|\mathcal{A}_1\mathsf{F}_1 + \mathcal{A}_2\mathsf{F}_2 + |\lambda_2|\mathcal{A}_4\mathsf{F}_3 + \mathcal{A}_5\mathsf{F}_4}{1 - |\kappa_1|\mathcal{A}_0 - |\kappa_2|\mathcal{A}_3},\\ \widehat{r}_2 &= \frac{|\lambda_1|\mathcal{B}_1\mathsf{F}_1 + \mathcal{B}_2\mathsf{F}_2 + |\lambda_2|\mathcal{B}_4\mathsf{F}_3 + \mathcal{B}_5\mathsf{F}_4}{1 - |\kappa_1|\mathcal{B}_0 - |\kappa_2|\mathcal{B}_3}.\end{aligned}$$

In order to verify the hypotheses of Lemma 3.4, we decompose the operator $\mathcal{T}$ into four operators $\mathcal{T}_{1,1}, \mathcal{T}_{1,2}, \mathcal{T}_{2,1}$ and $\mathcal{T}_{2,2}$ on M_{r^*} as follows:

$$\begin{aligned}\mathcal{T}_{1,1}(x,y)(t) &= -\kappa_1 \int_a^t \frac{(t-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds - \lambda_1 \int_a^t \frac{(t-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} h(s,x(s),y(s))ds\\ &\quad + \int_a^t \frac{(t-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \phi(s,x(s),y(s))ds,\end{aligned}$$

$$\mathcal{T}_{1,2}(x,y)(t)$$

$$
\begin{aligned}
= \quad & f_1(t)\Big[\kappa_1\int_a^b \frac{(b-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds+\lambda_1\int_a^b \frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds \\
& -\int_a^b \frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\phi(s,x(s),y(s))ds\Big]+f_6(t)\Big[\kappa_1\int_a^b \frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)}x(s)ds \\
& +\lambda_1\int_a^b \frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)}h(s,x(s),y(s))ds-\int_a^b \frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)}\phi(s,x(s),y(s))ds\Big] \\
& +f_2(t)\Big[-\kappa_1\sum_{i=1}^{n-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds \\
& -\lambda_1\sum_{i=1}^{n-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds \\
& +\sum_{i=1}^{n-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\phi(s,x(s),y(s))ds\Big]+f_3(t)\Big[\kappa_2\int_a^b \frac{(b-s)^{p_2-1}}{\Gamma(p_2)}y(s)ds \\
& +\lambda_2\int_a^b \frac{(b-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}u(s,x(s),y(s))ds-\int_a^b \frac{(b-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\psi(s,x(s),y(s))ds\Big] \\
& +f_4(t)\Big[\kappa_2\int_a^b \frac{(b-s)^{p_2-2}}{\Gamma(p_2-1)}y(s)ds+\lambda_2\int_a^b \frac{(b-s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2+p_2-1)}u(s,x(s),y(s))ds \\
& -\int_a^b \frac{(b-s)^{p_2+q_2-2}}{\Gamma(p_2+q_2-1)}\psi(s,x(s),y(s))ds\Big]+f_5(t)\Big[-\kappa_2\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2-1}}{\Gamma(p_2)}y(s)ds \\
& -\lambda_2\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}u(s,x(s),y(s))ds \\
& +\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\psi(s,x(s),y(s))ds\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_{2,1}(x,y)(t) \quad = \quad & -\kappa_2\int_a^t \frac{(t-s)^{p_2-1}}{\Gamma(p_2)}y(s)ds-\lambda_2\int_a^t \frac{(t-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}u(s,x(s),y(s))ds \\
& +\int_a^t \frac{(t-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\psi(s,x(s),y(s))ds,
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{T}_{2,2}(x,y)(t) \\
= \quad & g_1(t)\Big[\kappa_1\int_a^b \frac{(b-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds+\lambda_1\int_a^b \frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds \\
& -\int_a^b \frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\phi(s,x(s),y(s))ds\Big]+g_2(t)\Big[\kappa_1\int_a^b \frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)}x(s)ds \\
& +\lambda_1\int_a^b \frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)}h(s,x(s),y(s))ds-\int_a^b \frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)}\phi(s,x(s),y(s))ds\Big] \\
& +g_6(t)\Big[-\kappa_1\sum_{i=1}^{n-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds
\end{aligned}
$$

$$-\lambda_1\sum_{i=1}^{n-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds$$
$$+\sum_{i=1}^{n-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\phi(s,x(s),y(s))ds\Big]+g_4(t)\Big[\kappa_2\int_a^b\frac{(b-s)^{p_2-1}}{\Gamma(p_2)}y(s)ds$$
$$+\lambda_2\int_a^b\frac{(b-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}u(s,x(s),y(s))ds-\int_a^b\frac{(b-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\psi(s,x(s),y(s))ds\Big]$$
$$+g_5(t)\Big[\kappa_2\int_a^b\frac{(b-s)^{p_2-2}}{\Gamma(p_2-1)}y(s)ds+\lambda_2\int_a^b\frac{(b-s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2+p_2-1)}u(s,x(s),y(s))ds$$
$$-\int_a^b\frac{(b-s)^{p_2+q_2-2}}{\Gamma(p_2+q_2-1)}\psi(s,x(s),y(s))ds\Big]+g_3(t)\Big[-\kappa_2\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2-1}}{\Gamma(p_2)}y(s)ds$$
$$-\lambda_2\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}u(s,x(s),y(s))ds$$
$$+\sum_{i=1}^{n-2}\alpha_i\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\psi(s,x(s),y(s))ds\Big].$$

Notice that $\mathcal{T}_1(x,y)(t)=\mathcal{T}_{1,1}(x,y)(t)+\mathcal{T}_{1,2}(x,y)(t)$ and $\mathcal{T}_2(x,y)(t)=\mathcal{T}_{2,1}(x,y)(t)+\mathcal{T}_{2,2}(x,y)(t)$ on M_{r^*}. To verify condition (a) of Lemma 3.4, we use (3.23) to show that $\mathcal{T}M_{r^*}\subset M_{r^*}$. Setting $x=(x_1,x_2), y=(y_1,y_2), \hat{x}=(\hat{x}_1,\hat{x}_2)$ and $\hat{y}=(\hat{y}_1,\hat{y}_2)\in M_{r^*}$, and using condition (3.21), we obtain

$$|\mathcal{T}_{1,1}(x,y)(t)+\mathcal{T}_{1,2}(\hat{x},\hat{y})(t)|$$
$$\leq \sup_{t\in[a,b]}\Big\{|\kappa_1|\int_a^t\frac{(t-s)^{p_1-1}}{\Gamma(p_1)}|x(s)|ds+|\lambda_1|\int_a^t\frac{(t-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\mathsf{F}_1ds$$
$$+\int_a^t\frac{(t-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\mathsf{F}_2ds+|f_1(t)|\Big[|\kappa_1|\int_a^b\frac{(b-s)^{p_1-1}}{\Gamma(p_1)}|\hat{x}(s)|ds$$
$$+|\lambda_1|\int_a^b\frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\mathsf{F}_1ds+\int_a^b\frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\mathsf{F}_2ds\Big]$$
$$+|f_6(t)|\Big[|\kappa_1|\int_a^b\frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)}|\hat{x}(s)|ds+|\lambda_1|\int_a^b\frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)}\mathsf{F}_1ds$$
$$+\int_a^b\frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)}\mathsf{F}_2ds\Big]+|f_2(t)|\Big[|\kappa_1|\sum_{i=1}^{n-2}|\beta_i|\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1-1}}{\Gamma(p_1)}|\hat{x}(s)|ds$$
$$+|\lambda_1|\sum_{i=1}^{n-2}|\beta_i|\int_a^{\eta_i}\frac{(\eta_i-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\mathsf{F}_1ds+\sum_{i=1}^{n-2}|\beta_i|\int_a^{\eta_i}\frac{(\eta_i-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\mathsf{F}_2ds\Big]$$
$$+|f_3(t)|\Big[|\kappa_2|\int_a^b\frac{(b-s)^{p_2-1}}{\Gamma(p_2)}|\hat{y}(s)|ds+|\lambda_2|\int_a^b\frac{(b-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}\mathsf{F}_3ds$$
$$+\int_a^b\frac{(b-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\mathsf{F}_4ds\Big]+|f_4(t)|\Big[|\kappa_2|\int_a^b\frac{(b-s)^{p_2-2}}{\Gamma(p_2-1)}|\hat{y}(s)|ds$$

$$
\begin{aligned}
&+|\lambda_2|\int_a^b \frac{(b-s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2+p_2-1)}\mathsf{F}_3 ds + \int_a^b \frac{(b-s)^{p_2+q_2-2}}{\Gamma(p_2+q_2-1)}\mathsf{F}_4 ds\Big]\\
&+|f_5(t)|\Big[|\kappa_2|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2-1}}{\Gamma(p_2)}|\hat{y}(s)|ds\\
&+|\lambda_2|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}\mathsf{F}_3 ds + \sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\mathsf{F}_4 ds\Big]\Big\}\\
\leq\ & |\kappa_1|\mathcal{A}_0 r^* + |\lambda_1|\mathcal{A}_1\mathsf{F}_1 + \mathcal{A}_2\mathsf{F}_2 + |\kappa_2|\mathcal{A}_3 r^* + |\lambda_2|\mathcal{A}_4\mathsf{F}_3 + \mathcal{A}_5\mathsf{F}_4 \leq r^*.
\end{aligned}
$$

In a similar manner, we can find that

$$
|\mathcal{T}_{2,1}(x,y)(t)+\mathcal{T}_{2,2}(\hat{x},\hat{y})(t)| \leq |\kappa_1|\mathcal{B}_0 r^* + |\lambda_1|\mathcal{B}_1\mathsf{F}_1 + \mathcal{B}_2\mathsf{F}_2 + |\kappa_2|\mathcal{B}_3 r^* + |\lambda_2|\mathcal{B}_4\mathsf{F}_3 + \mathcal{B}_5\mathsf{F}_4 \leq r^*.
$$

It clearly follows from the above two inequalities that $\mathcal{T}_1(x,y)+\mathcal{T}_2(\hat{x},\hat{y}) \in M_{r^*}$. Next we show that the operator $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is compact and continuous, which means that the condition (b) of Lemma 3.4 is satisfied. Continuity of $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ follows from that of h,ϕ,u,ψ. For each $(x,y)\in M_{r^*}$, we have

$$
\begin{aligned}
&\|\mathcal{T}_{1,1}(x,y)\| \leq \sup_{t\in[a,b]}|\mathcal{T}_{1,1}(x,y)(t)|\\
\leq\ & \sup_{t\in[a,b]}\Big| -\kappa_1\int_a^t \frac{(t-s)^{p_1-1}}{\Gamma(p_1)}x(s)ds - \lambda_1\int_a^t\frac{(t-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}h(s,x(s),y(s))ds\\
&+\int_a^t\frac{(t-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\phi(s,x(s),y(s))ds\Big|\\
\leq\ & |\kappa_1|\frac{(b-a)^{p_1}r^*}{\Gamma(p_1+1)} + |\lambda_1|\frac{(b-a)^{\gamma_1+p_1}\mathsf{F}_1}{\Gamma(\gamma_1+p_1+1)} + \frac{(b-a)^{p_1+q_1}\mathsf{F}_2}{\Gamma(p_1+q_1+1)} = \mathcal{S}_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\mathcal{T}_{2,1}(x,y)\| \leq \sup_{t\in[a,b]}|\mathcal{T}_{2,1}(x,y)(t)|\\
\leq\ & \sup_{t\in[a,b]}\Big| -\kappa_2\int_a^t \frac{(t-s)^{p_2-1}}{\Gamma(p_2)}y(s)ds - \lambda_2\int_a^t\frac{(t-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}u(s,x(s),y(s))ds\\
&+\int_a^t\frac{(t-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\psi(s,x(s),y(s))ds\Big|\\
\leq\ & |\kappa_2|\frac{(b-a)^{p_2}r^*}{\Gamma(p_2+1)} + |\lambda_2|\frac{(b-a)^{\gamma_2+p_2}\mathsf{F}_3}{\Gamma(\gamma_2+p_2+1)} + \frac{(b-a)^{p_2+q_2}\mathsf{F}_4}{\Gamma(p_2+q_2+1)} = \mathcal{S}_2,
\end{aligned}
$$

which leads to

$$
\|(\mathcal{T}_{1,1},\mathcal{T}_{2,1})(x,y)\| \leq \mathcal{S}_1+\mathcal{S}_2.
$$

Thus the set $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})M_{r^*}$ is uniformly bounded. Furthermore, we show that the set $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})M_{r^*}$ is equicontinuous. For $a\leq t_1<t_2\leq b$ and for any $(x,y)\in M_{r^*}$, we

obtain

$$
\begin{aligned}
&|\mathcal{T}_{1,1}(x,y)(t_2) - \mathcal{T}_{1,1}(x,y)(t_1)| \\
&\le \sup_{t\in[a,b]} \Big| -\kappa_1 \int_a^{t_1} \frac{(t_2-s)^{p_1-1} - (t_1-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds - \kappa_1 \int_{t_1}^{t_2} \frac{(t_2-s)^{p_1-1}}{\Gamma(p_1)} x(s)ds \\
&- \lambda_1 \int_a^{t_1} \frac{(t_2-s)^{\gamma_1+p_1-1} - (t_1-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} h(s,x(s),y(s))ds \\
&- \lambda_1 \int_{t_1}^{t_2} \frac{(t_2-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)} h(s,x(s),y(s))ds \\
&+ \int_a^{t_1} \frac{(t_2-s)^{p_1+q_1-1} - (t_1-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \phi(s,x(s),y(s))ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)} \phi(s,x(s),y(s))ds \Big| \\
&\le \frac{|\kappa_1| r}{\Gamma(p_1+1)} \Big(|(t_2-a)^{p_1} - (t_1-a)^{p_1}| + 2(t_2-t_1)^{p_1} \Big) \\
&+ \frac{|\lambda_1| \mathsf{F}_1}{\Gamma(\gamma_1+p_1+1)} \Big(|(t_2-a)^{\gamma_1+p_1} - (t_1-a)^{\gamma_1+p_1}| + 2(t_2-t_1)^{\gamma_1+p_1} \Big) \\
&+ \frac{\mathsf{F}_2}{\Gamma(p_1+q_1+1)} \Big(|(t_2-a)^{p_1+q_1} - (t_1-a)^{p_1+q_1}| + 2(t_2-t_1)^{p_1+q_1} \Big).
\end{aligned}
$$

Similarly, we can get

$$
\begin{aligned}
|\mathcal{T}_{2,1}(x,y)(t_2) - \mathcal{T}_{2,1}(x,y)(t_1)| & \\
\le \frac{|\kappa_2| r}{\Gamma(p_2+1)} & \Big(|(t_2-a)^{p_2} - (t_1-a)^{p_2}| + 2(t_2-t_1)^{p_2} \Big) \\
+ \frac{|\lambda_2| \mathsf{F}_3}{\Gamma(\gamma_2+p_2+1)} & \Big(|(t_2-a)^{\gamma_2+p_2} - (t_1-a)^{\gamma_2+p_2}| + 2(t_2-t_1)^{\gamma_2+p_2} \Big) \\
+ \frac{\mathsf{F}_4}{\Gamma(p_2+q_2+1)} & \Big(|(t_2-a)^{p_2+q_2} - (t_1-a)^{p_2+q_2}| + 2(t_2-t_1)^{p_2+q_2} \Big).
\end{aligned}
$$

Hence, $|(\mathcal{T}_{1,1},\mathcal{T}_{2,1})(x,y)(t_2) - (\mathcal{T}_{1,1},\mathcal{T}_{2,1})(x,y)(t_1)|$ tends to zero as $t_1 \to t_2$ independent of $(x,y) \in M_{r^*}$. Therefore, the set $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})M_{r^*}$ is equicontinuous. Thus, by the Arzelà–Ascoli theorem, the operator $(\mathcal{T}_{1,1},\mathcal{T}_{2,1})$ is compact on M_{r^*}. Next we show that the operator $(\mathcal{T}_{1,2},\mathcal{T}_{2,2})$ is a contraction satisfying condition (c) of Lemma 3.4. For $(x_1,y_1),(x_2,y_2) \in M_{r^*},\ t \in [a,b]$, we have

$$
\begin{aligned}
&\|\mathcal{T}_{1,2}(x_1,y_1) - \mathcal{T}_{1,2}(x_2,y_2)\| = \sup_{t\in[a,b]} |\mathcal{T}_{1,2}(x_1,y_1)(t) - \mathcal{T}_{1,2}(x_2,y_2)(t)| \\
&\le \sup_{t\in[a,b]} \Big\{ |f_1(t)| \Big[|\kappa_1| \int_a^b \frac{(b-s)^{p_1-1}}{\Gamma(p_1)} |x_1(s) - x_2(s)| ds
\end{aligned}
$$

$$+|\lambda_1|\int_a^b \frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\Big|h(s,x_1(s),y_1(s))-h(s,x_2(s),y_2(s))\Big|ds$$
$$+\int_a^b \frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\Big|\phi(s,x_1(s),y_1(s))-\phi(s,x_2(s),y_2(s))\Big|ds\Big]$$
$$+|f_6(t)|\Big[|\kappa_1|\int_a^b \frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)}|x_1(s)-x_2(s)|ds$$
$$+|\lambda_1|\int_a^b \frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)}\Big|h(s,x_1(s),y_1(s))-h(s,x_2(s),y_2(s))\Big|ds$$
$$+\int_a^b \frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)}\Big|\phi(s,x_1(s),y_1(s))-\phi(s,x_2(s),y_2(s))\Big|ds\Big]$$
$$+|f_2(t)|\Big[|\kappa_1|\sum_{i=1}^{n-2}|\beta_i|\int_a^{\eta_i} \frac{(\eta_i-s)^{p_1-1}}{\Gamma(p_1)}|x_1(s)-x_2(s)|ds$$
$$+|\lambda_1|\sum_{i=1}^{n-2}|\beta_i|\int_a^{\eta_i} \frac{(\eta_i-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\Big|h(s,x_1(s),y_1(s))-h(s,x_2(s),y_2(s))\Big|ds$$
$$+\sum_{i=1}^{n-2}|\beta_i|\int_a^{\eta_i} \frac{(\eta_i-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\Big|\phi(s,x_1(s),y_1(s))-\phi(s,x_2(s),y_2(s))\Big|ds\Big]$$
$$+|f_3(t)|\Big[|\kappa_2|\int_a^b \frac{(b-s)^{p_2-1}}{\Gamma(p_2)}|y_1(s)-y_2(s)|ds$$
$$+|\lambda_2|\int_a^b \frac{(b-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}\Big|u(s,x_1(s),y_1(s))-u(s,x_2(s),y_2(s))\Big|ds$$
$$+\int_a^b \frac{(b-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\Big|\psi(s,x_1(s),y_1(s))-\psi(s,x_2(s),y_2(s))\Big|ds\Big]$$
$$+|f_4(t)|\Big[|\kappa_2|\int_a^b \frac{(b-s)^{p_2-2}}{\Gamma(p_2-1)}|y_1(s)-y_2(s)|ds$$
$$+|\lambda_2|\int_a^b \frac{(b-s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2+p_2-1)}\Big|u(s,x_1(s),y_1(s))-u(s,x_2(s),y_2(s))\Big|ds$$
$$+\int_a^b \frac{(b-s)^{p_2+q_2-2}}{\Gamma(p_2+q_2-1)}\Big|\psi(s,x_1(s),y_1(s))-\psi(s,x_2(s),y_2(s))\Big|ds\Big]$$
$$+|f_5(t)|\Big[|\kappa_2|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i} \frac{(\xi_i-s)^{p_2-1}}{\Gamma(p_2)}|y_1(s)-y_2(s)|ds$$
$$+|\lambda_2|\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i} \frac{(\xi_i-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}\Big|u(s,x_1(s),y_1(s))-u(s,x_2(s),y_2(s))\Big|ds$$
$$+\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i} \frac{(\xi_i-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\Big|\psi(s,x_1(s),y_1(s))-\psi(s,x_2(s),y_2(s))|ds\Big]\Big\}$$

$$
\begin{aligned}
&\leq \Big\{|\kappa_1|\mathcal{Q}_0\|x_1-x_2\| + |\lambda_1|L_1\mathcal{Q}_1\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big)\\
&\quad + L_2\mathcal{Q}_2\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big) + |\kappa_2|\mathcal{A}_3\|y_1-y_2\|\\
&\quad + |\lambda_2|L_3\mathcal{A}_4\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big) + L_4\mathcal{A}_5\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big)\Big\}\\
&\leq \Big(|\kappa_1|\mathcal{Q}_0 + |\lambda_1|L_1\mathcal{Q}_1 + L_2\mathcal{Q}_2 + |\kappa_2|\mathcal{A}_3 + |\lambda_2|L_3\mathcal{A}_4 + L_4\mathcal{A}_5\Big)\\
&\quad \times\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big)\\
&= \Big(|\kappa_1|\mathcal{Q}_0 + |\lambda_1|L_1\mathcal{Q}_1 + L_2\mathcal{Q}_2 + \Delta_2\Big)\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big), \qquad (3.24)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\mathcal{T}_{2,2}(x_1,y_1) - \mathcal{T}_{2,2}(x_2,y_2)\|\\
&\leq \Big(|\kappa_1|\mathcal{B}_0 + |\lambda_1|L_1\mathcal{B}_1 + L_2\mathcal{B}_2 + |\kappa_2|\mathcal{Q}_3 + |\lambda_2|L_3\mathcal{Q}_4 + L_4\mathcal{Q}_5\Big)\\
&\quad \times\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big)\\
&= \Big(\overline{\Delta}_1 + |\kappa_2|\mathcal{Q}_3 + |\lambda_2|L_3\mathcal{Q}_4 + L_4\mathcal{Q}_5\Big)\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big). \qquad (3.25)
\end{aligned}
$$

It follows from (3.24) and (3.25) that

$$
\begin{aligned}
&\|(\mathcal{T}_{1,2},\mathcal{T}_{2,2})(x_1,y_1) - (\mathcal{T}_{1,2},\mathcal{T}_{2,2})(x_2,y_2)\|\\
&\leq \Big[\Big(|\kappa_1|\mathcal{Q}_0 + |\lambda_1|L_1\mathcal{Q}_1 + L_2\mathcal{Q}_2 + \Delta_2\Big)\\
&\quad + \Big(\overline{\Delta}_1 + |\kappa_2|\mathcal{Q}_3 + |\lambda_2|L_3\mathcal{Q}_4 + L_4\mathcal{Q}_5\Big)\Big]\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big),
\end{aligned}
$$

which, in view of (3.22), implies that the operator $(\mathcal{T}_{1,2},\mathcal{T}_{2,2})$ is a contraction. Therefore, the condition (c) of Lemma 3.4 is satisfied. Thus, we deduce by the conclusion of Lemma 3.4 that the system (1.1)–(1.2) has at least one solution on $[a,b]$. □

Example 3.6. Consider the problem (3.12)–(3.13) with

$$
\begin{cases}
h(t,x(t),y(t)) = \dfrac{1}{\sqrt{2500+t}}\Big(2\cos x(t) + \dfrac{2|y(t)|}{1+|y(t)|}\Big),\\
\phi(t,x(t),y(t)) = \dfrac{t^2+2}{270}\Big(\dfrac{(x(t)+5)^2}{7+(x(t)+5)^2} + y(t) + \ln 3\Big),\\
u(t,x(t),y(t)) = \dfrac{1}{8+t}\Big(\dfrac{\tan^{-1}x(t)}{12} + \dfrac{y(t)+13}{12}\Big),\\
\psi(t,x(t),y(t)) = \dfrac{e^{-t}}{14\sqrt{100+t^6}}\Big(\sin x(t) + \cos y(t) + \dfrac{7}{40}\Big).
\end{cases} \qquad (3.26)
$$

Observe that

$$
|h(t,x_1,y_1) - h(t,x_2,y_2)| \leq \frac{1}{25}\Big(|x_1-x_2| + |y_1-y_2|\Big),
$$

$$|\phi(t,x_1,y_1)-\phi(t,x_2,y_2)| \le \frac{1}{90}\Big(|x_1-x_2|+|y_1-y_2|\Big),$$
$$|u(t,x_1,y_1)-u(t,x_2,y_2)| \le \frac{1}{96}\Big(|x_1-x_2|+|y_1-y_2|\Big),$$
$$|\psi(t,x_1,y_1)-\psi(t,x_2,y_2)| \le \frac{1}{140}\Big(|x_1-x_2|+|y_1-y_2|\Big).$$

With the given data, we find that $\mathcal{Q}_0 \simeq 0.767776$, $\mathcal{Q}_1 \simeq 0.001045$, $\mathcal{Q}_2 \simeq 0.153414$, $\mathcal{Q}_3 \simeq 0.610335$, $\mathcal{Q}_4 \simeq 0.003852$, $\mathcal{Q}_5 \simeq 0.143089$, $\Delta_2 \simeq 0.037588$, $\overline{\Delta}_1 \simeq 0.022760$, and $(|\kappa_1|\mathcal{Q}_0+|\lambda_1|L_1\mathcal{Q}_1+L_2\mathcal{Q}_2+\Delta_2)+(\overline{\Delta}_1+|\kappa_2|\mathcal{Q}_3+|\lambda_2|L_3\mathcal{Q}_4+L_4\mathcal{Q}_5) \simeq 0.141887 < 1$. As the hypothesis of Theorem 3.5 holds true, therefore its conclusion applies to the coupled boundary value problem (3.12)–(3.13) with $h(t,x(t),y(t))$, $\phi(t,x(t),y(t))$, $u(t,x(t),y(t))$ and $\psi(t,x(t),y(t))$ given by (3.26).

4 Uniqueness of Solutions

In this section we apply Banach's fixed point theorem to prove the uniqueness of solutions for the system (1.1)–(1.2). Before proceeding for this result, let us introduce the following notation:

$$\mathcal{M} = \mathcal{M}_1+\mathcal{M}_2,\ \mathcal{M}_1=|\lambda_1|\mathcal{N}_1\mathcal{A}_1+\mathcal{N}_2\mathcal{A}_2,\ \mathcal{M}_2=|\lambda_2|\mathcal{N}_3\mathcal{A}_4+\mathcal{N}_4\mathcal{A}_5, \tag{4.1}$$
$$\overline{\mathcal{M}} = \overline{\mathcal{M}}_1+\overline{\mathcal{M}}_2,\ \overline{\mathcal{M}}_1=|\lambda_1|\mathcal{N}_1\mathcal{B}_1+\mathcal{N}_2\mathcal{B}_2,\ \overline{\mathcal{M}}_2=|\lambda_2|\mathcal{N}_3\mathcal{B}_4+\mathcal{N}_4\mathcal{B}_5, \tag{4.2}$$
$$\mathcal{N}_1 = \sup_{t\in[a,b]}|h(t,0,0)|<\infty,\ \mathcal{N}_2=\sup_{t\in[a,b]}|\phi(t,0,0,)|<\infty,$$
$$\mathcal{N}_3 = \sup_{t\in[a,b]}|u(t,0,0,)|<\infty, \mathcal{N}_4=\sup_{t\in[a,b]}|\psi(t,0,0,)|<\infty. \tag{4.3}$$

Theorem 4.1. *Assume that the condition* $(\mathcal{H}_2)$ *holds. If*

$$\Omega+\overline{\Omega}<1, \tag{4.4}$$

where Ω *and* $\overline{\Omega}$ *are given by* (3.14) *and* (3.15) *respectively, then the system* (1.1)–(1.2) *has a unique solution on* $[a,b]$.

Proof. Setting $\varrho > \dfrac{\mathcal{M}+\overline{\mathcal{M}}}{1-\Omega-\overline{\Omega}}$, where $\Omega, \overline{\Omega}, \mathcal{M}$ and $\overline{\mathcal{M}}$ are given by (3.14), (3.15), (4.1) and (4.2) respectively, we first show that $\mathcal{T}S_\varrho \subset S_\varrho$, where $S_\varrho=\{(x,y)\in\mathcal{X}\times\mathcal{X} : \|(x,y)\|\le\varrho\}$, and the operator $\mathcal{T}$ is given by (3.1).

Using the assumption $(\mathcal{H}_2)$ together with (4.3), for $(x,y)\in S_\varrho,\ t\in[a,b]$, we have

$$\begin{aligned}
|h(t,x(t),y(t))| &\le |h(t,x(t),y(t))-h(t,0,0)|+|h(t,0,0)|\\
&\le L_1(\|x\|+\|y\|)+\mathcal{N}_1 \le L_1\varrho+\mathcal{N}_1,\\
|\phi(t,x(t),y(t))| &\le |\phi(t,x(t),y(t))-\phi(t,0,0)|+|\phi(t,0,0)| \le L_2\varrho+\mathcal{N}_2,\\
|u(t,x(t),y(t))| &\le |u(t,x(t),y(t))-u(t,0,0)|+|u(t,0,0)| \le L_3\varrho+\mathcal{N}_3,
\end{aligned}$$

$$|\psi(t,x(t),y(t))| \leq |\psi(t,x(t),y(t))-\psi(t,0,0)|+|\psi(t,0,0)| \leq L_4\varrho+\mathcal{N}_4.$$

In view of (3.14) and (4.1), we obtain

$$\begin{aligned}|\mathcal{T}_1(x,y)(t)| &\leq |\kappa_1|\|x\|\mathcal{A}_0+|\lambda_1|(L_1\varrho+\mathcal{N}_1)\mathcal{A}_1+(L_2\varrho+\mathcal{N}_2)\mathcal{A}_2\\ &\quad +|\kappa_2|\|y\|\mathcal{A}_3+|\lambda_2|(L_3\varrho+\mathcal{N}_3)\mathcal{A}_4+(L_4\varrho+\mathcal{N}_4)\mathcal{A}_5\\ &\leq \big(|\kappa_1|\mathcal{A}_0+|\lambda_1|L_1\mathcal{A}_1+L_2\mathcal{A}_2+|\kappa_2|\mathcal{A}_3+|\lambda_2|L_3\mathcal{A}_4+L_4\mathcal{A}_5\big)\varrho\\ &\quad +\big(|\lambda_1|\mathcal{N}_1\mathcal{A}_1+\mathcal{N}_2\mathcal{A}_2+|\lambda_2|\mathcal{N}_3\mathcal{A}_4+\mathcal{N}_4\mathcal{A}_5\big)\\ &= (\Delta_1+\Delta_2)\varrho+(\mathcal{M}_1+\mathcal{M}_2)\\ &= \Omega\varrho+\mathcal{M},\end{aligned}$$

whence

$$\|\mathcal{T}_1(x,y)\| \leq \Omega\varrho+\mathcal{M}. \tag{4.5}$$

In a similar way, one can obtain by using (3.15) and (4.2) that

$$\begin{aligned}|\mathcal{T}_2(x,y)(t)| &\leq \Big(|\kappa_1|\mathcal{B}_0+|\lambda_1|L_1\mathcal{B}_1+L_2\mathcal{B}_2+|\kappa_2|\mathcal{B}_3+|\lambda_2|L_3\mathcal{B}_4+L_4\mathcal{B}_5\Big)\varrho\\ &\quad +\Big(|\lambda_1|\mathcal{N}_1\mathcal{B}_1+\mathcal{N}_2\mathcal{B}_2+|\lambda_2|\mathcal{N}_3\mathcal{B}_4+\mathcal{N}_4\mathcal{B}_5\Big)\\ &= (\overline{\Delta}_1+\overline{\Delta}_2)\varrho+(\overline{\mathcal{M}}_1+\overline{\mathcal{M}}_2)\\ &= \overline{\Omega}\varrho+\overline{\mathcal{M}},\end{aligned}$$

which leads to

$$\|\mathcal{T}_2(x,y)\| \leq \overline{\Omega}\varrho+\overline{\mathcal{M}}. \tag{4.6}$$

From the inequalities (4.5) and (4.6), we get

$$\|\mathcal{T}(x,y)\| \leq (\Omega\varrho+\mathcal{M})+(\overline{\Omega}\varrho+\overline{\mathcal{M}}) \leq (\Omega+\overline{\Omega})\varrho+(\mathcal{M}+\overline{\mathcal{M}}) \leq \varrho,$$

which shows that $\mathcal{T}S_\varrho \subset S_\varrho$.

Now, for any $(x_1,y_1),(x_2,y_2)\in\mathcal{X}\times\mathcal{X}, t\in[a,b]$, it follows by using $(\mathcal{H}_2)$, (3.14) and (3.15) that

$$\begin{aligned}&\|\mathcal{T}_1(x_1,y_1)-\mathcal{T}_1(x_2,y_2)\| = \sup_{t\in[a,b]}|\mathcal{T}_1(x_1,y_1)(t)-\mathcal{T}_1(x_2,y_2)(t)|\\ &\leq \sup_{t\in[a,b]}\Big\{|\kappa_1|\int_a^t\frac{(t-s)^{p_1-1}}{\Gamma(p_1)}|x_1(s)-x_2(s)|ds\\ &+|\lambda_1|\int_a^t\frac{(t-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\big|h(s,x_1(s),y_1(s))-h(s,x_2(s),y_2(s))\big|ds\\ &+\int_a^t\frac{(t-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\big|\phi(s,x_1(s),y_1(s))-\phi(s,x_2(s),y_2(s))\big|ds\end{aligned}$$

$$+ |f_1(t)|\Big[|\kappa_1| \int_a^b \frac{(b-s)^{p_1-1}}{\Gamma(p_1)}|x_1(s) - x_2(s)|ds$$

$$+ |\lambda_1| \int_a^b \frac{(b-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\big|h(s, x_1(s), y_1(s)) - h(s, x_2(s), y_2(s))\big|ds$$

$$+ \int_a^b \frac{(b-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\big|\phi(s, x_1(s), y_1(s)) - \phi(s, x_2(s), y_2(s))\big|ds\Big]$$

$$+ |f_6(t)|\Big[|\kappa_1| \int_a^b \frac{(b-s)^{p_1-2}}{\Gamma(p_1-1)}|x_1(s) - x_2(s)|ds$$

$$+ |\lambda_1| \int_a^b \frac{(b-s)^{\gamma_1+p_1-2}}{\Gamma(\gamma_1+p_1-1)}\big|h(s, x_1(s), y_1(s)) - h(s, x_2(s), y_2(s))\big|ds$$

$$+ \int_a^b \frac{(b-s)^{p_1+q_1-2}}{\Gamma(p_1+q_1-1)}\big|\phi(s, x_1(s), y_1(s)) - \phi(s, x_2(s), y_2(s))\big|ds\Big]$$

$$+ |f_2(t)|\Big[|\kappa_1| \sum_{i=1}^{n-2} |\beta_i| \int_a^{\eta_i} \frac{(\eta_i-s)^{p_1-1}}{\Gamma(p_1)}|x_1(s) - x_2(s)|ds$$

$$+ |\lambda_1| \sum_{i=1}^{n-2} |\beta_i| \int_a^{\eta_i} \frac{(\eta_i-s)^{\gamma_1+p_1-1}}{\Gamma(\gamma_1+p_1)}\big|h(s, x_1(s), y_1(s)) - h(s, x_2(s), y_2(s))\big|ds$$

$$+ \sum_{i=1}^{n-2} |\beta_i| \int_a^{\eta_i} \frac{(\eta_i-s)^{p_1+q_1-1}}{\Gamma(p_1+q_1)}\big|\phi(s, x_1(s), y_1(s)) - \phi(s, x_2(s), y_2(s))\big|ds\Big]$$

$$+ |f_3(t)|\Big[|\kappa_2| \int_a^b \frac{(b-s)^{p_2-1}}{\Gamma(p_2)}|y_1(s) - y_2(s)|ds$$

$$+ |\lambda_2| \int_a^b \frac{(b-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}\big|u(s, x_1(s), y_1(s)) - u(s, x_2(s), y_2(s))\big|ds$$

$$+ \int_a^b \frac{(b-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\big|\psi(s, x_1(s), y_1(s)) - \psi(s, x_2(s), y_2(s))\big|ds\Big]$$

$$+ |f_4(t)|\Big[|\kappa_2| \int_a^b \frac{(b-s)^{p_2-2}}{\Gamma(p_2-1)}|y_1(s) - y_2(s)|ds$$

$$+ |\lambda_2| \int_a^b \frac{(b-s)^{\gamma_2+p_2-2}}{\Gamma(\gamma_2+p_2-1)}\big|u(s, x_1(s), y_1(s)) - u(s, x_2(s), y_2(s))\big|ds$$

$$+ \int_a^b \frac{(b-s)^{p_2+q_2-2}}{\Gamma(p_2+q_2-1)}\big|\psi(s, x_1(s), y_1(s)) - \psi(s, x_2(s), y_2(s))\big|ds\Big]$$

$$+ |f_5(t)|\Big[|\kappa_2| \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{p_2-1}}{\Gamma(p_2)}|y_1(s) - y_2(s)|ds$$

$$+ |\lambda_2| \sum_{i=1}^{n-2} |\alpha_i| \int_a^{\xi_i} \frac{(\xi_i-s)^{\gamma_2+p_2-1}}{\Gamma(\gamma_2+p_2)}\big|u(s, x_1(s), y_1(s)) - u(s, x_2(s), y_2(s))\big|ds$$

$$+\sum_{i=1}^{n-2}|\alpha_i|\int_a^{\xi_i}\frac{(\xi_i-s)^{p_2+q_2-1}}{\Gamma(p_2+q_2)}\big|\psi(s,x_1(s),y_1(s))-\psi(s,x_2(s),y_2(s))|ds\big]\Big\}$$
$$\leq\Big\{|\kappa_1|\mathcal{A}_0\|x_1-x_2\|+|\lambda_1|L_1\mathcal{A}_1\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big)$$
$$+L_2\mathcal{A}_2\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big)+|\kappa_2|\mathcal{A}_3\|y_1-y_2\|$$
$$+|\lambda_2|L_3\mathcal{A}_4\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big)+L_4\mathcal{A}_5\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big)\Big\}$$
$$\leq\Omega\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big).$$

Similarly, we can find that

$$\begin{aligned}\|\mathcal{T}_2(x_1,y_1)-\mathcal{T}_2(x_2,y_2)\| &= \sup_{t\in[a,b]}|\mathcal{T}_2(x_1,y_1)(t)-\mathcal{T}_2(x_2,y_2)(t)|\\ &\leq \Big(\overline{\Delta}_1+\overline{\Delta}_2\Big)\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big)\\ &= \overline{\Omega}\Big(\|x_1-x_2\|+\|y_1-y_2\|\Big).\end{aligned}$$

Consequently, we obtain

$$\|\mathcal{T}(x_1,y_1)-\mathcal{T}(x_2,y_2))\|\leq(\Omega+\overline{\Omega})(\|x_1-x_2\|+\|y_1-y_2\|),$$

which implies that the operator $\mathcal{T}$ is a contraction by the assumption (4.4)). Hence, by Banach's fixed point theorem, the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of the system (1.1)–(1.2) on $[a,b]$. □

Example 4.2. Consider the system (3.12)–(3.13) with

$$\begin{cases} h(t,x(t),y(t))=\dfrac{1}{\sqrt{64+t^2}}\Big(\tan^{-1}x(t)+\ln 7+y(t)\Big),\\ \phi(t,x(t),y(t))=\dfrac{t^2}{70\sqrt{4+t^6}}\Big(\dfrac{|x(t)|}{1+|x(t)|}+\sin y(t)+29\Big),\\ u(t,x(t),y(t))=\dfrac{e^{-t}+2}{30}\Big(x(t)+\cos y(t)\Big),\ \ t\in[0,1],\\ \psi(t,x(t),y(t))=\dfrac{t^4+4}{\sqrt{t^2+3600}}\Big(\sin x(t)+\dfrac{\tan^{-1}y(t)}{\sqrt{1+t^2}}\Big)+\dfrac{1}{4}.\end{cases}\tag{4.7}$$

It is easy to verify that $(\mathcal{H}_2)$ holds true with $L_1=1/8, L_2=1/140, L_3=1/10$, and $L_4=1/12$. Using these values together with the data of Example 3.3, we find that $\Omega+\overline{\Omega}\simeq 0.243797<1$. Thus, all the conditions of Theorem 4.1 are satisfied. Hence, by the conclusion of Theorem 4.1, the coupled system (3.12)–(3.13) with $h(t,x(t),y(t))$, $\phi(t,x(t),y(t))$, $u(t,x(t),y(t))$ and $\psi(t,x(t),y(t))$ given by (4.7) has a unique solution on $[0,1]$.

Remark 4.3. Fixing $\lambda_1=0=\lambda_2$ in the results of this paper, we obtain the ones for a nonlinear Langevin type system involving mixed Riemann-Liouville and Caputo fractional derivatives, supplemented with coupled multipoint boundary conditions.

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